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Direct and Inverse Scattering Problems in the Time Domain

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A Scattering in non-dispersive, lossless media

1. Imbedding approach

1.1 Basic equations

The Maxwell equations (source free region)

$$\boxed{\begin{aligned} \nabla \times \bar{E}(\bar{r},t) &= -\partial_t \bar{B}(\bar{r},t) \\ \nabla \times \bar{H}(\bar{r},t) &= \partial_t \bar{D}(\bar{r},t) \end{aligned}}$$

Constitutive relations
(lossless non-dispersive dielectric, variation only in z)

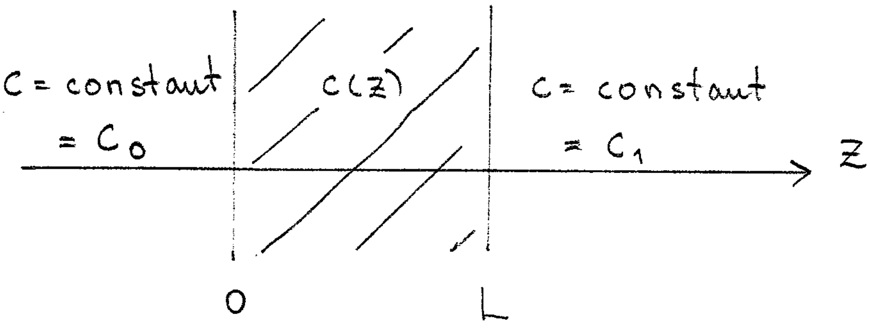
$$\boxed{\begin{aligned} \bar{D}(\bar{r},t) &= \epsilon(z)\bar{E}(\bar{r},t) \\ \bar{B}(\bar{r},t) &= \mu_0 \bar{H}(\bar{r},t) \end{aligned}}$$

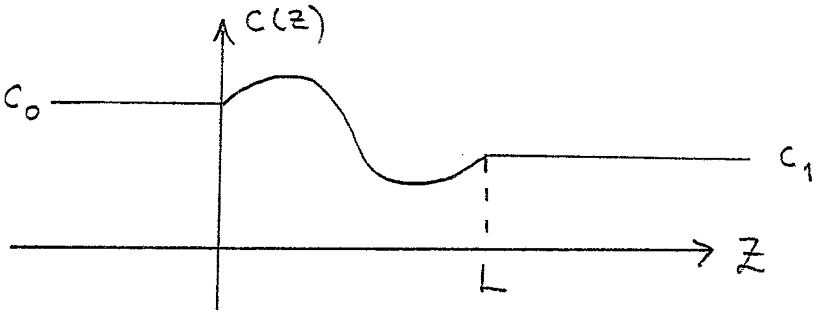
Phase velocity $c(z) = 1/\sqrt{\epsilon(z)\mu_0}$

Assume all fields depend only on (z,t) and are transverse to $\hat{z}$.

The electric field $E = u(z,t)$ satisfies

$$\boxed{\partial_z^2\, u(z,t) - c^{-2}(z)\,\partial_t^2\, u(z,t) = 0}$$





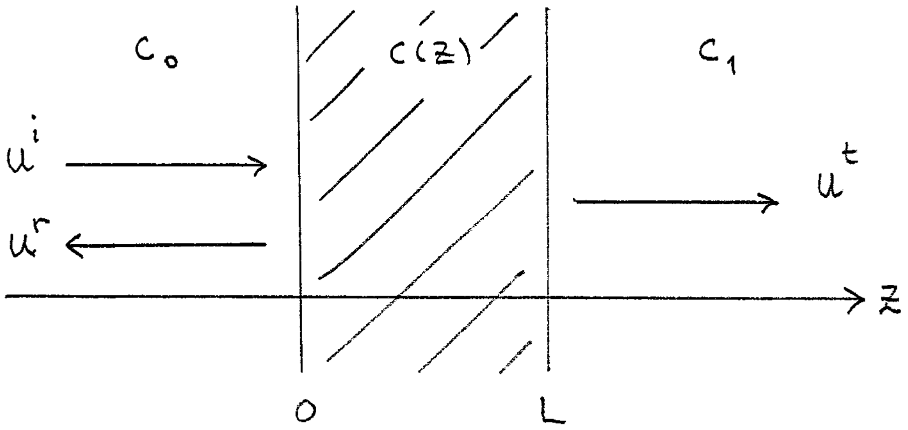
Assume $c(z)$ is continuous (within the slab cont. diff.)

General solution of the wave equation in a region where $c(z) = \text{constant} = c$

$$u(z,t) = f(t - z/c) + g(t + z/c)$$

right going left going waves

1.2 Scattering setup

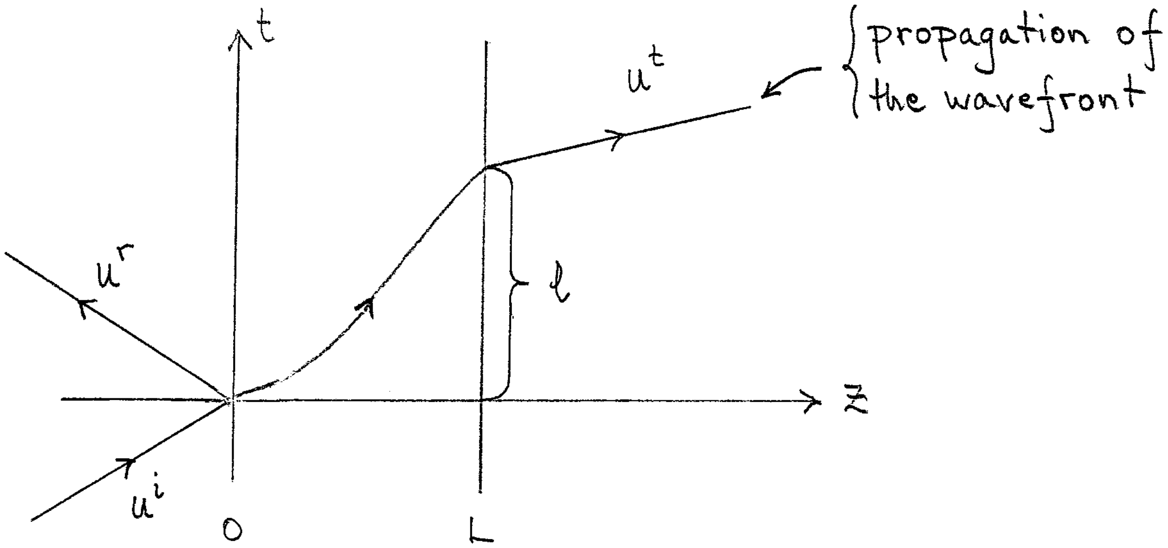


Solution:

$$u(z,t) = \begin{cases} u^i(t - z/c_0) + u^r(t + z/c_0) &, \quad z < 0 \\ u^t(t - \ell - (z-L)/c_1) &, \quad z > L \end{cases}$$

$$\ell = \int_0^L \frac{dz}{c(z)}$$ (time the wavefront takes to travel through the slab)

In a space-time diagram

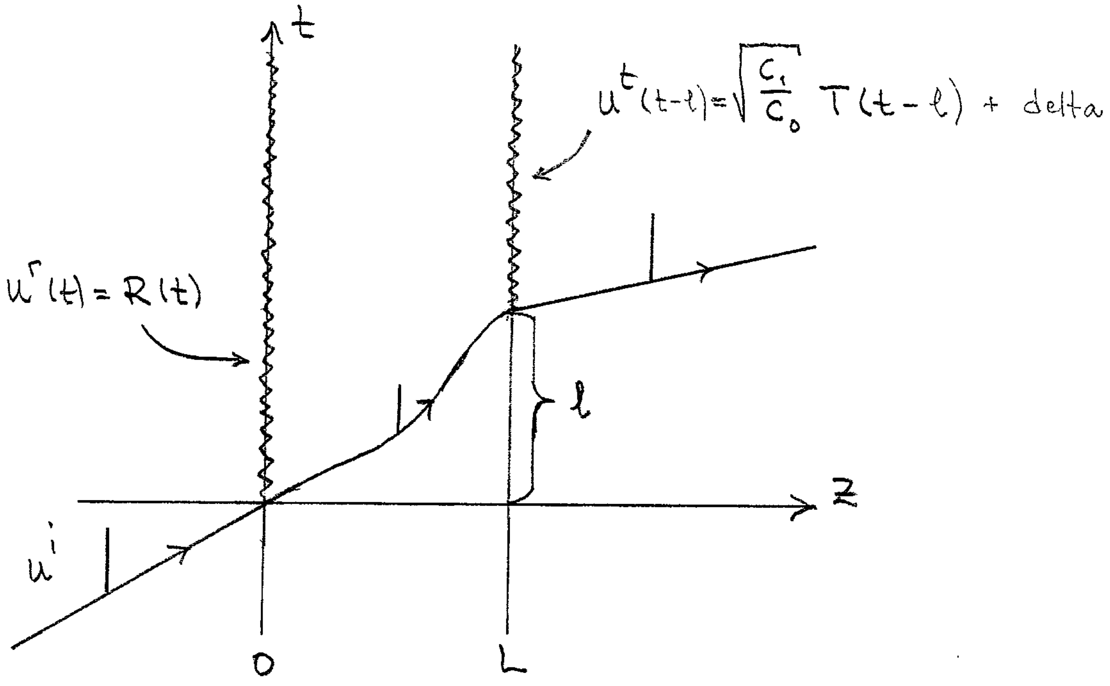


Reflection and transmission kernels

Relation between u^r, u^t and u^i [general property of linear, hyperbolic PDE with time independent coefficients, Duhamel's principle, see also below]

$$\begin{cases} u^r(t) = \int_{-\infty}^{t} R(t-t')\, u^i(t')\, dt' \\ u^t(t) = \sqrt{\frac{c_1}{c_0}} \left\{ u^i(t) + \int_{-\infty}^{t} T(t-t')\, u^i(t')\, dt' \right\} \end{cases}$$

Space - time again with $u^i(t - z/c_0) = \delta(t - z/c_0)$



Note! The kernels R and T are independent of the incident field. They are determined by the properties of the slab, i.e. by $c(z)$.
(More about this later.)

1.3 Wave splitting

Introduce the following transformation:

$$u^{\pm}(z,t) = \frac{1}{2}\left[u(z,t) \mp c(z) \int_{-\infty}^{t} u_z(z,t')\,dt' \right]$$

$$= \frac{1}{2}\left[u(z,t) \mp c(z)\, \partial_t^{-1} u_z(z,t) \right]$$

This can be viewed as a change of basis from $\begin{pmatrix} u \\ u_z \end{pmatrix}$ to $\begin{pmatrix} u^+ \\ u^- \end{pmatrix}$.

$$\boxed{\begin{pmatrix} u^+ \\ u^- \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -c\partial_t^{-1} \\ 1 & +c\partial_t^{-1} \end{pmatrix} \begin{pmatrix} u \\ u_z \end{pmatrix} = P \begin{pmatrix} u \\ u_z \end{pmatrix}}$$

P has a formal inverse: $P^{-1} = \begin{pmatrix} 1 & 1 \\ -c^{-1}\partial_t & c^{-1}\partial_t \end{pmatrix}$

This transformation defines the <u>wave splitting</u>

Note that the total field $\underline{\underline{u(z,t) = u^+(z,t) + u^-(z,t)}}$

In a region where $c =$ constant the wave splitting projects out the right and left going waves.

Proof: The general solution in such a region is:

$$u(z,t) = f(t - z/c) + g(t+z/c) \quad , \quad (c = \text{constant})$$

$$u^{\pm}(z,t) = \frac{1}{2}\left[u(z,t) \mp c \int_{-\infty}^{t} u_z(z,t')\,dt' \right]$$

$$= \frac{1}{2}\Big\{ f(t-z/c) + g(t+z/c)$$

$$\mp \int_{-\infty}^{t} \left[-f'(t'-z/c) + g'(t'+z/c)\right] dt' \Big\}$$

$$= \frac{1}{2}\Big\{ f(t-z/c) \pm f(t-z/c)$$

$$+ g(t+z/c) \mp g(t+z/c) \Big\}$$

$$\begin{cases} u^{+}(z,t) = f(t-z/c) & \text{(right going)} \\ u^{-}(z,t) = g(t+z/c) & \text{(left going)} \end{cases}$$

In a region where c is non-constant let the wave splitting <u>define</u> left and right going waves.

1.4 PDE (dynamics) for $u^\pm$

Rewrite the wave eq. $\left[\partial_z^2 - c^{-2}(z)\partial_t^2\right] u(z,t) = 0$ as

$$\boxed{\partial_z \begin{pmatrix} u \\ u_z \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ c^{-2}(z)\partial_t^2 & 0 \end{pmatrix} \begin{pmatrix} u \\ u_z \end{pmatrix} = D \begin{pmatrix} u \\ u_z \end{pmatrix}}$$

Formal differentiation on $\begin{pmatrix} u^+ \\ u^- \end{pmatrix} = P \begin{pmatrix} u \\ u_z \end{pmatrix}$ gives

$$\partial_z \begin{pmatrix} u^+ \\ u^- \end{pmatrix} = \partial_z \left[P \begin{pmatrix} u \\ u_z \end{pmatrix} \right] = P_z \begin{pmatrix} u \\ u_z \end{pmatrix} + P \underbrace{\partial_z \begin{pmatrix} u \\ u_z \end{pmatrix}}_{D\binom{u}{u_z}}$$

$$= P_z P^{-1} \begin{pmatrix} u^+ \\ u^- \end{pmatrix} + P D P^{-1} \begin{pmatrix} u^+ \\ u^- \end{pmatrix}$$

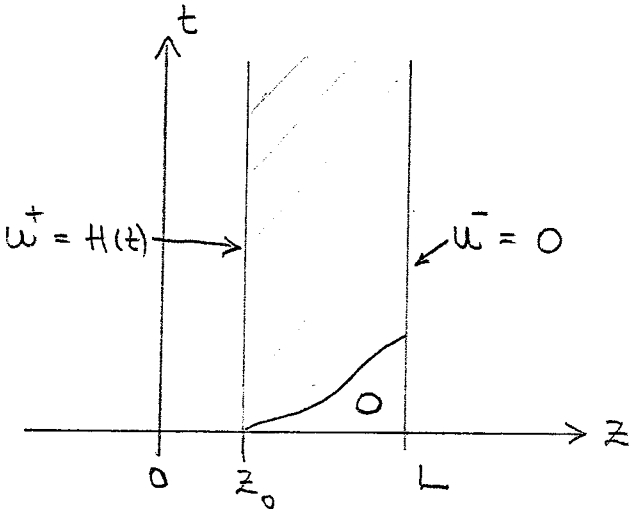
$$\boxed{\partial_z \begin{pmatrix} u^+ \\ u^- \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} u^+ \\ u^- \end{pmatrix}}$$

New PDE, equivalent to the wave eq.

$$\begin{cases} \alpha = -c^{-1}\partial_t + \frac{1}{2} c'/c \\ \beta = -\frac{1}{2} c'/c \\ \gamma = \beta \\ \delta = c^{-1}\partial_t + \frac{1}{2} c'/c \end{cases}$$

1.5 Duhamel's principle (relation between $u^\pm$)

Canonical problem for a subsection $[z_0, L]$ of the slab $[0, L]$.



$$\left\{ \begin{array}{lll} \partial_z \begin{pmatrix} u^+ \\ u^- \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} u^+ \\ u^- \end{pmatrix} & & z_0 < z < L \ , \ t > 0 \\ \text{Initial cond.} & u^\pm(z, 0) = 0 \ , & z_0 < z < L \\ & & \text{Heaviside fcn.} \\ \text{Bound. cond.} & \left\{ \begin{array}{l} u^+(z_0, t) = H(t) \\ u^-(L, t) = 0 \end{array} \right. & (\text{no sources } z > L) \end{array} \right.$$

This is a well-posed problem. Call the solution $U^\pm(z,t)$.

Causality implies that $U^\pm(z,t) = 0$ for $t < \ell(z) = \int_{z_0}^{z} \bar{c}^{-1}(z') dz'$

$\left(\text{Recall} \quad \ell = \int_0^L \bar{c}^{-1}(z) dz \right)$

The solutions $U^{\pm}(z,t)$ are continuously differentiable everywhere, except along the characteristic curve $t = \ell(z)$.

Causality holds: $U^{\pm}(z,t) = 0 \qquad t < \ell(z)$

Important observation:

If the boundary condition at z_0, $H(t)$, is replaced by $\lambda H(t-t_0)$ the solutions are: $\lambda U^{\pm}(z, t-t_0)$ (due to linearity, unique solvability and time independent coefficients in the PDE).

The full problem:

$$\begin{cases} \partial_z \begin{pmatrix} u^+ \\ u^- \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} u^+ \\ u^- \end{pmatrix}, \quad z_0 < z < L, \quad t > 0 \\ \text{Initial cond.} \quad u^{\pm}(z,0) = 0, \quad z_0 < z < L \\ \text{Bound. cond.} \quad \begin{cases} u^+(z_0, t) = f(t) H(t) \\ u^-(L,t) = 0 \qquad (\text{no sources } z > L) \end{cases} \end{cases}$$

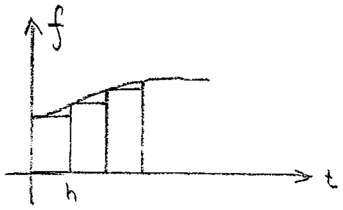
$f(t)$ is an arbitrary continuously differentiable function.

Approximate $f(t)$, $t > 0$.

$$f(t) \approx f_0 H(t) + \sum_{k=1}^{\infty} (f_k - f_{k-1}) H(t - kh) \quad , \quad t > 0$$

where (the sum is always finite for a fixed t)

$$\begin{cases} t_k = hk \\ f_k = f(t_k) \end{cases} \quad , \quad k = 0, 1, 2, \ldots$$



Corresponding approximate solution

$$u^{\pm}(z,t) \approx f_0 \; U^{\pm}(z,t) + \sum_{k=1}^{\infty} (f_k - f_{k-1}) \; U^{\pm}(z, t - kh)$$

$$= f_0 \; U^{\pm}(z,t) + \sum_{k=1}^{\infty} \frac{f_k - f_{k-1}}{h} \; U^{\pm}(z, t - kh)\, h$$

Let $h \to 0$: $$u^{\pm}(z,t) = f(0) \; U^{\pm}(z,t) + \int_0^t f'(t') \; U^{\pm}(z, t - t')\, dt'$$

$$\uparrow \int_0^{t - \ell(z)}$$

Especially $z = z_0$

$$u^{-}(z_0,t) = f(0)\, U^{-}(z_0,t) + \int_0^t f'(t')\, U^{-}(z_0, t - t')\, dt'$$

$$= \int_0^t f(t')\, \partial_t U^{-}(z_0, t - t')\, dt' \qquad (\text{since } U^{-}(z, \ell(z)^{+}) = 1$$

Denote $R(z,t) = \partial_t U^{-}(z,t)$

Additional notes

Properties of $U^{\pm}(z,t)$

1) Causality $U^{\pm}(z,t)=0 \qquad t < \ell(z) = \int_{z_0}^{z} \frac{dz'}{c(z')}$

2) Any discontinuity of $U^{\pm}(z,t)$ is along the characteristics $t = \pm \ell(z) + t_0 \qquad (t_0 \text{ constant})$

3) The discontinuity along $t = \ell(z)$ satisfies

$$\frac{d}{dz}\left(U^{+}(z,\ell(z)^{+})\right) = \partial_z U^{+}(z,\ell(z)^{+}) + \frac{1}{c(z)}\partial_t U^{+}(z,\ell(z)^{+})$$

The dynamics gives

$$\frac{d}{dz}\left(U^{+}(z,\ell(z))\right) = \frac{c'(z)}{2c(z)}\left(U^{+}(z,\ell(z)^{+}) - U^{-}(z,\ell(z)^{+})\right)$$

Since $[U^{-}(z,\ell(z))]=0$ we get

$$\frac{d}{dz}[U^{+}(z,\ell(z))] = \frac{c'(z)}{2c(z)}[U^{+}(z,\ell(z))]$$

ODE with solution

$$[U^{+}(z,\ell(z))] = [U^{+}(z_0,\ell(z_0))]\, e^{\int_{z_0}^{z}\frac{c'(z)}{2c(z')}dz'}$$

$$= \sqrt{\frac{c(z)}{c(z_0)}}\,[U^{+}(z_0,0)]$$

Transmitted field

The solution $$u^{\pm}(z,t) = f(0)\, U^{\pm}(z,t) + \int_0^{t-\ell(z)} f'(t')\, U^{\pm}(z,t-t')\, dt'$$

Integrate by parts:

$$u^{\pm}(z,t) = f(t-\ell(z))\, U^{\pm}(z,\ell(z)) + \int_0^{t-\ell(z)} \partial_t U^{\pm}(z,t-t')\, f(t')\, dt'$$

Specifically, $z = L$

$$u^{+}(L, t+\ell(L)) = \sqrt{\frac{c_1}{c(z_0)}}\; u^{+}(z_0,t) + \int_0^{t} \partial_t U^{+}(L, t+\ell(L)-t')\, u^{+}(z_0,t')\, dt'$$

$$= \sqrt{\frac{c_1}{c(z_0)}} \left\{ u^{+}(z_0,t) + \int_0^{t} T(z_0, t-t')\, u^{+}(z_0,t')\, dt' \right\}$$

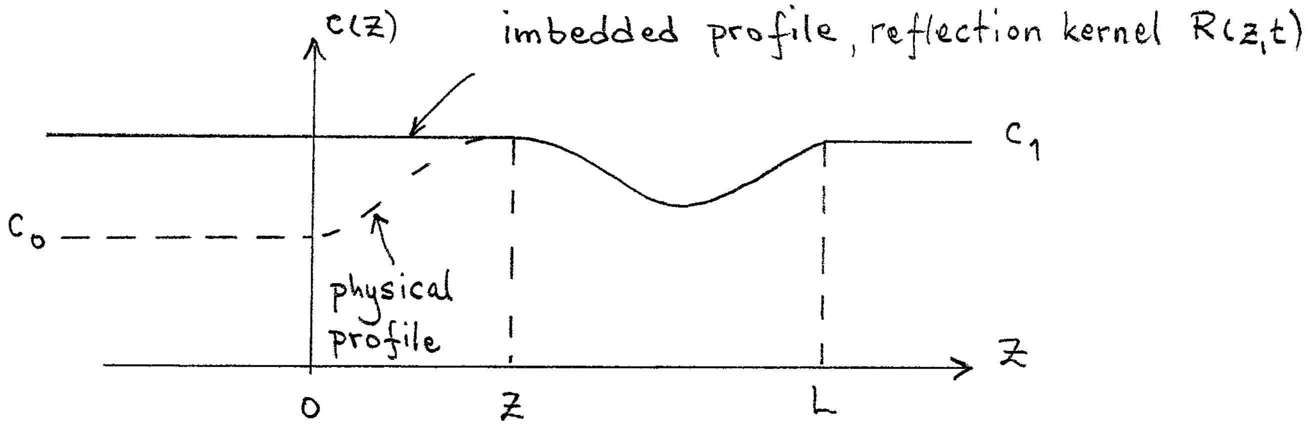
where $$T(z_0,t) = \sqrt{\frac{c(z_0)}{c_1}}\; \partial_t U^{+}(L, t+\ell(L))$$

General relation between u^+ and u^- $(z_0 \to z)$

$$\boxed{u^-(z,t) = \int_0^t R(z,t-t')\, u^+(z,t')\, dt'}$$

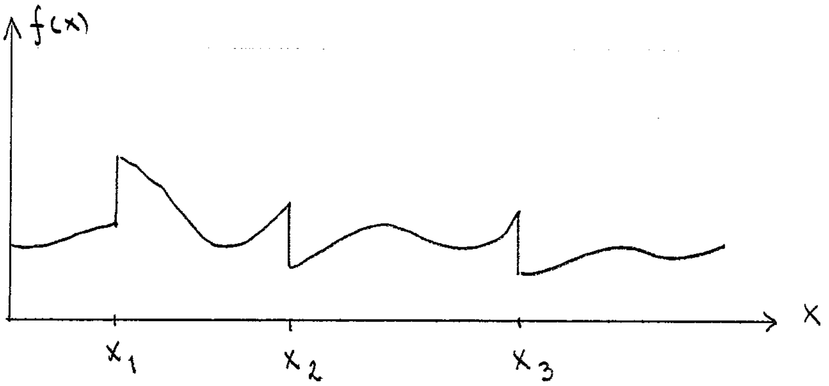
$$\begin{cases} \text{Special case } z=0 \\ R(0,t) \text{ is the "physical" reflection kernel} \end{cases}$$

For a general z inside the slab $(0,L)$ we can give an imbedding interpretation of $R(z,t)$, being the reflection kernel of a fictitious slab (z,L) with continuous $c(z)$ at the front edge.



1.6 Differentiation of jumps

Let $f(x)$ be a function with finite discontinuities at $\{x_i\}_{i=1}^{N}$



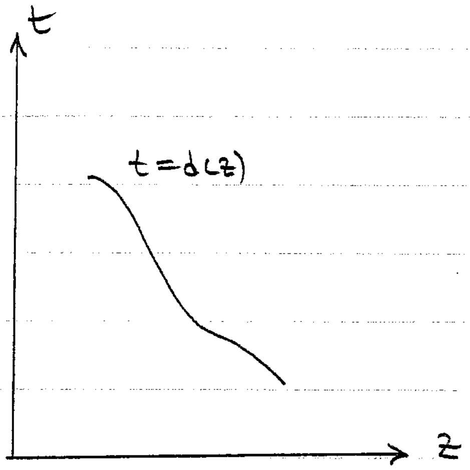
Denote the jump with $[f(x_i)] = f(x_i^+) - f(x_i^-)$

$$\frac{d}{dx} f(x) = f'(x) + \sum_{i=1}^{N} \delta(x-x_i)\,[f(x_i)]$$

where $f'(x)$ is the "ordinary derivative" outside the discontinuities.

Let $R(z,t)$ have a finite jump discontinuity at $t = d(z)$

(assume $d(z)$ monotone)



Denote the jump by

$$[R(z,d(z))] = \\ = R(z,d(z)^+) - R(z,d(z)^-)$$

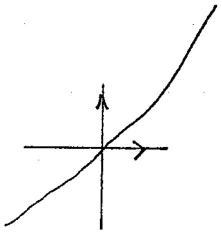
$$\frac{d}{dt} R(z,t) = \underbrace{R_t(z,t)}_{\text{"ordinary der."}} + [R(z,d(z))]\,\delta(t - d(z))$$

Differentiate w.r.t. z

$$\frac{d}{dz} R(z,t) = R_z(z,t) + \underbrace{\left(R(z^+,d(z)) - R(z^-,d(z))\right)}_{\mp[R(z,d(z))]} \underbrace{\delta(z - d^{-1}(t))}_{|d'(z)|\delta(t-d(z))}$$

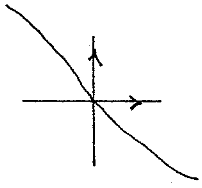
$\uparrow$ sign dep. on $d' \gtrless 0$

$d' > 0$



– sign

$d' < 0$



+ sign

$$\frac{d}{dz} R(z,t) = R_z(z,t) - d'(z)\,[R(z,d(z))]\,\delta(t - d(z))$$

1.7 Integro-differential equation for $R(z,t)$
(Imbedding equation)

Assume R differentiable, except possibly on the curve $t = d(z)$, and $t = 0$.

Denote $u^-(z,t) = \int_{-\infty}^{t} R(z,t-t')\,u^+(z,t')\,dt'$

by $u^- = R * u^+$ and $[R] = R(z,d(z)^+) - R(z,d(z)^-)$

Differentiate w.r.t. z and use

$$\partial_z \begin{pmatrix} u^+ \\ u^- \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} u^+ \\ u^- \end{pmatrix} \qquad \left\{ \begin{aligned} \alpha &= -\tfrac{1}{c}\partial_t + \tfrac{1}{2}c'/c \\ \beta &= \gamma = -\tfrac{1}{2}c'/c \\ \delta &= \tfrac{1}{c}\partial_t + \tfrac{1}{2}c'/c \end{aligned} \right.$$

$$\begin{aligned} u^-_z = \partial_z[R * u^+] &= R_z * u^+ + R * u^+_z - [R]\,d'(z)\,u^+(t-d(z)) \\ &= R_z * u^+ + R * [\alpha u^+ + \beta u^-] - [R]\,d'\,u^+(t-d) \\ &= R_z * u^+ - \tfrac{1}{c} R * u^+_t + \tfrac{1}{2}c'/c\{R * u^+ - R * (R * u^+)\} \\ &\quad - [R]\,d'\,u^+(t-d) \\ &= R_z * u^+ - \tfrac{1}{c} R_t * u^+ - \tfrac{1}{c} R(0)\,u^+ - \tfrac{1}{c}[R]\,u^+(t-d) \\ &\quad + \tfrac{1}{2}c'/c\{R * u^+ - R * (R * u^+)\} - [R]\,d'\,u^+(t-d) \end{aligned}$$

On the other hand

$$u^-_z = \gamma u^+ + \delta u^- = -\tfrac{1}{2} c'/c\, u^+ + \tfrac{1}{c} u^-_t + \tfrac{1}{2} c'/c\, R * u^+$$

$$= \tfrac{1}{2} c'/c \left\{ -u^+ + R * u^+ \right\} + \tfrac{1}{c} \left\{ R(0) u^+ + [R]\, u^+(t-d) + R_t * u^+ \right\}$$

Balance terms!

$$\begin{cases} u^+(z, t-d(z)) \left\{ \frac{2}{c} + d'(z) \right\} [R] = 0 \\ u^+(z,t) \left\{ \frac{2}{c} R(z,0) - \frac{1}{2} c'/c \right\} = 0 \\ \left\{ R_z - \frac{2}{c} R_t - \frac{1}{2} c'/c\, R * R \right\} * u^+ = 0 \end{cases}$$

The imbedding equation

$$\boxed{\begin{aligned} &\partial_z R(z,t) = \frac{2}{c(z)} \partial_t R(z,t) + \frac{1}{2} \frac{c'(z)}{c(z)} \int_0^t R(z,t-t') R(z,t') dt' \\ &R(z,0) = \tfrac{1}{4} c'(z) \\ &R(L,t) = 0 \qquad \text{(no slab, no reflection)} \end{aligned}}$$

Characteristic curves: $t = d(z)$ with $d'(z) = -2/c(z)$

Note:

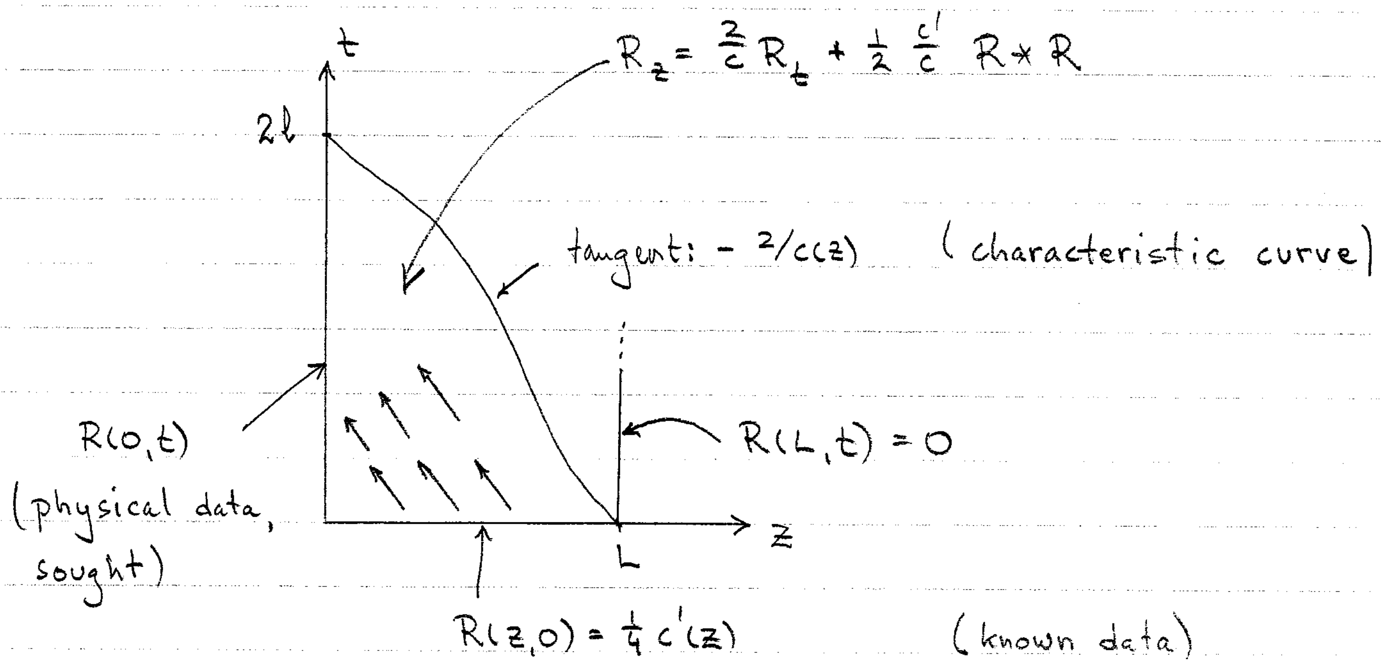
This is a non-linear integro-differential equation.

It is associated with Riccati ODE eq.

This imbedding equation for $R(z,t)$ solves both the direct and the inverse problem.

Problem	Known (Indata)	Sought
Direct	$c(z)$	$R(0,t)$
Inverse	$R(0,t)$ ("physical" refl. kernel)	$c(z)$

Direct problem   c(z) known

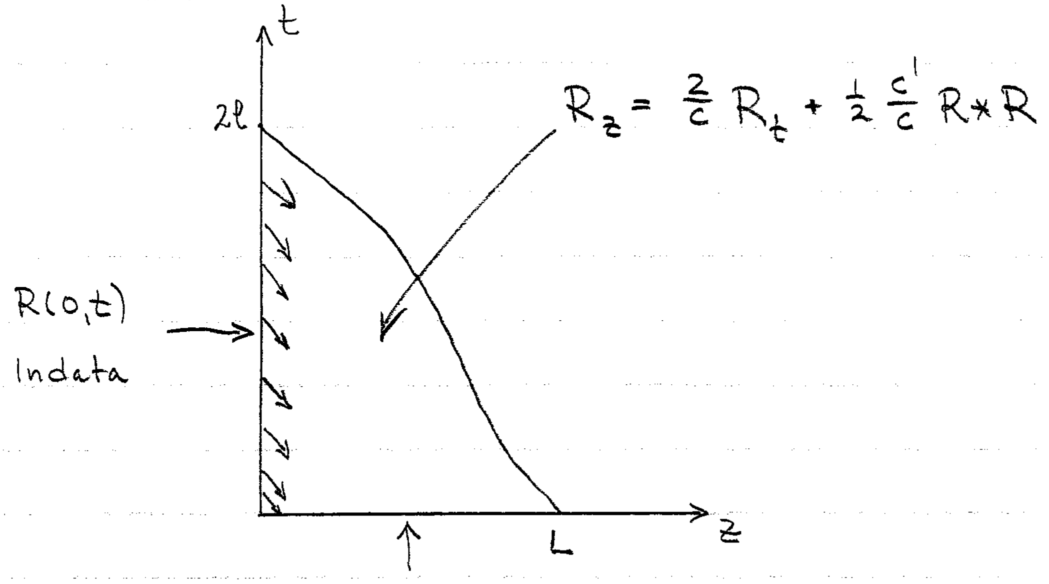


The boundary value of R at t=0 can, due to the directional derivative, be propagated to the t axis (z=0). The value of R at z=0 is the physical reflection kernel.

Note: To obtain the solution R(0,t) for times greater than one round trips ($t > 2\ell$) an analogous technique has to be adopted.

Inverse problem $R(0,t)$, $t \in (0, 2\ell)$ known

$\uparrow$ one round trip



$R(z,0) = \frac{1}{4} c'(z)$ (sought)

Propagate the indata $R(0,t)$ down to the z axis and read off $c'(z) \Rightarrow c(z)$

Note: Since $c(z)$ determines the coefficients in the equation, the equation itself is unknown, just the form is known.

1.8 Travel time coordinate transformation

"R-equation" revisited:

$$\begin{cases} R_z(z,t) = \frac{2}{c(z)} R_t(z,t) + \frac{1}{2}\frac{c'(z)}{c(z)} \int_0^t R(z,t-t')R(z,t')dt' , & t>0 \\ R(z,0) = \frac{1}{4} c'(z) \\ R(L,t) = 0 \\ R(0,t) = \text{"physical" reflection kernel} \end{cases}$$

Change of independent coordinates (<u>travel time coordinates</u>)

$$\begin{cases} x = x(z) = \int_0^z \frac{dz'}{c(z')} \Big/ \ell \qquad \left(\ell = \int_0^L \frac{dz}{c(z)} \right) \\ s = t/\ell \\ R(z,t) \to R(x,s)/\ell \end{cases}$$

$$\partial_z = \frac{dx}{dz}\partial_x = \frac{1}{c(z)\ell}\partial_x \qquad \partial_t = \frac{ds}{dt}\partial_s = \frac{1}{\ell}\partial_s$$

This transformation transforms the inhomogeneous region $z \in (0,L)$ to $x \in (0,1)$ and, forthermore, one round trip $t = 2\ell$ becomes $s = 2$.

New form of the R-equation

$$\begin{cases} R_x(x,s) = 2R_s(x,s) - \frac{1}{2}A(x)\int_0^s R(x,s-s')R(x,s')\,ds' \quad , \quad s>0 \\ R(x,0) = -\frac{1}{4}A(x) \qquad 0 \leq x \leq 1 \\ R(1,s) = 0 \quad , \quad s>0 \end{cases}$$

$$A(x) = -\ell\, c'(z(x)) = \frac{\ell}{2}\,\frac{\epsilon'(z(x))}{\epsilon(z(x))}\,c(z(x))$$

and $R(0,s)$ related to the "physical" refl. kernel.

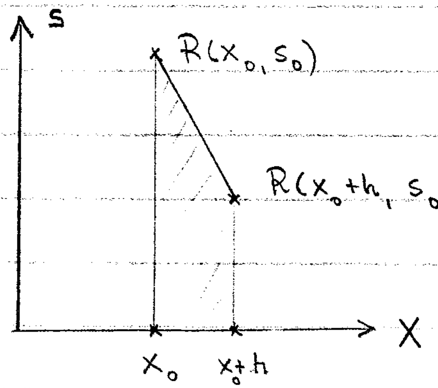
The characteristic curves are now straight lines $\left(s = d(x)\,,\ d'(x) = -2\right)$

Rewrite as

$$\frac{d}{dx}R(x,s-2x) = -\frac{1}{2}A(x)\,(R*R)(x,s-2x)$$

Integration from x_0 to x_0+h and take $s = s_0 + 2x_0$

$$R(x_0+h, s_0-2h) - R(x_0,s_0) = -\frac{1}{2}\int_{x_0}^{x_0+h} A(x)\,(R*R)(x, s_0+2(x_0-x))\,dx$$



More details on the direct problem

Discretize: in x : $x_i = ih$; $i = 0, 1, \ldots, N$ $(Nh = 1)$

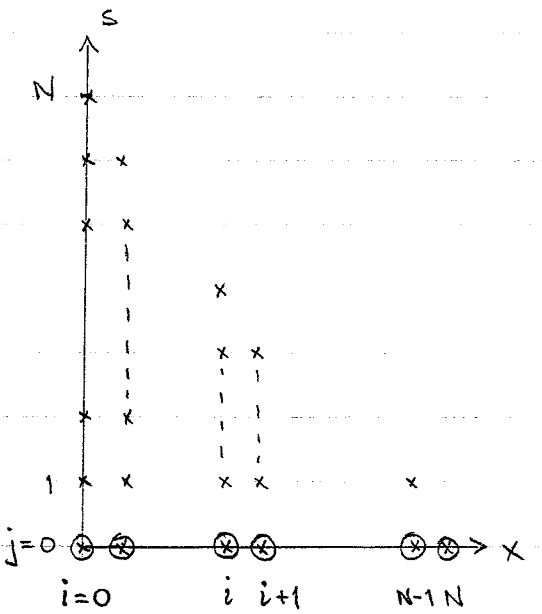
in s : $s_j = 2jh$; $j = 0, 1, \ldots, N$

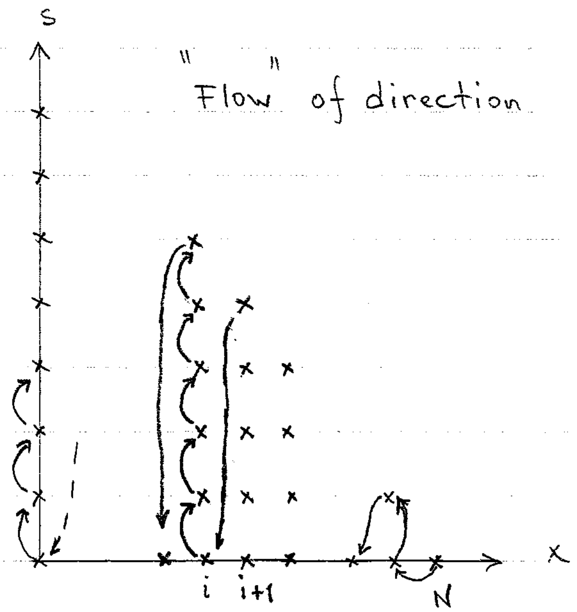
$$R_{i,j} \equiv R(x_i, s_j) \quad ; \quad A_i \equiv A(x_i)$$

Algorithm with trapezoidal role

$$\begin{cases} R_{i,j} = R_{i+1,j-1} + \frac{h}{4} \left\{ A_{i+1} (R*R)_{i+1,j-1} + A_i \underbrace{(R*R)_{i,j}}_{= 2h \sum_{k=1}^{j} R_{i,j-k} R_{i,k}} \right\} + \mathcal{O}(h^3) \\ R_{i,0} = -A_i/4 \end{cases}$$

contains a term $R_{i,j}$





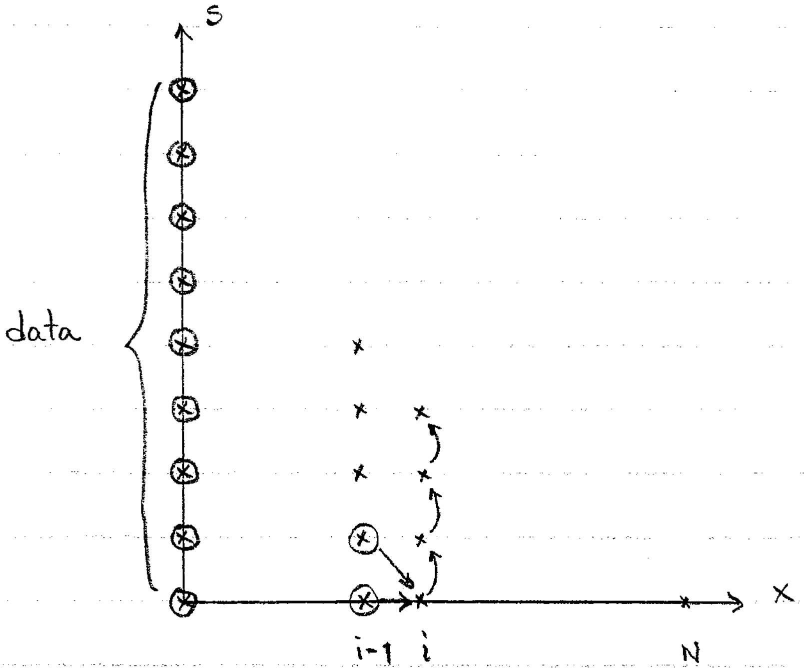
⊗ known values of R

More details on the inverse problem

Solve the algorithm in the opposite direction

$$\begin{cases} R_{i,j} = R_{i-1,j+1} - \frac{h}{4}\left\{ A_i \underset{\uparrow}{(R*R)_{i,j}} + A_{i-1} (R*R)_{i-1,j+1} \right\} + O(h^3) \\ A_i = -4R_{i,0} \\ R_{0,j} = \text{your data} \end{cases}$$

contains a term $R_{i,j}$



$j=0$: $-A_i/4 = R_{i-1,1} - \frac{h}{4} A_{i-1} \, 2h \, R_{i-1,1} R_{i-1,0}$

gives A_i, then fill "the ladder" for i up to top.

1.9 Weak scattering approximation

If multiple scattering can be ignored. (weak scatterer)

discard the convolution term in the R-equation

$$R_x = 2R_s - \tfrac{1}{2} A \; R*R$$

That is $R^0_x(x,s) = 2R^0_s(x,s)$

or $\frac{d}{dx} R^0(x, s-2x) = 0$

$R^0(x, s-2x) =$ constant function of $x = R^0(0,s)$

Let $s = 2x$

$$\underline{-\tfrac{1}{4}A^0(x)} = R^0(x,0) = \underline{R^0(0,2x)}$$

Direct problem: $A(x) \rightarrow R^0(0,s) = -\frac{1}{4} A(s/2)$

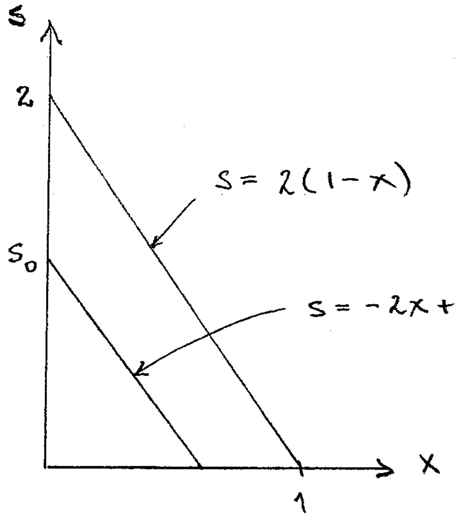
Inverse problem: $R(0,s) \rightarrow \overset{0}{A}(x) = -4R(0,2x)$

1.10 Propagation of singularities (Wave fronts)

Let $R(x,s)$ satisfy

$$\begin{cases} R_x(x,s) = 2R_s(x,s) - \frac{1}{2}A(x)\int_0^s R(x,s-s')R(x,s')\,ds', & s>0,\ 0<x<1 \\ R(x,0) = -\frac{1}{4}A(x), & 0 \leq x \leq 1 \\ R(1,s) = 0, & s>0 \end{cases}$$

If $R(x,s)$ is discontinuous along a curve $s = d(x)$, this curve must be a characteristic curve of the PDE, i.e $s = -2x + s_0$ (s_0 constant)



Denote the finite jump discontinuity by a square bracket.

$$[R(x, s_0-2x)] = R(x, s_0-2x+0) - R(x, s_0-2x-0)$$

Evaluate the PDE above and below the discontinuity

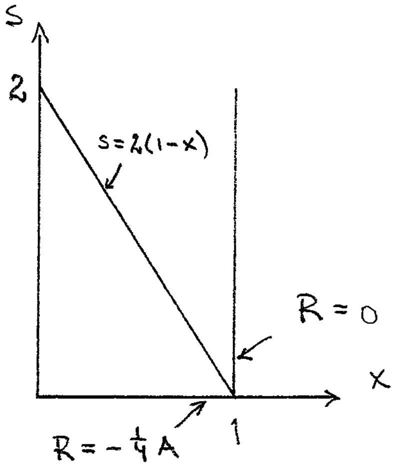
$$[R_x(x, s_0-2x)] = 2[R_s(x, s_0-2x)] + \underset{\substack{\uparrow \\ \text{conv. always cont.}}}{0}$$

$$\frac{\partial}{\partial x}[R(x, s_0-2x)] - 2\frac{\partial}{\partial s}[R(x, s_0-2x)] = 0$$

$$\frac{d}{dx}[R(x, s_0-2x)] = 0$$

$$[R(x, s_0-2x)] = \text{constant}$$

Evaluate this constant at $x = s_0/2$



If $A(x)$ is assumed to be continuous in $0 < x < 1$ there is no discontinuity in $R(x,s)$ along $s = s_0 - 2x$

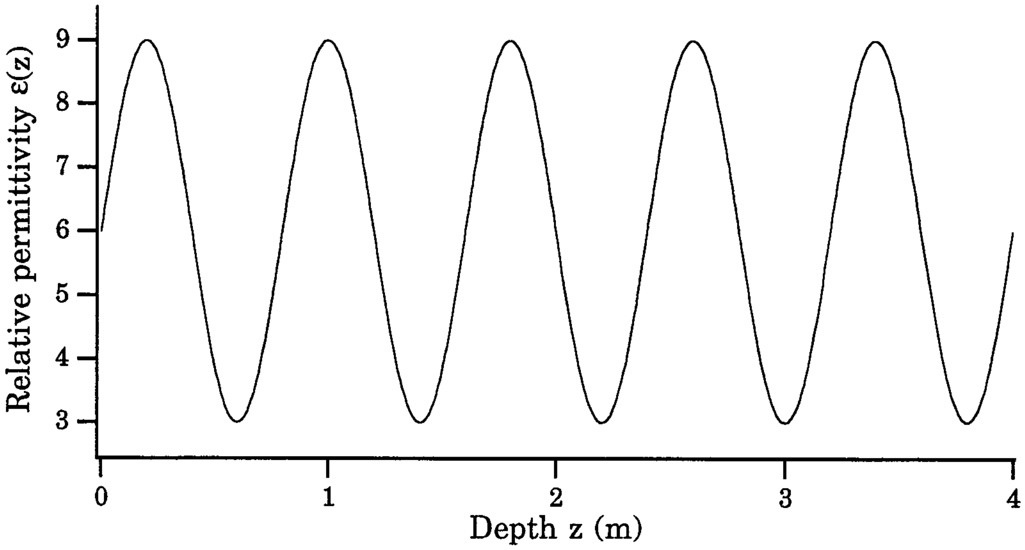
If $A(1^-) \neq A(1^+) = 0$ then $R(x,s)$ is discontinuous along $s = 2 - 2x$

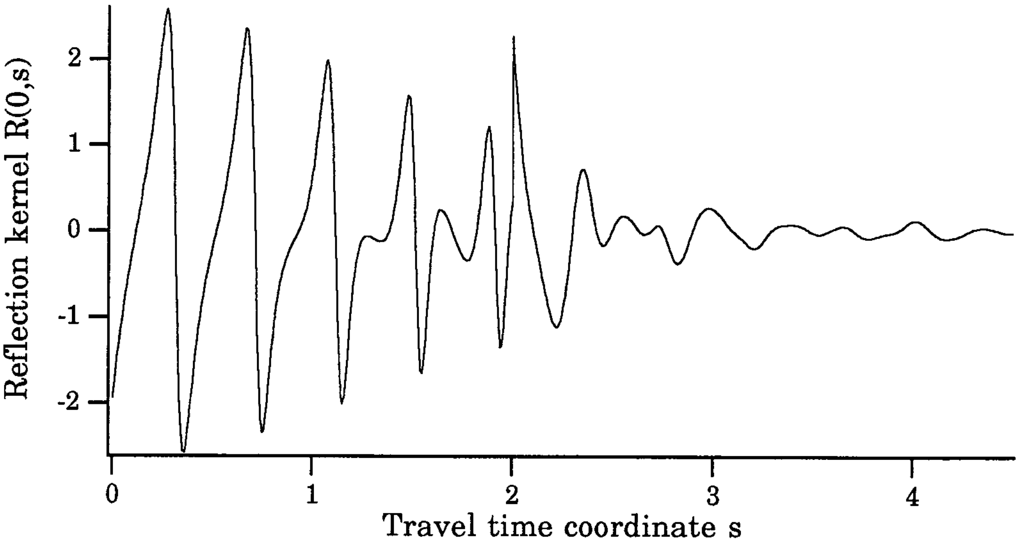
$$[R(x, 2(1-x)] = [R(1,0)] = 0 - (-\tfrac{1}{4} A(1^-)) = \tfrac{1}{4} A(1^-) = \text{const.}$$

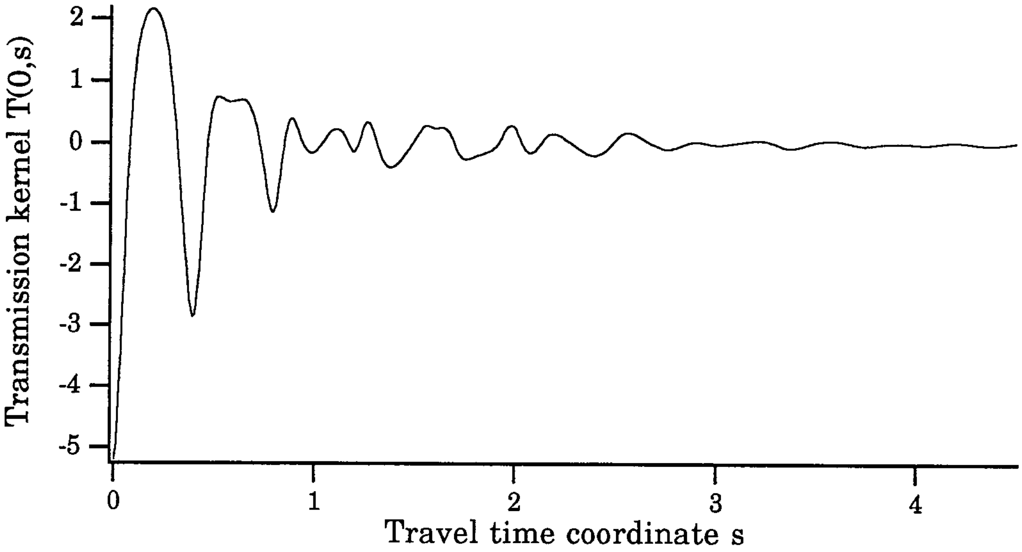
Conclusion: The kernel $R(x,s)$ is continuous everywhere, except along the line $s = 2(1-x)$, which is one round trip in the slab $[x,1]$. The finite jump discontinuity is

$$[R(x, 2(1-x))] = \tfrac{1}{4} A(1^-) = \text{constant}$$

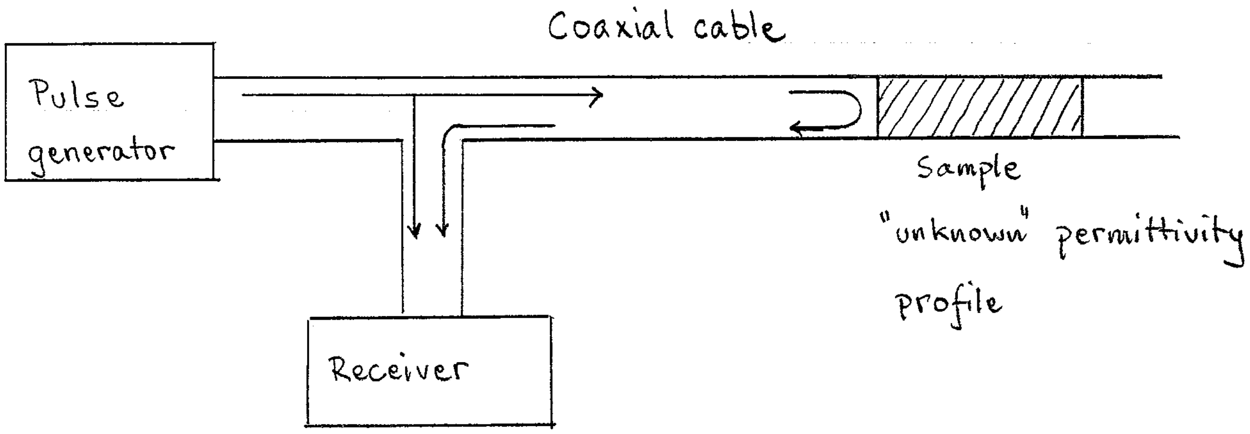
1.11 Numerical example







1.12 Experimental setup



Deconvolution

In the experiment one measures E^i and E^r. The input to the algorithm is the kernel $R(t)$

Relation:

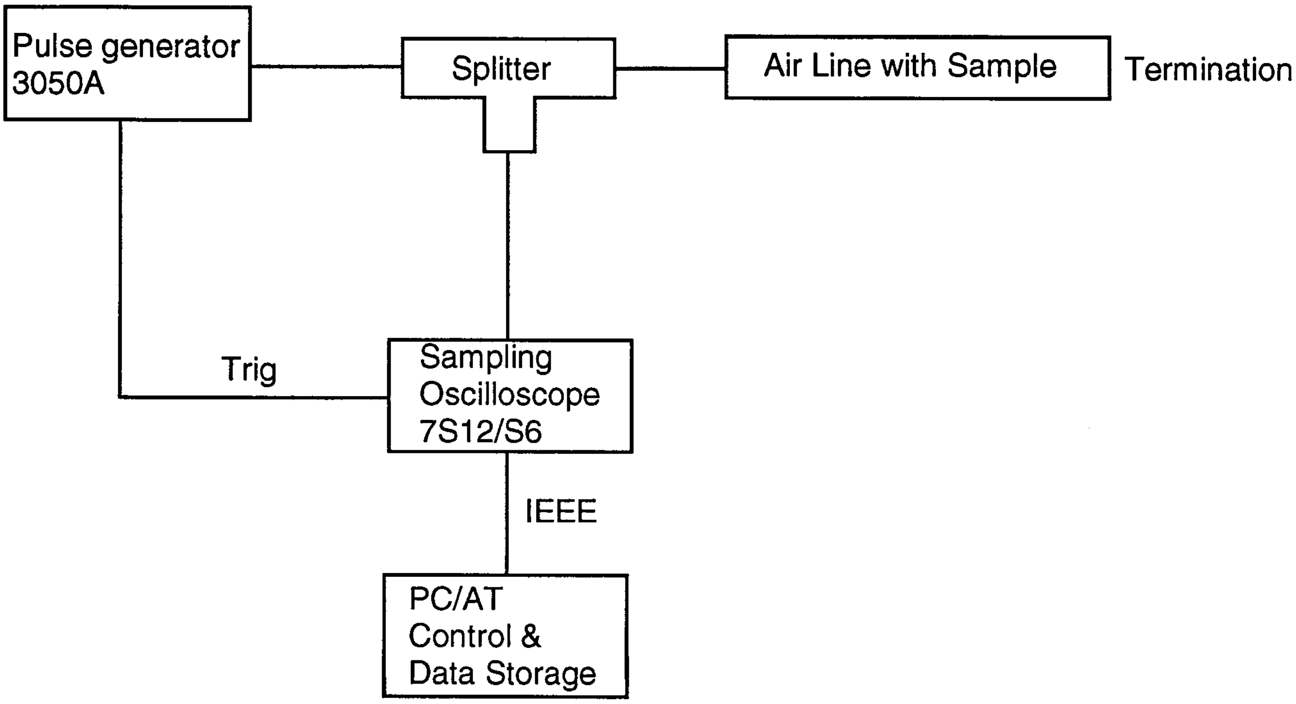
$$E^r(t) = \int_{-\infty}^{t} R(t-t')\, E^i(t')\, dt'$$

To get $R(t)$ from $E^r(t)$ and $E^i(t)$ is called

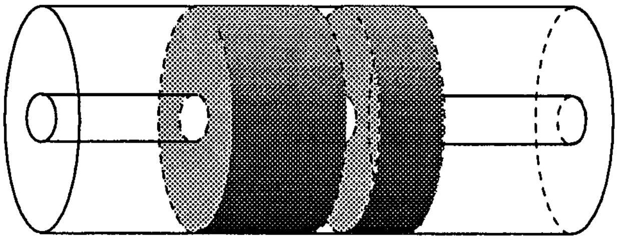
deconvolution

The problems encountered in deconvolution are not addressed here (stability problems etc.).

Experimetal setup

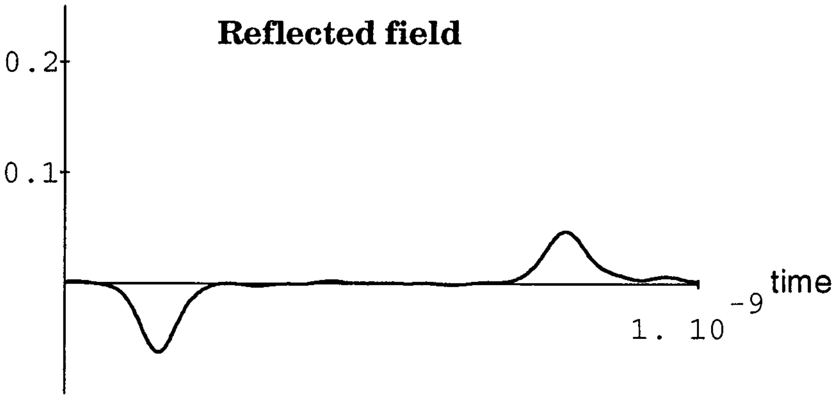


Sample holder

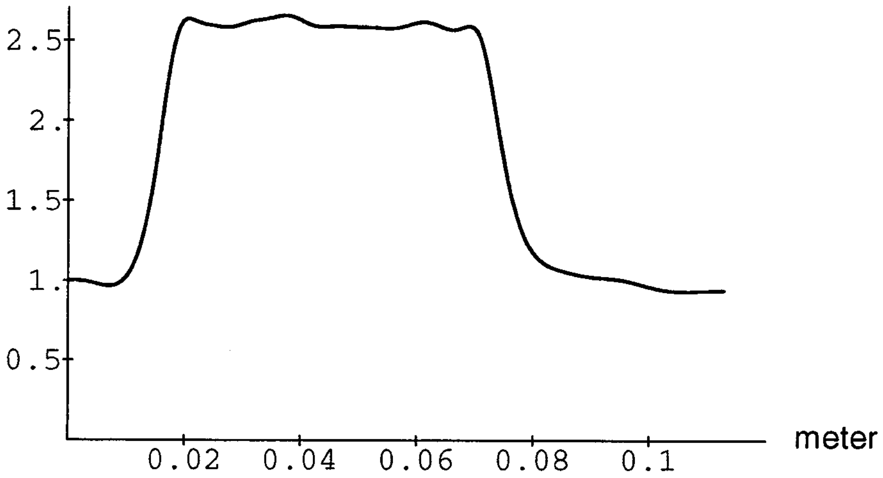


Sample: Teflon 6.00 cm long, Rel. permittivity 2.64

Incident field

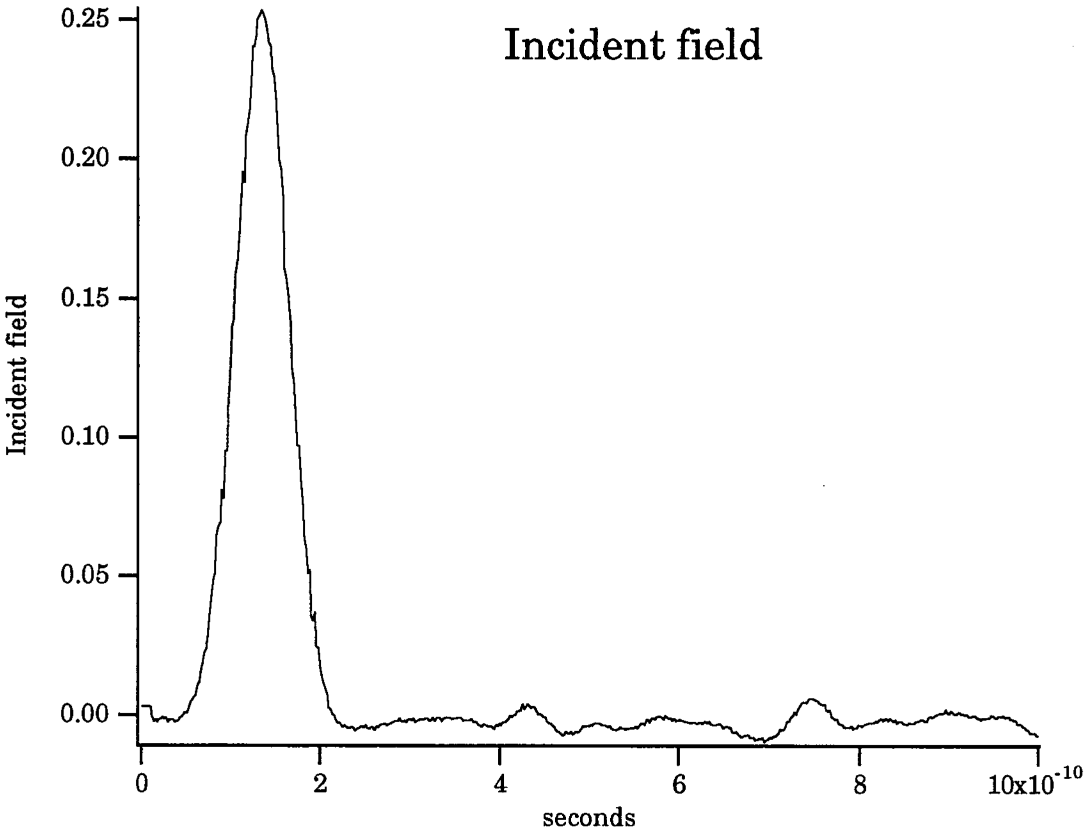

Reflected field

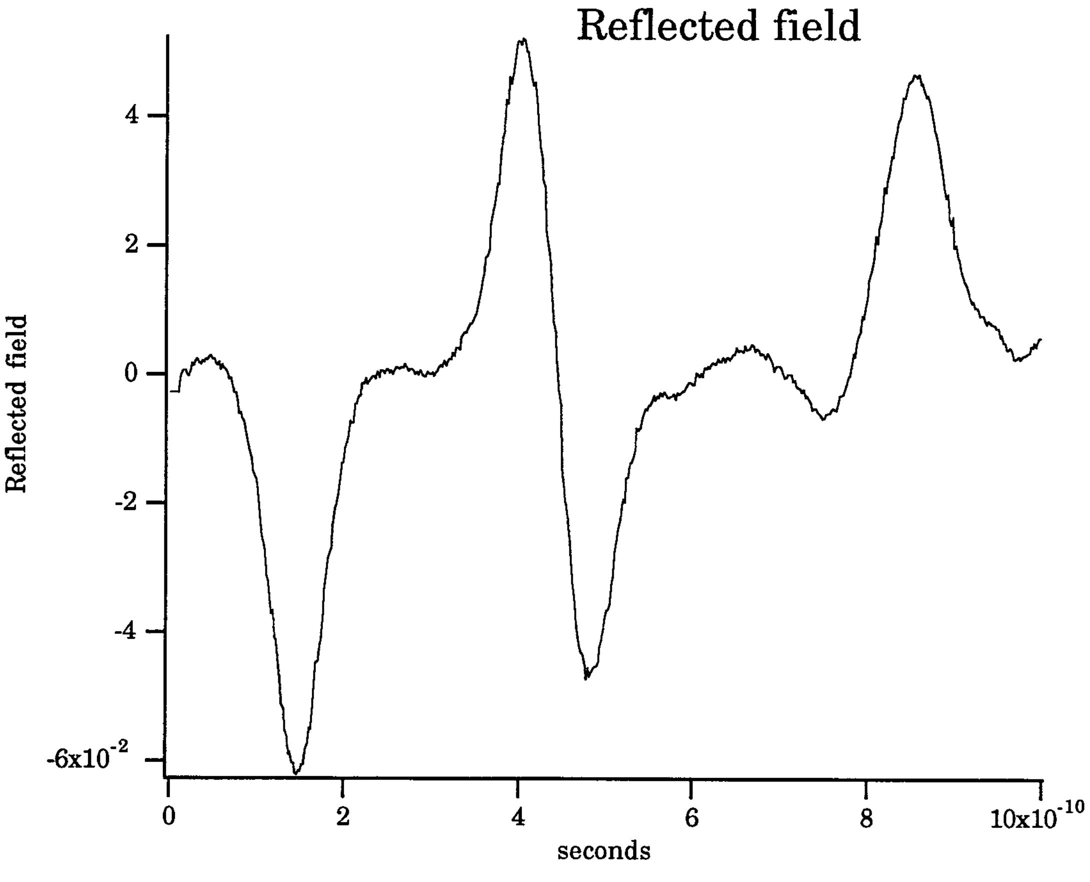
Reconstruction of relative permittivity



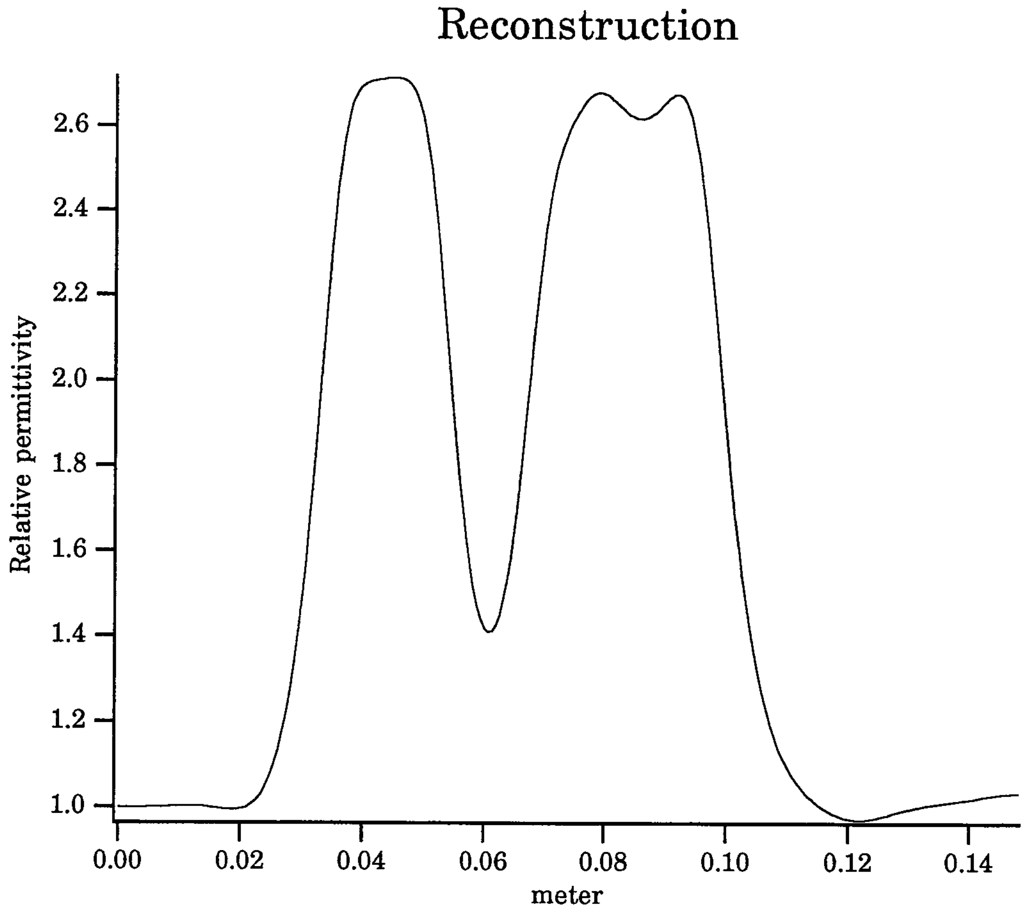
(31

Experiment






Reflection kernel
Reflection kernel
2
1
0
-1
-2x10^9
0.0
0.2
0.4
0.6
0.8
1.0
1.2x10^-9
seconds


Reconstruction
Relative permittivity
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
meter

2. Green functions approach

2.1 Basic equations

In the previous sections the direct and inverse scattering problems were solved using the imbedding equation.

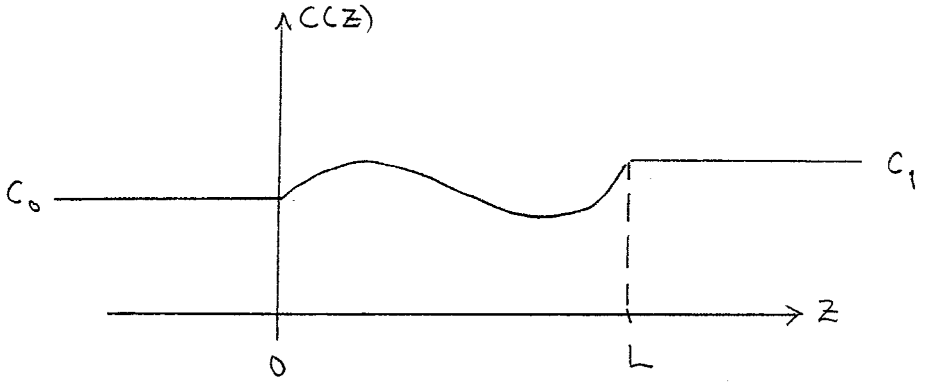
This section contains an alternative (and numerically more attractive) formulation.

The imbedding method did not address the internal fields inside the slab. With this method, based upon Green functions, the direct and the inverse scattering problems are solved and the internal field can be computed.

As before, the wave equation holds

$$\boxed{\partial_z^2 u(z,t) - c^{-2}(z)\,\partial_t^2 u(z,t) = 0}$$

and c(z) is assumed continuous

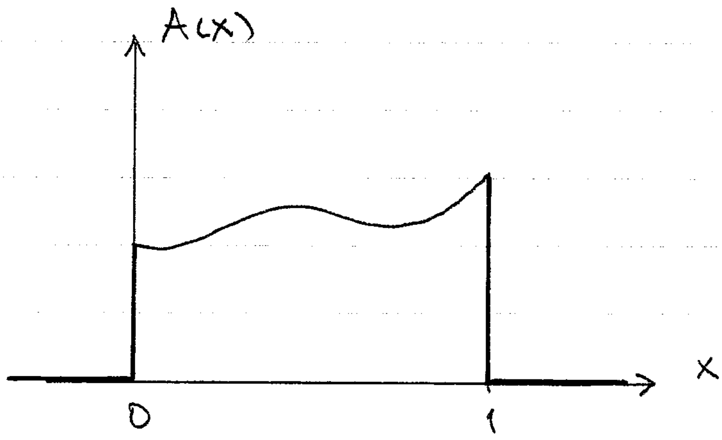


As above, it is convenient, but not necessary, to introduce <u>travel time coordinates</u>

$$\begin{cases} x = x(z) = \dfrac{1}{\ell} \displaystyle\int_0^z \dfrac{dz'}{c(z')} \\ s = t/\ell \\ u(z,t) \rightarrow u(x,s) \end{cases} \qquad \left(\ell = \int_0^L \frac{dz}{c(z)} \right)$$

The wave equation becomes

$$\boxed{\begin{aligned} &\partial_x^2 u(x,s) - \partial_s^2 u(x,s) + A(x)\,\partial_x u(x,s) = 0 \\ &A(x) = -\partial_x \ln\big(c(z(x))\big) = \frac{1}{2}\frac{\epsilon'(z(x))}{\epsilon(z(x))}\,c(z(x)) \end{aligned}}$$



2.2 Wave splitting

We adopt the same concept of wave splitting as before, i.e.

$$u^{\pm}(x,s) = \frac{1}{2}\left\{ u(x,s) \mp \partial_s^{-1} u_x(x,s) \right\}$$
$$= \frac{1}{2}\left\{ u(x,s) \mp \int_{-\infty}^{s} \partial_x u(x,s')\, ds' \right\}$$

The splitting does the same job as before; it projects out the left and right moving parts of the field $u(x,s)$, respectively.

$u^{\pm}$ satisfies as before

$$\partial_x \begin{pmatrix} u^+ \\ u^- \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} u^+ \\ u^- \end{pmatrix}$$

$$\begin{cases} \alpha = -\partial_s - \frac{1}{2} A(x) \\ \beta = \gamma = \frac{1}{2} A(x) \\ \delta = \partial_s - \frac{1}{2} A(x) \end{cases}$$

As before

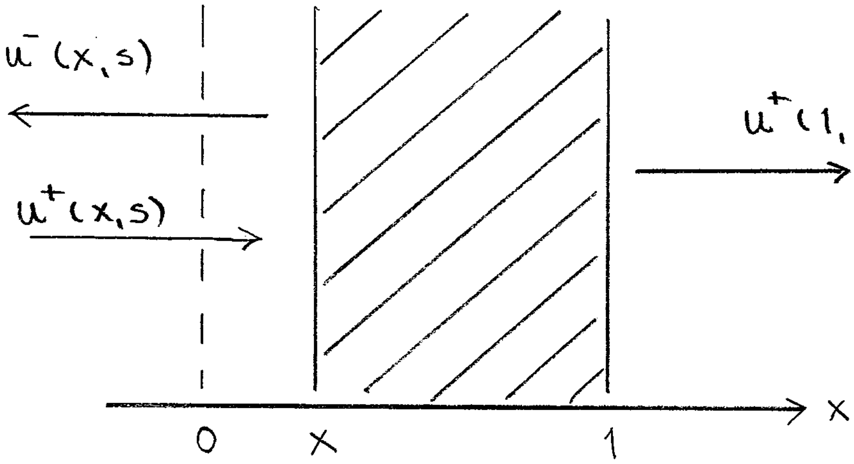
$$u(x,s) = u^+(x,s) + u^-(x,s)$$
$$u_x(x,s) = u_s^-(x,s) - u_s^+(x,s)$$

2.3 Relation between $u^\pm$

In section 1, $u^\pm$ were related on the edges of a subsection $[x,1]$ of the slab, i.e. (cf. p. 11 & 4)

$$\begin{cases} u^-(x,s) = \int_{-\infty}^{s} R(x,s-s')\, u^+(x,s')\, ds' \\ u^+(1,s+1-x) = t(x,1)\left\{ u^+(x,s) + \int_{-\infty}^{s} T(x,s-s')\, u^+(x,s')\, ds' \right\} \end{cases}$$

$$t(x,1) = \exp\left[-\tfrac{1}{2}\int_x^1 A(x')\,dx'\right]$$

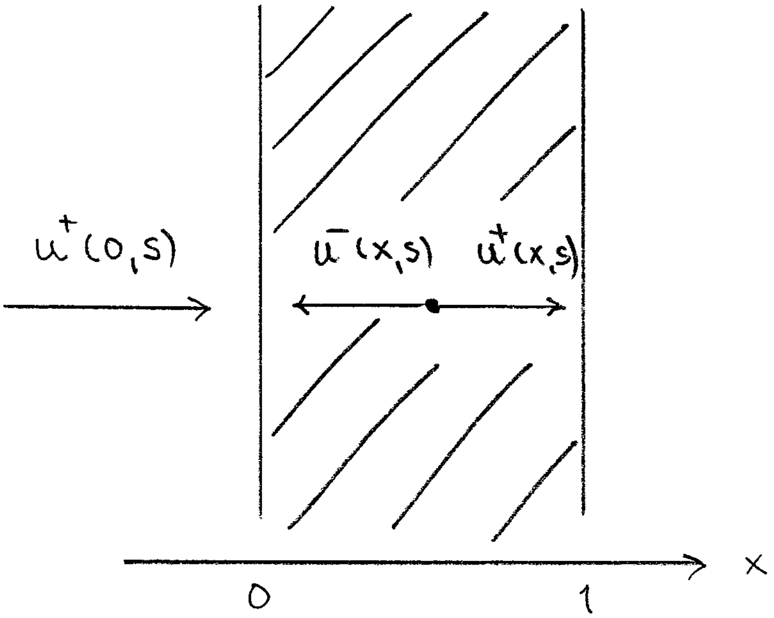


Physical kernels: $\begin{pmatrix} R(0,s) \\ T(0,s) \end{pmatrix}$

In this section we will relate the fields $u^{\pm}$ at a point x <u>inside</u> the slab to the fields on the edges of the <u>physical</u> slab.

Specifically (no excitation from the right),

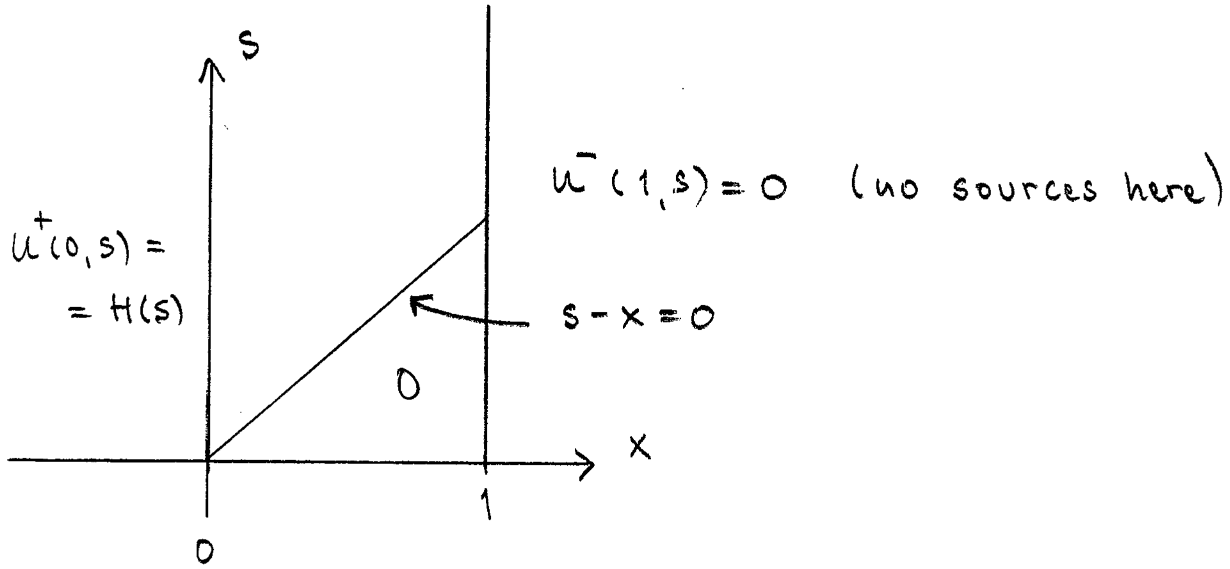
$$\begin{cases} u^{+}(x,s) & \text{as a function of } u^{+}(0,s) \\ u^{-}(x,s) & \text{— '' —} \end{cases}$$



To find this relation between $u^{\pm}(x,s)$ and $u^{-}(0,s)$ use the Duhamel's principle.

2.4 Duhamel's principle (cf. p. 8 – 11)

Canonical problem



$$\begin{cases} \partial_x \begin{pmatrix} u^+ \\ u^- \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} u^+ \\ u^- \end{pmatrix} , \quad 0 < x < 1, \quad s > 0 \\ \text{Initial condition:} \quad u^\pm(x,0) = 0 , \quad 0 < x < 1 \\ \text{Boundary conditions:} \quad \begin{cases} u^+(0,s) = H(s) \\ u^-(1,s) = 0 \end{cases} \end{cases}$$

This is a well-posed problem. Call the unique solution $U(x,s)$.

Causality implies that $U(x,s) = 0$ for $s < x$.

Apply the wave splitting: $U^{\pm}(x,s)$

The solutions $U^{\pm}(x,s)$ are continuously differentiable everywhere, except along the characteristic curve $s=x$.

<u>Causality</u>: $U^{\pm}(x,s)=0 \qquad s<x$

<u>Linearity</u> and <u>invariance under time translations</u>:
If $H(s)$ is replaced by $\lambda H(s-s_0)$
then the solutions are $\lambda U^{\pm}(x,s-s_0)$.

<u>The full problem</u>:

$$\begin{cases} \partial_x \begin{pmatrix} u^+ \\ u^- \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} u^+ \\ u^- \end{pmatrix} , \quad 0<x<1, \quad s>0 \\ \text{Initial cond.} \quad u^{\pm}(x,0)=0 , \quad 0<x<1 \\ \text{Boundary cond.} \quad \begin{cases} u^+(0,s) = f(s)H(s) \\ u^-(1,s) = 0 \end{cases} \end{cases}$$

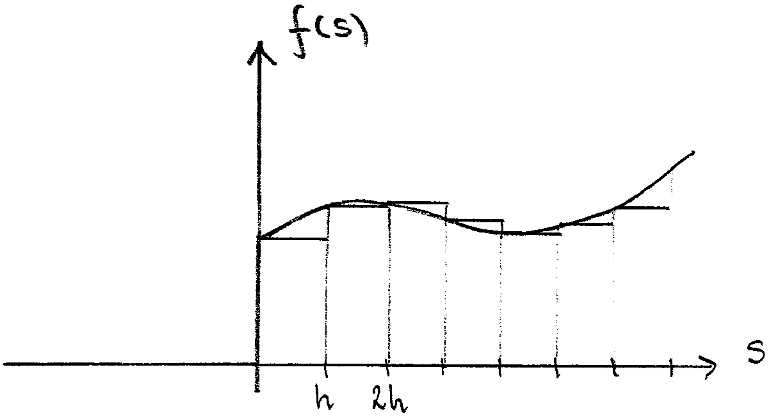
$f(s)$ is an arbitrary continuously differentiable function.

Approximate $f(s)$, $s > 0$

$$f(s) = f_0 H(s) + \sum_{k=1}^{\infty} (f_k - f_{k-1}) H(s - kh) \quad , \quad s > 0$$

where $f_k = f(kh)$.

Note that sum is always finite for fixed s.



Corresponding approximate solution

$$u^{\pm}(x, s) \approx f_0 U^{\pm}(x, s) + \sum_{k=1}^{\infty} (f_k - f_{k-1}) U^{\pm}(x, s - kh)$$

$$= f_0 U^{\pm}(x, s) + \sum_{k=1}^{\infty} \frac{f_k - f_{k-1}}{h} U^{\pm}(x, s - kh) h$$

In the limit $h \to 0$

$$u^{\pm}(x,s) = f(0)\, U^{\pm}(x,s) + \int_0^{\infty} f'(s')\, U^{\pm}(x, s-s')\, ds'$$

$$= f(0)\, U^{\pm}(x,s) + \int_0^{s-x} f'(s')\, U^{\pm}(x, s-s')\, ds'$$

Integrate by parts:

$$\boxed{u^{\pm}(x,s) = f(s-x)\, U^{\pm}(x,x) + \int_0^{s-x} \partial_s U^{\pm}(x, s-s')\, f(s')\, ds'}$$

Propagation of singularity arguments show:

$$\begin{cases} U^{+}(x,x) = t(0,x) = \exp\left[-\frac{1}{2}\int_0^{x} A(x')\, dx'\right] \\ U^{-}(x,x) = 0 \end{cases}$$

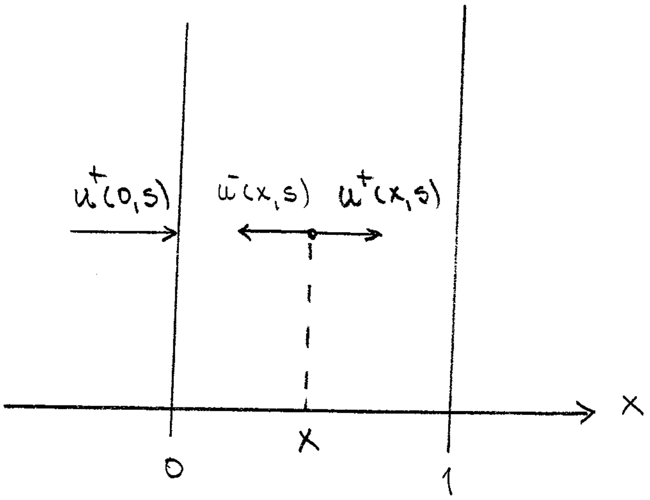
Introduce the notation (Green functions $G^{\pm}(x,s)$)

$$\begin{cases} G^{+}(x,s) = \partial_s U^{+}(x, s+x) \, (t(0,x))^{-1} \\ G^{-}(x,s) = \partial_s U^{-}(x, s+x) \, (t(0,x))^{-1} \end{cases}$$

then

$$\begin{cases} u^{+}(x, s+x) = t(0,x) \left\{ u^{+}(0,s) + \int_{-\infty}^{s} G^{+}(x, s-s') \, u^{+}(0,s') \, ds' \right\} \\ u^{-}(x, s+x) = t(0,x) \int_{-\infty}^{s} G^{-}(x, s-s') \, u^{+}(0,s') \, ds' \end{cases}$$

This is the formal relation between the excitation $u^{+}(0,s)$ on the boundary, $z=0$, and the internal field $u^{+}(x,s)$. Note that $u(x,s) = u^{+}(x,s) + u^{-}(x,s)$



2.5 Connection to the scattering kernels

Let $x=0$ and $x=1$ in the Green's functions representation on the previous page ($t(0,0)=1$)

$$\begin{cases} u^+(0,s) = u^+(0,s) + \int_0^s G^+(0,s-s')\, u^+(0,s')\, ds' \\ u^-(0,s) = \int_0^s G^-(0,s-s')\, u^+(0,s')\, ds' \end{cases}$$

$$\begin{cases} u^+(1,s+1) = t(0,1)\left\{ u^+(0,s) + \int_0^s G^+(1,s-s')\, u^+(0,s')\, ds' \right\} \\ u^-(1,s+1) = t(0,1) \int_0^s G^-(1,s-s')\, u^+(0,s')\, ds' \end{cases}$$

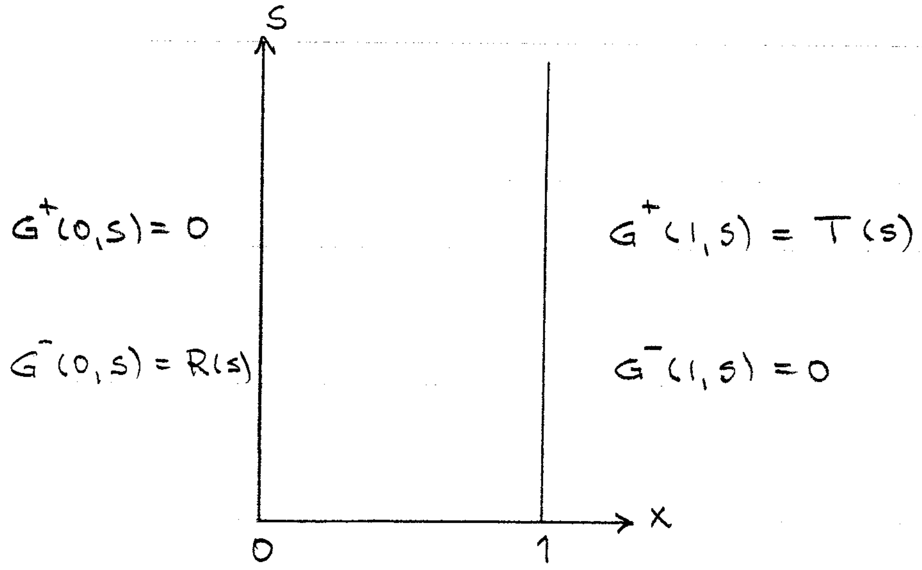
Compare with the scattering operator formulation on p. 37 with $x=0$ (omit the 0 in R and T kernels here)

$$\begin{cases} u^-(0,s) = \int_0^s R(s-s')\, u^+(0,s')\, ds' \\ u^+(1,s+1) = t(0,1)\left\{ u^+(0,s) + \int_0^s T(s-s')\, u^+(0,s')\, ds' \right\} \end{cases}$$

Direct comparison and $u^-(1,s)=0$ (no sources $x>1$) give

$$\boxed{\begin{array}{ll} G^+(0,s)=0 & G^-(1,s)=0 \\ G^-(0,s) = R(s) = \text{Physical reflection kernel} & \\ G^+(1,s) = T(s) = \text{Physical transmission kernel} & \end{array}}$$

These are the boundary conditions on $G^{\pm}(x,s)$



There is also a more general connection between $G^{\pm}(x,s)$ and $R(x,s)$

$$t(0,x)\int_{-\infty}^{s} G^{-}(x,s-s')\,u^{+}(0,s')\,ds' = u^{-}(x,s+x)$$

$$= \int_{-\infty}^{s+x} R(x,s+x-s')\,u^{+}(x,s')\,ds' = \int_{-\infty}^{s} R(x,s-s'')\,u^{+}(x,s''+x)\,ds''$$

$$= \int_{-\infty}^{s} R(x,s-s'')\,t(0,x)\Big[u^{+}(0,s'') + \int_{-\infty}^{s''} G^{+}(x,s''-s')\,u^{+}(0,s')\,ds'\Big]\,ds''$$

Since $\int_{-\infty}^{s} ds'' \int_{-\infty}^{s''} ds' = \int_{-\infty}^{s} ds' \int_{s'}^{s} ds''$ we get

$$\boxed{G^{-}(x,s) = R(x,s) + \int_{0}^{s} R(x,s-s')\,G^{+}(x,s')\,ds'}$$

Similarly,

$$\boxed{G^{+}(1,s) = G^{+}(x,s) + T(x,s) + \int_{0}^{s} T(x,s-s')\,G^{+}(x,s')\,ds'}$$

2.6 Differential equations for $G^{\pm}$

We have the dynamics (PDE) for $u^{\pm}$ (p. 36)

$$\partial_x \begin{pmatrix} u^+ \\ u^- \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} u^+ \\ u^- \end{pmatrix} \qquad (*)$$

$$\begin{cases} \alpha = -\partial_s - \frac{1}{2} A(x) \\ \beta = \gamma = \frac{1}{2} A(x) \\ \delta = \partial_s - \frac{1}{2} A(x) \end{cases}$$

and the relation between $u^{\pm}$ (p. 43)

$$\begin{cases} u^+(x, s+x) = t(0,x) \left\{ u^+(0,s) + \int_{-\infty}^{s} G^+(x, s-s')\, u^+(0,s')\, ds' \right\} \\ u^-(x, s+x) = t(0,x) \int_{-\infty}^{s} G^-(x, s-s')\, u^+(0,s')\, ds' \end{cases} \qquad (**)$$

Assume $G^{\pm}(x,s)$ everywhere differentiable except possibly along $s = d_{\pm}(x)$

Denote the jump in $G^{\pm}$ by $[G^{\pm}(x, d_{\pm}(x))]$

$$[G^{\pm}(x, d_{\pm}(x))] = G^{\pm}(x, d_{\pm}(x)^+) - G^{\pm}(x, d_{\pm}(x)^-)$$

Note that $\frac{d}{dx} t(0,x) = -\frac{1}{2} A(x)\, t(0,x)$

Differentiate $u^+(x, s+x)$ in $(**)$ w.r.t. x

$$\frac{d}{dx} u^+(x, s+x) = -\frac{1}{2} A(x) \cancelto{②}{u^+(x, s+x)}$$

$$+ t(0, x) \int_{-\infty}^{s} G^+_x(x, s-s') \, u^+(0, s') \, ds'$$

$$- t(0, x) \, d'_+(x) \left[G^+(x, d_+(x)) \right] u^+(0, s - d_+(x))$$

On the other hand from $(**)$ and the use of $(*)$

$$\frac{d}{dx} u^+(x, s+x) = u^+_x(x, s+x) + u^+_s(x, s+x)$$

$$= - \cancelto{①}{u^+_s(x, s+x)} - \frac{1}{2} A(x) \, u^+(x, s+x) + \frac{1}{2} A(x) \, u^-(x, s+x)$$

$$+ \cancelto{①}{u^+_s(x, s+x)} = -\frac{1}{2} A(x) \cancelto{②}{u^+(x, s+x)}$$

$$+ \frac{1}{2} A(x) \, t(0, x) \int_{-\infty}^{s} G^-(x, s-s') \, u^+(0, s') \, ds'$$

Balance terms!

$$\begin{cases} d'_+(x) = 0 \quad \Rightarrow \quad d_+(x) = \text{constant} \\[2ex] G^+_x(x, s) = \frac{1}{2} A(x) \, G^-(x, s) \end{cases}$$

Analogously for the $u^-(x, s+x)$ term; diff $(**)$ w.r.t. x

$$\frac{d}{dx} u^-(x, x+s) = -\tfrac{1}{2} A(x) \cancelto{①}{u^-}(x, s+x)$$

$$+ t(0,x) \int_{-\infty}^{s} G^-_x(x, s-s') u^+(0, s') ds'$$

$$- t(0,x) d_-'(x) [G^-(x, d_-(x))] u^+(0, s-d_-(x))$$

and from $(**)$ and $(*)$

$$\frac{d}{dx} u^-(x, x+s) = u^-_x(x, x+s) + u^-_s(x, x+s)$$

$$= \tfrac{1}{2} A(x) u^+(x, x+s) + u^-_s(x, x+s)$$

$$- \tfrac{1}{2} A(x) u^-(x, x+s) + u^-_s(x, x+s)$$

$$= \tfrac{1}{2} A(x) t(0,x) \left\{ u^+(0,s) + \int_{-\infty}^{s} G^+(x, s-s') u^+(0,s') ds' \right\}$$

$$- \tfrac{1}{2} A(x) \cancelto{①}{u^-}(x, x+s) + 2 t(0,x) \{ G^-(x,0) u^+(0,s)$$

$$+ [G^-(x, d_-(x))] u^+(0, s-d_-(x)) + \int_{-\infty}^{s} G_s(x, s-s') u^+(0,s') ds' \}$$

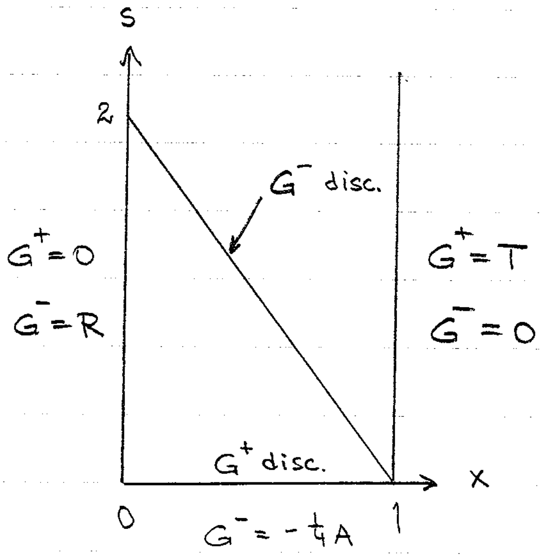
Balance terms!

$$\begin{cases} G^-_x(x,s) - 2 G^-_s(x,s) = \tfrac{1}{2} A(x) G^+(x,s) \\ G^-(x,0) = -\tfrac{1}{4} A(x) \\ d_-'(x) = -2 \end{cases}$$

In summary

$$\boxed{\begin{aligned} &G^+_x(x,s) = \tfrac{1}{2} A(x)\, G^-(x,s) \\ &G^-_x(x,s) - 2G^-_s(x,s) = \tfrac{1}{2} A(x)\, G^+(x,s) \\ &G^-(x,0) = -\tfrac{1}{4} A(x) \end{aligned}}$$

$$\left\{ \begin{array}{llll} G^+(x,s) & \text{is discontinuous} & \text{along} & s = \text{constant} \\ G^-(x,s) & \text{is discontinuous} & \text{along} & s = -2x + \text{constant} \end{array} \right.$$



Notice that the Green function equations satisfy a system of <u>linear PDE.</u> The imbedding equation was <u>non</u>-linear.

2.7 Propagation of singularities

G^+ is discontinuous at $s=0$

$$G^+_x(x,0) = \frac{1}{2}A(x)\,G^-(x,0) = -\frac{1}{8}A^2(x)$$

Integration gives

$$G^+(x,0) = -\frac{1}{8}\int_0^x A^2(x')\,dx'$$

G^- is discontinuous along $s = -2x + 2x_0$ x_0 constant

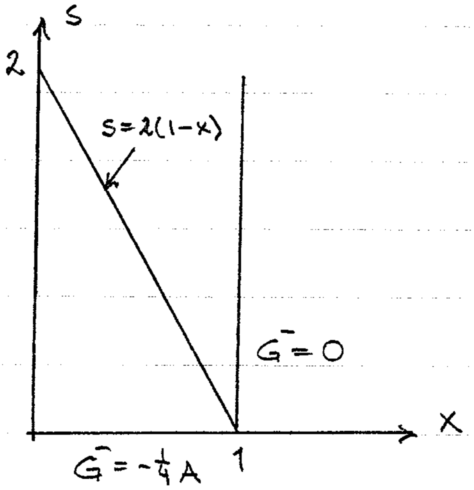
Evaluate PDE above and below discontinuity

$$\frac{\partial}{\partial x}\left[G^-(x,2(x_0-x))\right] - 2\frac{\partial}{\partial s}\left[G^-(x,2(x_0-x))\right] = \underset{\substack{\uparrow \\ G^+ \text{ cont. along} \\ s=2(x_0-x)}}{0}$$

$$\frac{d}{dx}\left[G^-(x,2(x_0-x))\right] = 0$$

$$\left[G^-(x,2(x_0-x))\right] = \text{constant}$$

Evaluate the constant at $x = x_0$



If $A(x)$ is continuous in $0 < x < 1$ then the only discontinuous line is $s = 2(1-x)$ $(x_0 = 1)$

$$[G^-(x, 2(1-x))] = [G^-(1,0)] = 0 - (-\tfrac{1}{4}A(1)) = \tfrac{1}{4}A(1) = \text{const.}$$

<u>Conclusion:</u> In the region $s > 0$, $0 < x < 1$

$G^\pm(x,s)$ is continuous everywhere except along

$s = 2(1-x)$ where

$$[G^-(x, 2(1-x))] = \tfrac{1}{4}A(1)$$

$G^\pm(x,s)$ have values at $s = 0$

$$\begin{cases} G^+(x,0) = -\frac{1}{8}\int_0^x A^2(x')\,dx' \\ G^-(x,0) = -\frac{1}{4}A(x) \end{cases}$$

Direct and inverse problems

Problem	known	sought
Direct	$A(x)$	$R(s) = G^{-}(0,s)$
Inverse	$R(s)$	$A(x)$

2.8 Numerical implementation

The differential equations

$$\begin{cases} \partial_x G^+ = \frac{1}{2} A G^- \\ \partial_x G^- - 2\partial_s G^- = \frac{1}{2} A G^+ \end{cases}$$

is rewritten as

$$\begin{cases} \frac{d}{dx}\left[G^+(x,s)\right] = \frac{1}{2} A(x)\, G^-(x,s) \\ \frac{d}{dx}\left[G^-(x,s-2x)\right] = \frac{1}{2} A(x)\, G^+(x,s-2x) \end{cases}$$

Integrate in x from x−h to x and reset s

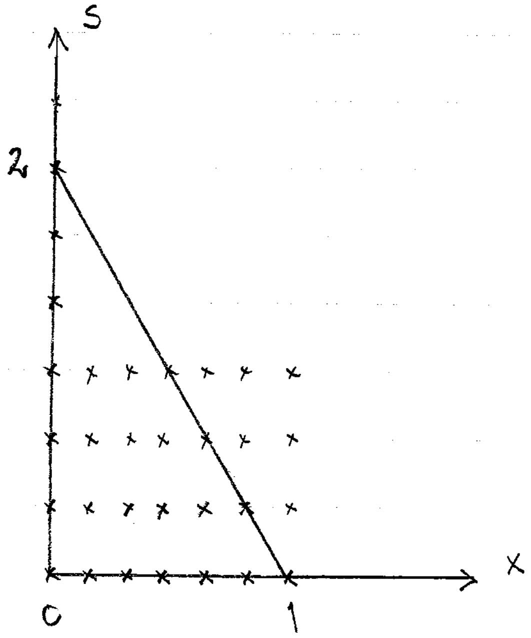
$$\begin{cases} G^+(x,s) - G^+(x-h,s) \\ \qquad = \frac{1}{2}\int_{x-h}^{x} A(x')\, G^-(x',s)\, dx' \\ G^-(x,s) - G^-(x-h,s+2h) \\ \qquad = \frac{1}{2}\int_{x-h}^{x} A(x')\, G^+(x',s+2(x-x'))\, dx' \end{cases}$$

Introduce the discretization in (x,s)-space

$h = 1/N$

$$\begin{cases} x_i = ih & i=0,1,\ldots,N \\ s_j = 2jh & j=0,1,\ldots \end{cases}$$

$$G^{\pm}_{ij} = G^{\pm}(x_i, s_j) \qquad A_i = A(x_i)$$



The trapezoidal rule gives $(\mathcal{O}(h^3)$ neglected$)$

$$G^+_{ij} - G^+_{i-1,j} = \frac{h}{4}\left[A_i G^-_{ij} + A_{i-1} G^-_{i-1,j}\right] \qquad (*)$$

$$G^-_{ij} - G^-_{i-1,j+1} = \frac{h}{4}\left[A_i G^+_{ij} + A_{i-1} G^+_{i-1,j+1}\right] \qquad (**)$$

$i=1,2,\ldots,N$
$j=0,1,\ldots$

The direct problem

$$\begin{cases} A_i \ , i = 0,1,\dots,N & \text{known} \\ R_j = R(2hj) = G^-(0,2hj) = G^-_{0,j} & , j = 0,1,2,\dots \end{cases}$$

Algorithm [(*) and (**) $i \to i+1; j \to j-1$]

$$\begin{cases} G^+_{i,j} = G^+_{i-1,j} + \frac{h}{4}\left[A_i G^-_{i,j} + A_{i-1} G^-_{i-1,j}\right] \\ G^-_{i,j} = G^-_{i+1,j-1} - \frac{h}{4}\left[A_{i+1} G^+_{i+1,j-1} + A_i G^+_{i,j}\right] \end{cases}$$

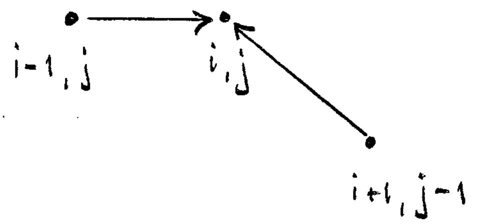
$i = 1,2,\dots,N \quad , \quad j = 1,2,3,\dots$

Solve for $G^{\pm}_{i,j}$

$$G^+_{i,j} = \left(1 + \frac{h^2}{16}A_i^2\right)^{-1}\left[G^+_{i-1,j} + \frac{h}{4}\left(A_i G^-_{i+1,j-1} + A_{i-1} G^-_{i-1,j}\right) - \frac{h^2}{16} A_i A_{i+1} G^+_{i+1,j-1}\right]$$

$$G^-_{i,j} = G^-_{i+1,j-1} - \frac{h}{4}\left[A_{i+1} G^+_{i+1,j-1} + A_i G^+_{i,j}\right]$$

$i = 1,2,3,\dots,N \ ; \ j = 1,2,3,\dots$



Molecule

Initial values

$$\begin{cases} G^+(x,0) = -\frac{1}{8}\int_0^x A^2(x')\,dx' \\ G^-(x,0) = -\frac{1}{4}A(x) \end{cases}$$

These initial values become upon discretization

$$\boxed{\begin{array}{ll} G^+_{i,0} = G^+_{i-1,0} - \frac{h}{16}\left(A^2_{i-1} + A^2_i\right) & , i = 1,2,\dots,N \\ G^-_{i,0} = -\frac{1}{4}A_i & ; i = 0,1,\dots,N \end{array}}$$

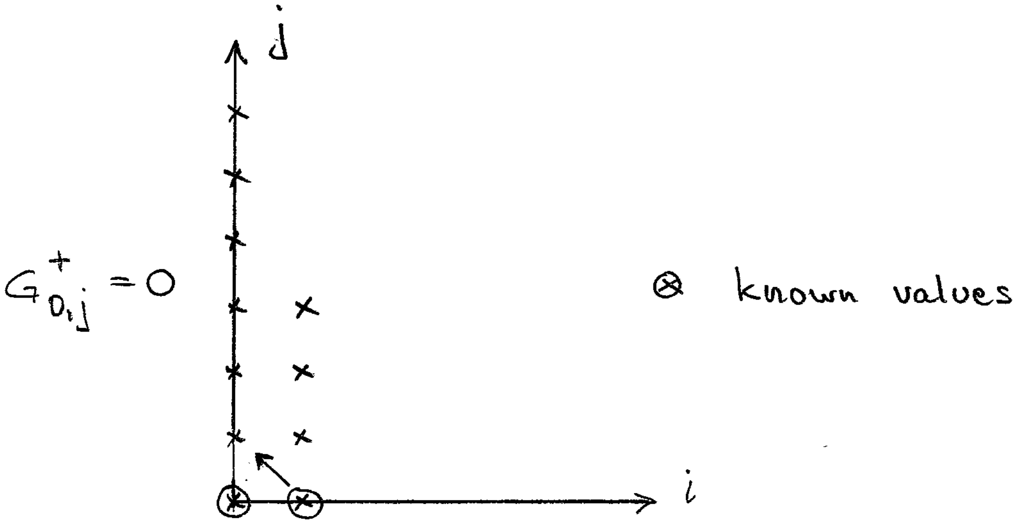
Final values

$$\begin{cases} G^+(0,s) = 0 \\ G^-(0,s) = R(s) \end{cases} \qquad \begin{cases} G^+(1,s) = T(s) \\ G^-(1,s) = 0 \end{cases}$$

This gives

$$\boxed{\begin{array}{ll} G^+_{0,j} = 0 & \\ G^-_{0,j} = R(2jh) & \\ & j = 0,1,2,\dots \\ G^+_{N,j} = T(2jh) & \\ G^-_{N,j} = 0 & \end{array}}$$

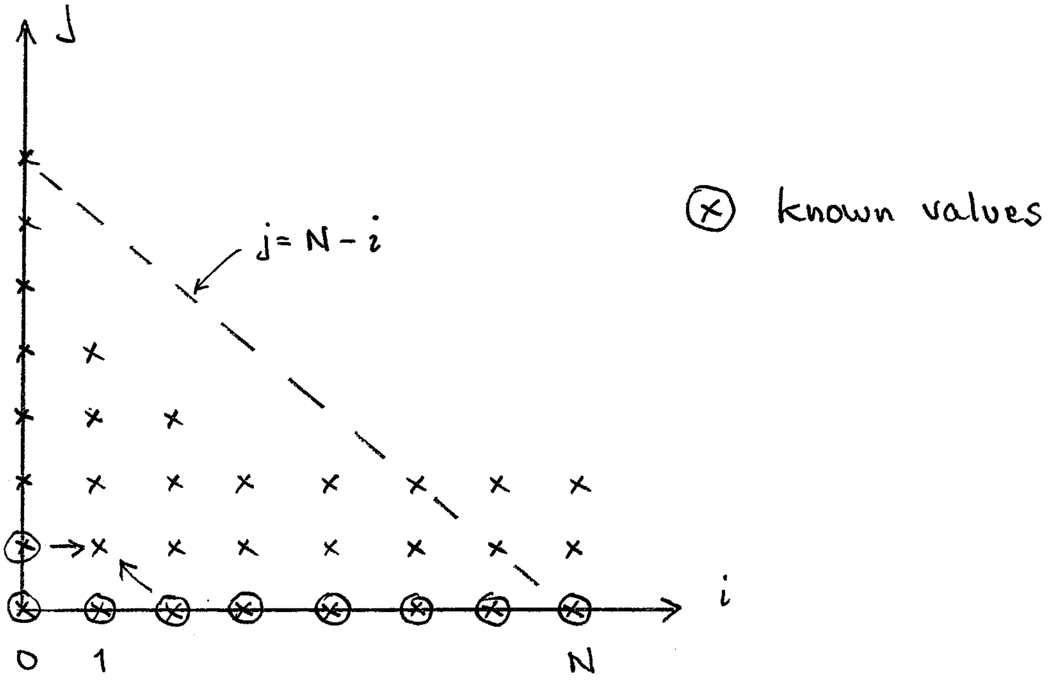
Special problems with i=0



The algorithm becomes

$$\boxed{G^{-}_{0,j} = G^{-}_{1,j-1} - \frac{h}{4} A_1 G^{+}_{1,j-1} \quad , j = 1, 2, 3, \ldots}$$

The algorithm proceeds as



Special measures have to be taken along the discontinuity $s = 2 - x$ in G^-

$$\left[G^-(x, 2(1-x)) \right] = \frac{1}{4} A(1)$$

implies

$$\left[G^-_{i,N-i} \right] = \frac{1}{4} A_N$$

The Green function algorithm is in general one order of magnitude faster than the algolithm based upon the imbedding approach. (no convolutions!)

The inverse problem

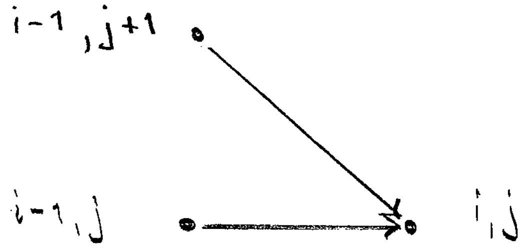
$$\begin{cases} \begin{cases} G^{+}_{0,j} = 0 \\ G^{-}_{0,j} = R(2jh) \end{cases} \quad j = 0,1,\dots,N & \text{one round trip known} \\ \\ A_i \; , \; i = 0,1,\dots,N & \text{sought} \end{cases}$$

Algorithm [(*) and (**) on p. 55]

$$\begin{cases} G^{+}_{i,j} = G^{+}_{i-1,j} + \frac{h}{4}\left[A_i G^{-}_{i,j} + A_{i-1} G^{-}_{i-1,j} \right] \\ G^{-}_{i,j} = G^{-}_{i-1,j+1} + \frac{h}{4}\left[A_i G^{+}_{i,j} + A_{i-1} G^{+}_{i-1,j+1} \right] \end{cases}$$

$$i = 1,2,3,\dots,N \; ; \; j = 0,1,2,\dots,N-1$$

$$\boxed{\begin{aligned} G^{+}_{i,j} &= \left(1 - \frac{h^2}{16} A_i^2\right)^{-1} \Big[G^{+}_{i-1,j} + \\ &\quad + \frac{h}{4}\left(A_i G^{-}_{i-1,j+1} + A_{i-1} G^{-}_{i-1,j} \right) + \frac{h^2}{16} A_i A_{i-1} G^{+}_{i-1,j+1} \Big] \\ G^{-}_{i,j} &= G^{-}_{i-1,j+1} + \frac{h}{4}\left[A_i G^{+}_{i,j} + A_{i-1} G^{+}_{i-1,j+1} \right] \\ & i = 1,2,3,\dots,N \; ; \; j = 0,1,2,3,\dots,N-1 \end{aligned}}$$



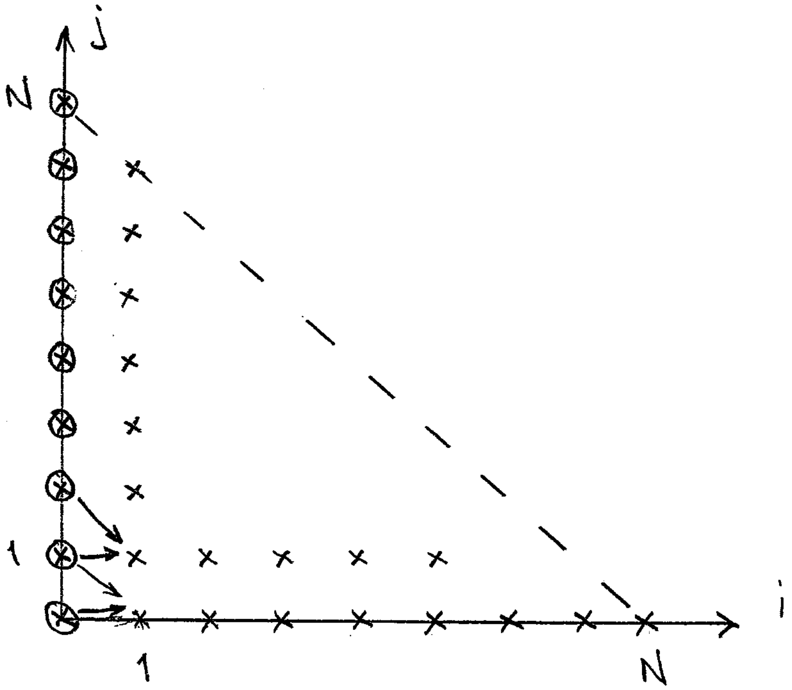
Molecule

The special case $j=0$

$$\boxed{\begin{aligned} G^+_{i,0} &= G^+_{i-1,0} - \frac{h}{16}\left(A_i^2 + A_{i-1}^2\right), \quad i = 1,2,\dots,N \\ G^-_{i,0} &= -\frac{1}{4} A_i, \quad i = 0,1,2,\dots,N \end{aligned}}$$

Initial values, $i=0$

$$\boxed{\begin{aligned} G^+_{0,j} &= 0 \\ G^-_{0,j} &= R(2jh) \end{aligned} \qquad j = 0,1,2,\dots,N}$$



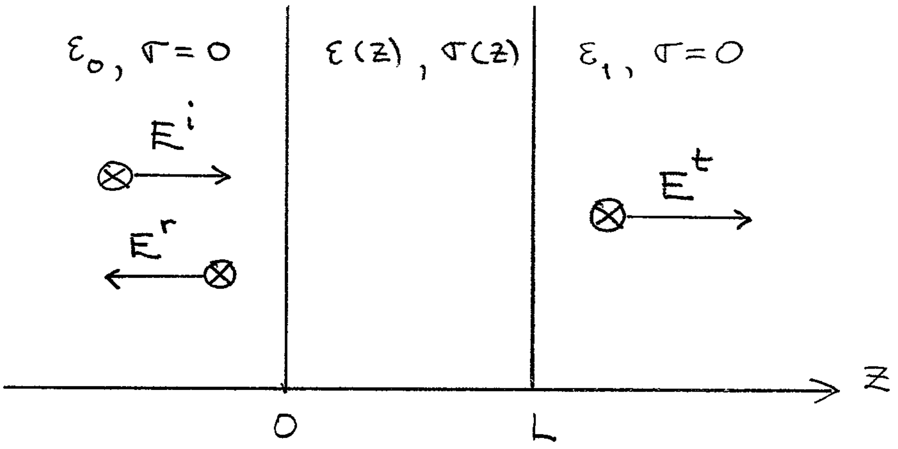
The unknown A_i satisfies

$$A_i\left(1 + h\,G^+_{i-1,0} - \frac{h^2}{16} A^2_{i-1}\right) - \frac{h^2}{16} A_i^3 = -4 G^-_{i-1,1} - h\,A_{i-1}\,G^+_{i-1,1} = \text{known}$$

B. Scattering in non-dispersive lossy media
Continuous permittivity profile

1. Imbedding approach

1.1. Basic equations



Transverse polarization and Maxwell's equations imply

$$\begin{cases} \partial_z^2 E(z,t) - c^{-2}(z)\,\partial_t^2 E(z,t) - b(z)\,\partial_t E(z,t) = 0 \\ c^{-2}(z) = \varepsilon(z)\mu_0 \\ b(z) = \sigma(z)\mu_0 \end{cases}$$

Assume $\varepsilon(z)$ continuous at $z = 0, L$, i.e

$$\varepsilon(0^+) = \varepsilon_0 \quad \text{and} \quad \varepsilon(L^-) = \varepsilon_1$$

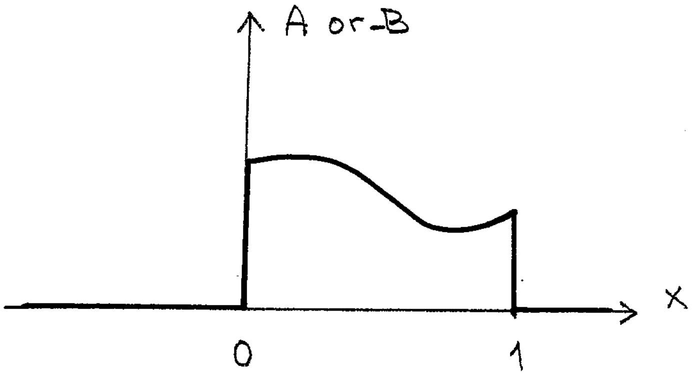
Introduce travel time coordinates

$$\begin{cases} x = x(z) = \int_0^z \ell^{-1} c^{-1}(z')\,dz' \\ s = t/\ell \\ u(x,s) = E(z,t) \end{cases} \qquad \left(\ell = \int_0^L c^{-1}(z)\,dz \right)$$

The PDE is then

$$\boxed{\begin{aligned} &\partial_x^2 u(x,s) - \partial_s^2 u(x,s) + A(x)\,\partial_x u(x,s) + B(x)\,\partial_s u(x,s) = 0 \\ &A(x) = -\partial_x \ln\left(c(z(x))\right) \\ &B(x) = -\ell\, b(z(x))\, c^2(z(x)) \end{aligned}}$$

$A(x)$, $B(x) \neq 0$ when $x \in (0,1)$, otherwise 0



$s = 2$ one round trip through the slab

The wave splitting is the same as before

$$u^{\pm}(x,s) = \frac{1}{2}\left\{ u(x,s) \mp \int_{-\infty}^{s} u_x(x,s')\,ds' \right\}$$

$$= \frac{1}{2}\left\{ u(x,s) \mp \partial_s^{-1} u_x(x,s) \right\}$$

In a homogeneous region, $\varepsilon(z) =$ constant and $\sigma = 0$, $A(x) = B(x) = 0$. Then, as in the lossless case, the wave splitting projects out the left- and right going parts of the field.

$$\begin{cases} u^{+}(x,s) = f(x-s) & \text{(right going)} \\ u^{-}(x,s) = g(x+s) & \text{(left going)} \end{cases}$$

$$u(x,s) = f(x-s) + g(x+s)$$

To find the PDE for the $u^{\pm}$ proceed as in the lossless case, Chapter A. 1.4, p 7.

Rewrite the wave equation as a first order system

$$\partial_x \begin{pmatrix} u \\ u_x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \partial_s^2 - B\partial_s & -A \end{pmatrix} \begin{pmatrix} u \\ u_x \end{pmatrix} = D \begin{pmatrix} u \\ u_x \end{pmatrix}$$

and

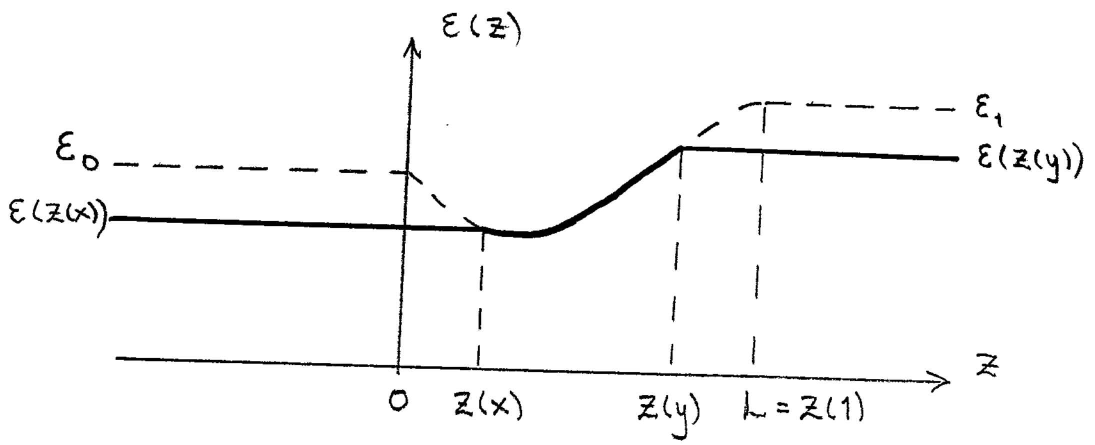
$$\begin{pmatrix} u^+ \\ u^- \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -\partial_s^{-1} \\ 1 & \partial_s^{-1} \end{pmatrix} \begin{pmatrix} u \\ u_x \end{pmatrix} = P \begin{pmatrix} u \\ u_x \end{pmatrix}$$

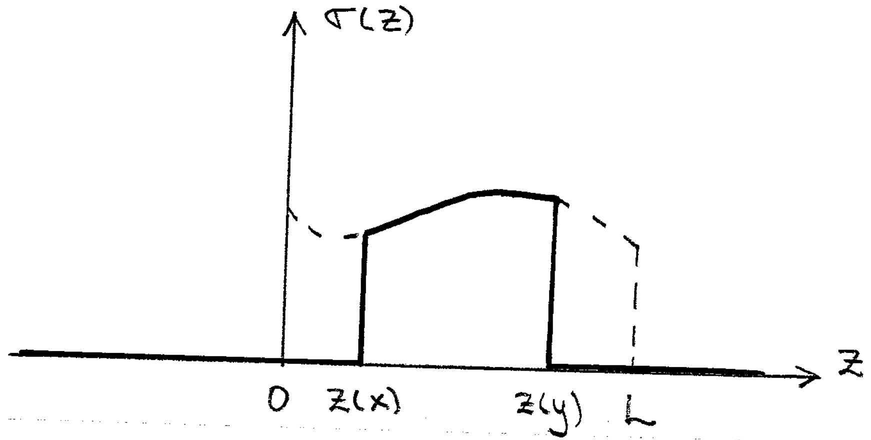
As before we get

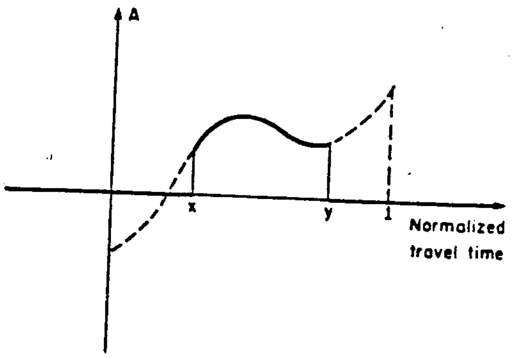
$$\partial_x \begin{pmatrix} u^+ \\ u^- \end{pmatrix} = P_x P^{-1} \begin{pmatrix} u^+ \\ u^- \end{pmatrix} + PDP^{-1} \begin{pmatrix} u^+ \\ u^- \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} u^+ \\ u^- \end{pmatrix}$$

$$\begin{cases} \alpha = -\partial_s - \frac{1}{2}(A - B) \\ \beta = \frac{1}{2}(A + B) \\ \gamma = \frac{1}{2}(A - B) \\ \delta = \partial_s - \frac{1}{2}(A + B) \end{cases}$$

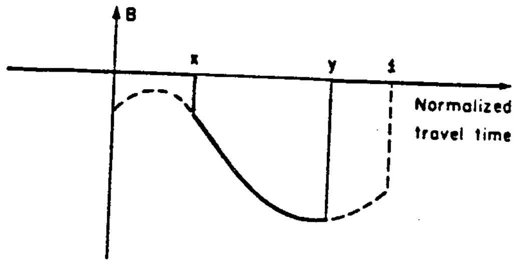
As in the lossless case
we adopt the imbedding technique, but now we allow both endpoints of the slab to vary.







One round trip
$s = 2(y - x)$.

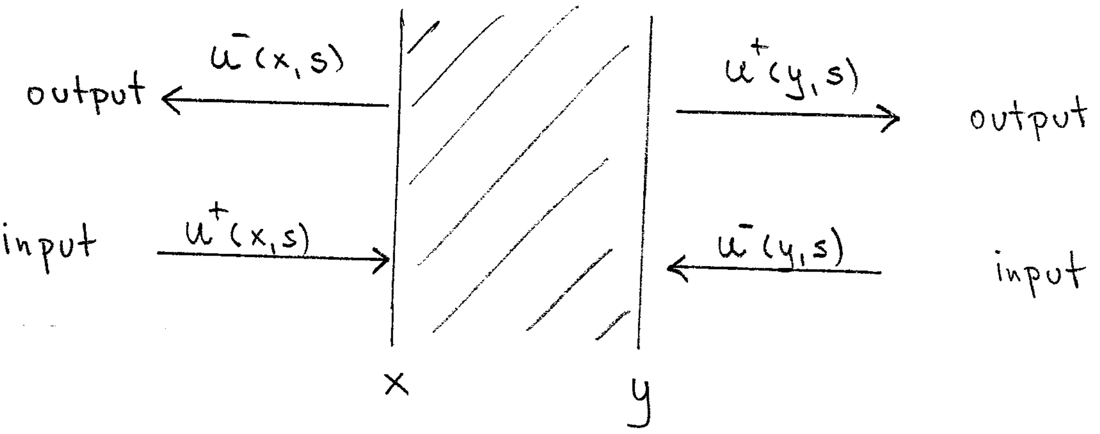


1.2 Scattering operator representation

The relation between $u^{\pm}(x,s)$ and $u^{\pm}(y,s)$ on both sides of the slab is in the general case with sources on both sides of the slab

$$\begin{pmatrix} u^+(y,s) \\ u^-(x,s) \end{pmatrix} = \begin{pmatrix} T^+(x,y) & R^-(x,y) \\ R^+(x,y) & T^-(x,y) \end{pmatrix} \begin{pmatrix} u^+(x,s) \\ u^-(y,s) \end{pmatrix}$$

output matrix operator input



Explicitly,

$$\begin{cases} u^+(y,s) = T^+(x,y)\,u^+(x,s) + R^-(x,y)\,u^-(y,s) \\ u^-(x,s) = R^+(x,y)\,u^+(x,s) + T^-(x,y)\,u^-(y,s) \end{cases}$$

R^+ reflection operator from the left (as in Chapter A)
R^- ———— " ———— right
T^+ transmission operator from the left (as in Chapter A)
T^- ———— " ———— right

Integral representations (Duhamel's principle)

$$\begin{cases} [R^+(x,y)\,u^+(x\,\cdot)](x,s) = \int_{-\infty}^{s} R^+(x,y,s-s')\,u^+(x,s')\,ds' \\ [R^-(x,y)\,u^-(y,\cdot)](y,s) = \int_{-\infty}^{s} R^-(x,y,s-s')\,u^-(y,s')\,ds' \\ [T^+(x,y)\,u^+(x,\cdot)](y,s+y-x) = t^-(x,y)\left\{ u^+(x,s) + \int_{-\infty}^{s} T(x,y,s-s')\,u^+(x,s')\,ds' \right\} \\ [T^-(x,y)\,u^-(y,\cdot)](x,s+y-x) = t^+(x,y)\left\{ u^-(y,s) + \int_{-\infty}^{s} T(x,y,s-s')\,u^-(y,s')\,ds' \right\} \end{cases}$$

$$t^{\pm}(x,y) = \exp\left\{ \pm \frac{1}{2} \int_x^y [A(x') \pm B(x')]\,dx' \right\}$$

$R^{\pm}(x,y,s)$	are the reflection kernels from left and right, respectively.
$T(x,y,s)$	is the transmission kernel. Notice that the integral representations of the operators $T^{\pm}(x,y)$ are the same (apart from a multiplicative factor). This is <u>reciprocity</u>!

1.3 The imbedding equations for $R^{\pm}(x,y,s)$ and $T(x,y,s)$

Scattering operator representation:

$$\begin{pmatrix} u^+(y,s) \\ u^-(x,s) \end{pmatrix} = \begin{pmatrix} T^+(x,y) & R^-(x,y) \\ R^+(x,y) & T^-(x,y) \end{pmatrix} \begin{pmatrix} u^+(x,s) \\ u^-(y,s) \end{pmatrix} = S(x,y) \begin{pmatrix} u^+(x,s) \\ u^-(y,s) \end{pmatrix}$$

Dynamics of $u^{\pm}$:

$$\partial_x \begin{pmatrix} u^+(x,s) \\ u^-(x,s) \end{pmatrix} = \begin{pmatrix} \alpha(x) & \beta(x) \\ \gamma(x) & \delta(x) \end{pmatrix} \begin{pmatrix} u^+(x,s) \\ u^-(x,s) \end{pmatrix}$$

Differentiate the scattering operator representation w.r.t. x

$$\begin{pmatrix} 0 \\ u^-_x(x,s) \end{pmatrix} = S_x(x,y) \begin{pmatrix} u^+(x,s) \\ u^-(y,s) \end{pmatrix} + S(x,y) \begin{pmatrix} u^+_x(x,s) \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ \gamma(x) & \delta(x) \end{pmatrix} \begin{pmatrix} u^+(x,s) \\ u^-(x,s) \end{pmatrix} = S_x(x,y) \begin{pmatrix} u^+(x,s) \\ u^-(y,s) \end{pmatrix}$$

$$+ S(x,y) \begin{pmatrix} \alpha(x) & \beta(x) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u^+(x,s) \\ u^-(x,s) \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ \gamma(x) & \delta(x) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ R^+(x,y) & T^-(x,y)\end{pmatrix}\begin{pmatrix} u^+(x,s) \\ u^-(y,s)\end{pmatrix} =$$

$$= S_x(x,y)\begin{pmatrix} u^+(x,s)\\ u^-(y,s)\end{pmatrix} + S(x,y)\begin{pmatrix}\alpha(x) & \beta(x)\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ R^+(x,y) & T^-(x,y)\end{pmatrix}\begin{pmatrix}u^+(x,s)\\u^-(y,s)\end{pmatrix},$$

$$S_x(x,y) = \begin{pmatrix} 0 & 0 \\ \gamma(x) & \delta(x) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ R^+(x,y) & T^-(x,y)\end{pmatrix}$$

$$- S(x,y)\begin{pmatrix}\alpha(x) & \beta(x)\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ R^+(x,y) & T^-(x,y)\end{pmatrix}$$

$$= -\begin{pmatrix} T^+(x,y) & 0 \\ R^+(x,y) & 1\end{pmatrix}\begin{pmatrix}\alpha(x) & \beta(x)\\ -\gamma(x) & -\delta(x)\end{pmatrix}\begin{pmatrix}1 & 0\\ R^+(x,y) & T^-(x,y)\end{pmatrix}$$

and similarly for $S_y(x,y)$.

In terms of the kernels $R^\pm(x,y,s)$ and $T(x,y,s)$

$$\begin{cases} [R^+(x,y)u^+(x,\cdot)](x,s) = \int_{-\infty}^{s} R^+(x,y,s-s')u^+(x,s')ds' \\ [R^-(x,y)u^-(y,\cdot)](y,s) = \int_{-\infty}^{s} R^-(x,y,s-s')u^-(y,s')ds' \\ [T^+(x,y)u^+(x,\cdot)](y,s+y-x) = t^-(x,y)\left\{u^+(x,s) + \int_{-\infty}^{s} T(x,y,s-s')u^+(x,s')ds'\right\} \\ [T^-(x,y)u^-(y,\cdot)](x,s+y-x) = t^+(x,y)\left\{u^-(y,s) + \int_{-\infty}^{s} T(x,y,s-s')u^-(y,s')ds'\right\} \end{cases}$$

we get (without derivation) 6 imbedding equations for the 3 kernels, $R^\pm$, T.

Imbedding equations

variation of the left endpoint x

1.
$$\begin{cases} R^+_x(x,y,s) = 2R^+_s(x,y,s) - B(x)\,R^+(x,y,s) - \\ \qquad -\frac{1}{2}\left[A(x)+B(x)\right]\int_0^s R^+(x,y,s-s')\,R^+(x,y,s')\,ds' \quad , s>0 \\ R^+(y,y,s) = 0 \quad , s>0 \\ R^+(x,y,0^+) = -\frac{1}{4}\left[A(x)-B(x)\right] \quad , 0<x<y<1 \end{cases}$$

2.
$$\begin{cases} T_x(x,y,s) = -\frac{1}{2}\left[A(x)+B(x)\right]\Big\{R^+(x,y,s) + \\ \qquad + \int_0^s T(x,y,s-s')\,R^+(x,y,s')\,ds'\Big\} \quad , s>0 \\ T(y,y,s) = 0 \quad , s>0 \end{cases}$$

3.
$$\begin{cases} R^-_x(x,y,s) = \left[A(x)+B(x)\right]\exp\left[\int_x^y B(x')\,dx'\right]\Big\{T(x,y,s-2(y-x)) + \\ \qquad + \frac{1}{2}\int_0^{s-2(y-x)} T(x,y,s-2(y-x)-s')\,T(x,y,s')\,ds'\Big\} \quad , s>2(y-x) \\ R^-(y,y,s) = 0 \quad , s>0 \end{cases}$$

Similarly, for a variation of the right endpoint y.

4. $$\begin{cases} R^+_y(x,y,s) = -[A(y)-B(y)]\exp[\int_x^y B(x')dx']\{T(x,y,s-2(y-x)) + \\ \qquad +\frac{1}{2}\int_0^{s-2(y-x)} T(x,y,s-2(y-x)-s')T(x,y,s')ds'\}, \quad s>2(y-x) \\ R^+(x,x,s)=0 \quad , \quad s>0 \end{cases}$$

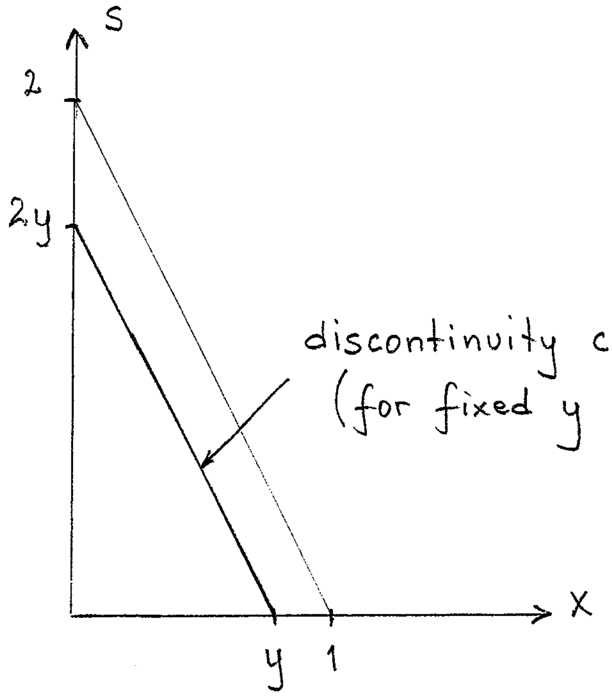
5. $$\begin{cases} T_y(x,y,s) = -\frac{1}{2}[A(y)-B(y)]\{R^-(x,y,s) + \\ \qquad + \int_0^s T(x,y,s-s')R^-(x,y,s')ds'\}, \quad s>0 \\ T(x,x,s)=0 \quad , \quad s>0 \end{cases}$$

6. $$\begin{cases} R^-_y(x,y,s) = -2R^-_s(x,y,s) + B(y)R^-(x,y,s) - \\ \qquad -\frac{1}{2}[A(y)-B(y)]\int_0^s R^-(x,y,s-s')R^-(x,y,s')ds' \quad , \quad s>0 \\ R^-(x,x,s)=0 \quad , \quad s>0 \\ R^-(x,y,0^+) = \frac{1}{4}[A(y)+B(y)] \quad , \quad 0<x<y<1. \end{cases}$$

1.4 Additional properties of $R^{\pm}(x,y,s)$

The kernels $R^{\pm}(x,y,s)$ are discontinuous along the curve $s = 2(y-x)$, cf. section A.1.10, p. 24
Propagation of singularities arguments show that

$$\left\{\begin{aligned} \left[R^{+}(x,y,s)\right]_{s=2(y-x)^{-}}^{s=2(y-x)^{+}} &= \tfrac{1}{4}\left[A(y)-B(y)\right]\exp\left[\int_x^y B(x')\,dx'\right] \\ \left[R^{-}(x,y,s)\right]_{s=2(y-x)^{-}}^{s=2(y-x)^{+}} &= -\tfrac{1}{4}\left[A(x)+B(x)\right]\exp\left[\int_x^y B(x')\,dx'\right] \end{aligned}\right.$$



discontinuity curve of $R^{\pm}(x,y,s)$
(for fixed y as a function of (x,s))

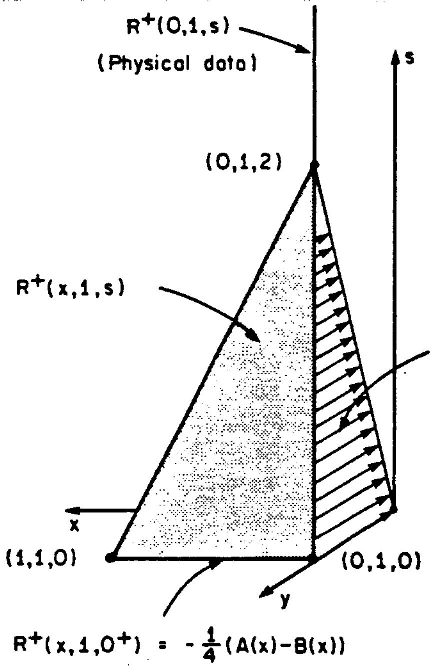
This discontinuity in $R^{\pm}$ reflects the difference in $\partial_z \varepsilon(z)$ and $\sigma(z)$ at the back wall of the slab (x,y).

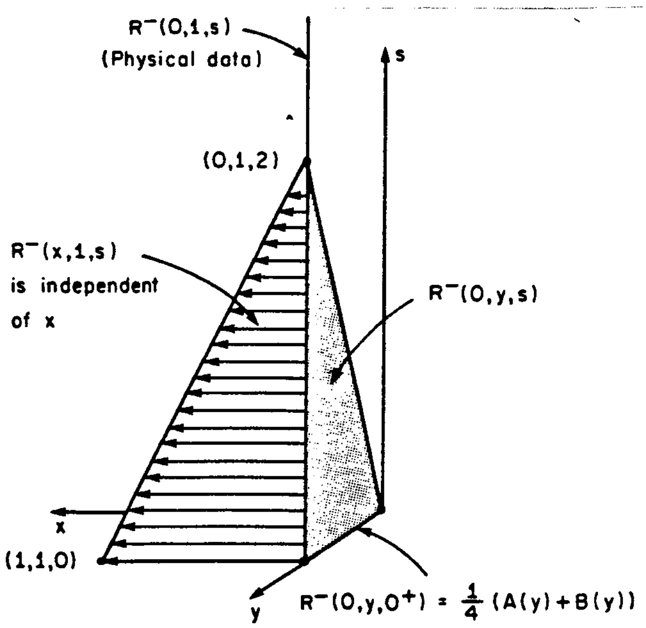
One can also show

$$\begin{cases} R^{+}(x,y,s) \text{ independent of } y \text{ for } s < 2(y-x) \\ R^{-}(x,y,s) \text{ —— '' —— } x \text{ —— '' ——} \end{cases}$$

or

$$\begin{cases} R^{+}(x,y,s) = R^{+}(x, x+\tfrac{s}{2}+0, s) \quad , \quad s < 2(y-x) \\ R^{-}(x,y,s) = R^{-}(y-\tfrac{s}{2}-0, y, s) \quad , \quad s < 2(y-x) \end{cases}$$





1.5 The direct problem

$$\begin{cases} \text{Given:} \quad \varepsilon(z)\,,\ \sigma(z) \ \Rightarrow \ x(z)\,,\ A(x)\,,\ B(x) \ \text{known} \\ \text{Find:} \quad R^{\pm}(0,1,s)\,,\ T(0,1,s) \quad \text{"physical" kernels} \end{cases}$$

1. $$\begin{cases} R^+_x = 2R^+_s - BR^+ - \frac{1}{2}[A+B]\, R^+ * R^+ \\ R^+(s=0^+) = -\frac{1}{4}[A-B] \end{cases}$$

As in the lossless case (p. 21) this equation can be solved for $R^+(0,1,s)$.

6. $$\begin{cases} R^-_y = -2R^-_s + BR^- - \frac{1}{2}[A-B]\, R^- * R^- \\ R^-(s=0^+) = \frac{1}{4}[A+B] \end{cases}$$

gives $R^-(0,1,s)$

$T(0,1,s)$ can be found from

2. $$T_x = -\frac{1}{2}[A+B]\left\{R^+ + T * R^+\right\}$$

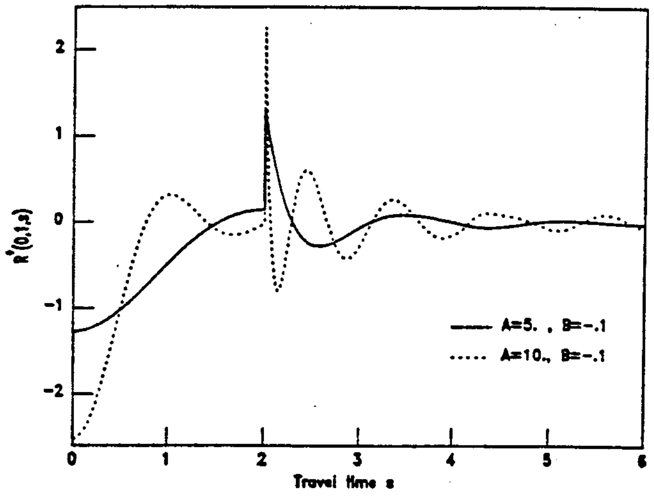
Examples

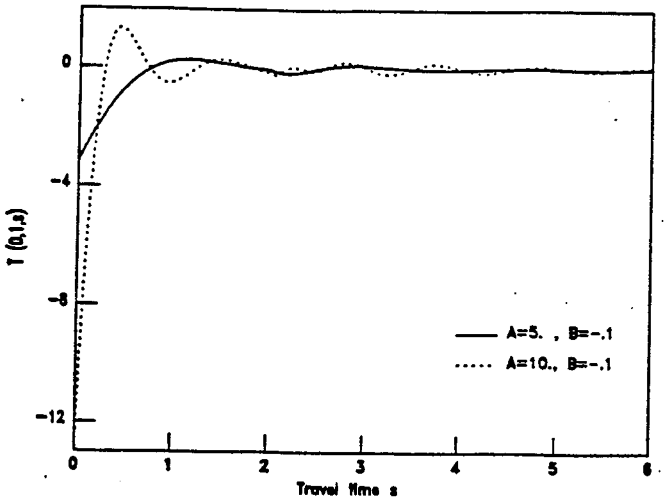
A and B constants

correspond to $\varepsilon(z) = \varepsilon(0)\left[1 - \frac{z}{L}(1 - e^{-A})\right]^{-2}$

$$\sigma(z) = -\sqrt{\frac{\varepsilon(0)}{\mu_0}}\,\frac{B(1-e^{-A})}{AL}\left[1 - \frac{z}{L}(1-e^{-A})\right]^{-2}$$

$R^{\pm}$ and T can be solved analytically





1.6 The propagator kernels $W(x,y,s)$ and $V^{\pm}(x,y,s)$

Back to the integral representations (sources on the left hand side, i.e. $u^-(y,s) = 0$)

$$\begin{cases} u^-(x,s) = \int_{-\infty}^{s} R^+(x,y,s-s')\, u^+(x,s')\, ds' \\ u^+(y, s+y-x) = t^-(x,y)\left\{ u^+(x,s) + \int_{-\infty}^{s} T(x,y,s-s')\, u^+(x,s')\, ds' \right\} \end{cases}$$

The second equation (Volterra eq. of the second kind) can be solved for $u^+(x,s)$

$$u^+(x,s) = [t^-(x,y)]^{-1} \left\{ u^+(y, s+y-x) + \int_{-\infty}^{s} W(x,y,s-s')\, u^+(y, s'+y-x)\, ds' \right\}$$

and $W(x,y,s)$ satisfies <u>the resolvent equation</u>

$$\boxed{T(x,y,s) + W(x,y,s) + \int_0^s T(x,y,s-s')\, W(x,y,s')\, ds' = 0 \quad , \; s>0}$$

This is also a Volterra equation of the second kind

$$T(x,y,s) \Longleftrightarrow W(x,y,s)$$

That is one-to-one correspondence between T and W.

Insert

$$u^+(x,s) = [t^-(x,y)]^{-1} \left\{ u^+(y, s+y-x) + \int_{-\infty}^{s} W(x,y,s-s')\, u^+(y, s'+y-x)\, ds' \right\}$$

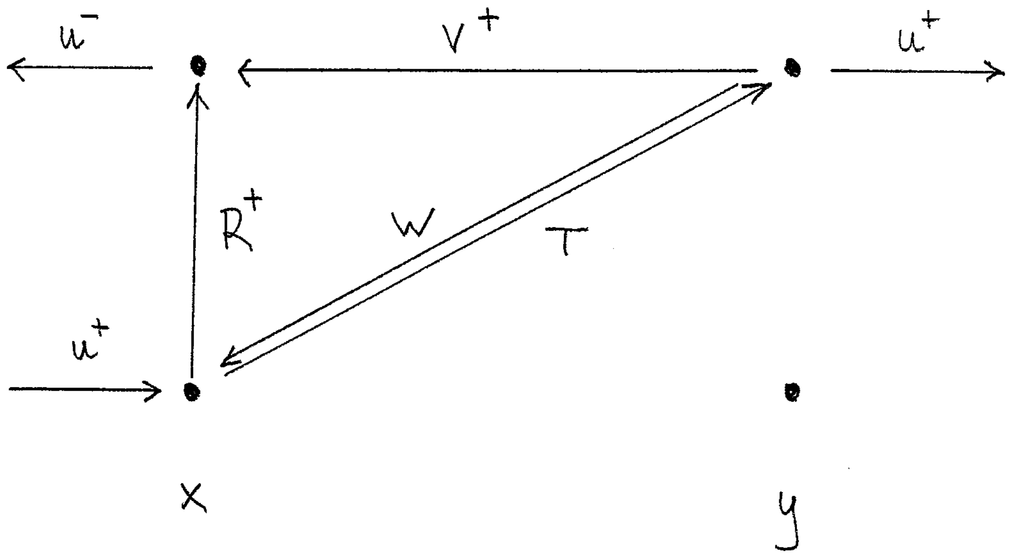
in the top equation on p. 77, and rewrite as

$$u^-(x,s) = [t^-(x,y)]^{-1} \int_{-\infty}^{s} V^+(x,y,s-s')\, u^+(y, s'+y-x)\, ds'$$

where

$$\boxed{V^+(x,y,s) = R^+(x,y,s) + \int_0^s R^+(x,y,s-s')\, W(x,y,s')\, ds'}$$

Relation between the kernels R^+, T, W and V^+



We get similar results for the case with sources on the right hand side, i.e. $u^+(x,s)=0$.

$$u^-(y,s) = \left[t^+(x,y)\right]^{-1} \left\{ u^-(x,s+y-x) + \int_{-\infty}^{s} W(x,y,s-s')\, u^-(x,s'+y-x)\, ds' \right\}$$

where $W(x,y,s)$ is defined on p. 77

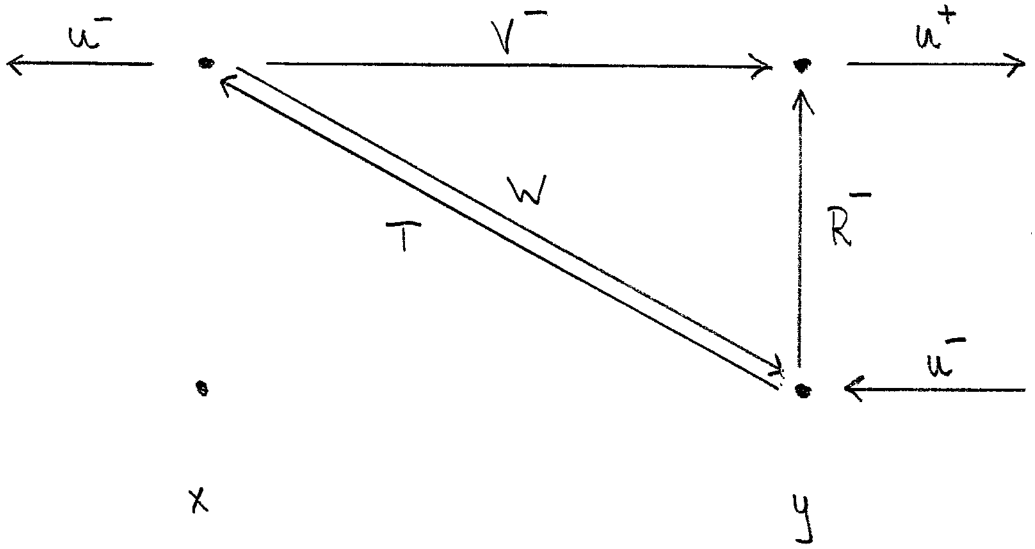
Furthermore,

$$u^+(y,s) = \left[t^+(x,y)\right]^{-1} \int_{-\infty}^{s} V^-(x,y,s-s')\, u^-(x,s'+y-x)\, ds'$$

where

$$\boxed{V^-(x,y,s) = R^-(x,y,s) + \int_0^s R^-(x,y,s-s')\, W(x,y,s')\, ds'}$$

and



1.7 Imbedding equations for the propagator kernels $W(x,y,s)$ and $V^{\pm}(x,y,s)$

From the resolvent equation $\quad T + W + T*W = 0$
we get by diff. w.r.t. x

$$T_x + W_x + T_x * W + T * W_x = 0$$

The imbedding equation for T : $\; T_x = -\frac{1}{2}[A+B][R^+ + T*R^+]$
gives

$$W_x = \tfrac{1}{2}[A+B][R^+ + \cancelto{\textcircled{1}}{T*R^+}] + \tfrac{1}{2}[A+B][\cancelto{\textcircled{1}}{R^+ * W} + \cancelto{\textcircled{1}}{(T*R^+)}*W]$$

$$- T * W_x = \tfrac{1}{2}[A+B]R^+ - T*W_x$$

Convolute with W :

$$\cancelto{\textcircled{1}}{W*W_x} = \tfrac{1}{2}[A+B]\, W*R^+ - \underbrace{W*(T*W_x)}$$

$$- T*W_x - \cancelto{\textcircled{1}}{W*W_x}$$

$$T*W_x + \tfrac{1}{2}[A+B]\, W*R^+ = 0$$

$$\begin{cases} W_x(x,y,s) = \frac{1}{2}\left[A(x)+B(x)\right]\left[R^+(x,y,s) + \int_0^s W(x,y,s-s')R^+(x,y,s')ds'\right] & , s>0 \\ W(y,y,s) = 0 & , s>0 \end{cases}$$

Similarly,

$$\begin{cases} W_y(x,y,s) = \frac{1}{2}[A(y)-B(y)]\left[R^-(x,y,s) + \int_0^s W(x,y,s-s')R^-(x,y,s')\,ds'\right], & s>0 \\ W(x,x,s) = 0, & s>0 \end{cases}$$

$$\begin{cases} V^+_x(x,y,s) = 2V^+_s(x,y,s) - B(x)V^+(x,y,s) + \\ \qquad + \frac{1}{2}[A(x)-B(x)]W(x,y,s), & s>0 \\ V^+(y,y,s) = 0, & s>0 \\ V^+(x,y,0^+) = -\frac{1}{4}[A(x)-B(x)], & x<y \end{cases}$$

$$\begin{cases} V^-_y(x,y,s) = -2V^-_s(x,y,s) + B(y)V^-(x,y,s) + \\ \qquad + \frac{1}{2}[A(y)+B(y)]W(x,y,s), & s>0 \\ V^-(x,x,s) = 0, & s>0 \\ V^-(x,y,0^+) = \frac{1}{4}[A(y)+B(y)], & x<y \end{cases}$$

The two final imbedding equations are

$$\begin{cases} V_y^+(x,y,s) = \frac{1}{2}[A(y) - B(y)] \int_0^s R^-(x,y,s') V^+(x,y,s-s') ds' = \\ \qquad = \frac{1}{2}[A(y) - B(y)] \int_0^s R^+(x,y,s') V^-(x,y,s-s') ds' , \quad 0 < s < 2(y-x) \\ \\ V_x^-(x,y,s) = \frac{1}{2}[A(x) + B(x)] \int_0^s R^+(x,y,s') V^-(x,y,s-s') ds' = \\ \qquad = \frac{1}{2}[A(x) + B(x)] \int_0^s R^-(x,y,s') V^+(x,y,s-s') ds' , \quad 0 < s < 2(y-x) \end{cases}$$

The final equalities come from $V^{\pm} = R^{\pm} + R^{\pm} * W$

From the definition of $V^{\pm}$ and imbedding equation of W we get

$$\begin{cases} W_x(x,y,s) = \frac{1}{2}(A(x) + B(x)) V^+(x,y,s) , & s > 0 \\ W_y(x,y,s) = \frac{1}{2}(A(y) - B(y)) V^-(x,y,s) , & s > 0 \end{cases}$$

1.8 Compact support of W and $V^\pm$

The imbedding equations for W and R^+ give

$$\begin{cases} W_x = \frac{1}{2}(A+B)\left(R^+ + R^+ * W\right) & (1) \\ R^+_x = 2R^+_s - BR^+ - \frac{1}{2}(A+B)\, R^+ * R^+ & (2) \end{cases}$$

Convolve (1) with R^+ and use (2)

$$W_x * R^+ = \frac{1}{2}(A+B)\left[R^+ * R^+ + \underbrace{(R^+ * W) * R^+}_{W * (R^+ * R^+)}\right]$$

$$W_x * R^+ = 2R^+_s - R^+_x - BR^+ + W * \left[2R^+_s - R^+_x - BR^+\right]$$

$$\partial_x\left(R^+ + W * R^+\right) - 2\partial_s\left(R^+ + W * R^+\right) + B\left(R^+ + W * R^+\right)$$

$$+ 2R^+(0)\, W = 0$$

Use (1) and $R^+(0) = -\frac{1}{4}(A-B)$

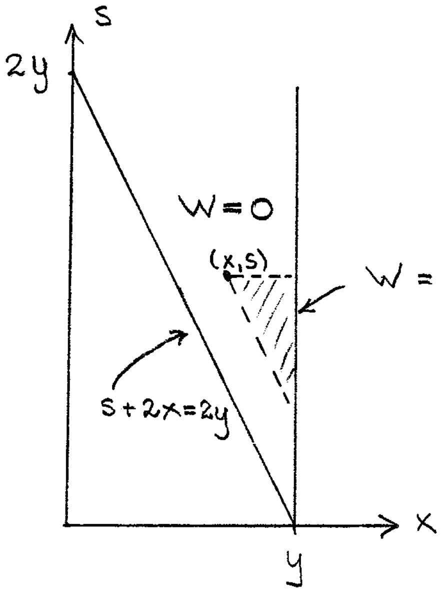
$$\partial_x\left(\frac{2W_x}{A+B}\right) - 2\partial_s\left(\frac{2W_x}{A+B}\right) + B\,\frac{2W_x}{A+B} - \frac{1}{2}(A-B)\,W = 0$$

Principle part: $\quad W_{xx} - 2W_{xs}$

Characteristic curves: $\begin{cases} s = \text{constant} \\ s + 2x = \text{constant} \end{cases}$

Cauchy data on $x = y$: $W(y,s) = W_x(y,s) = 0$

(follows from $W_x = \frac{1}{2}(A+B)(R^+ + W * R^+)$)

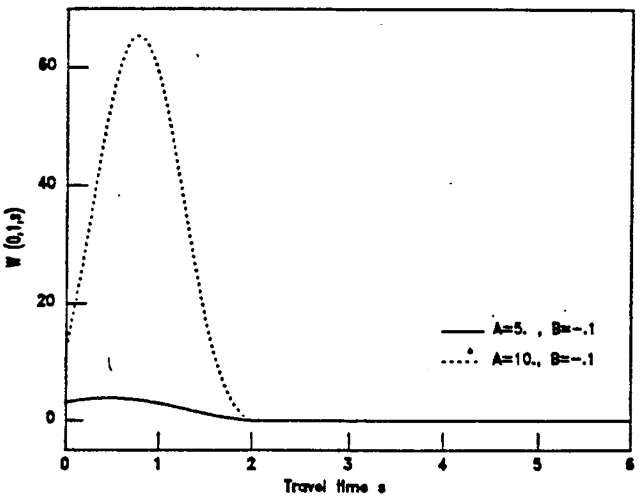


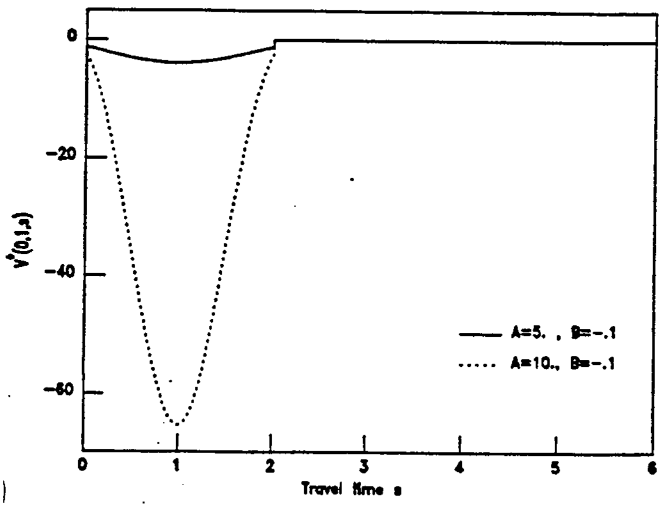
Uniqueness of the solution of the linear PDE for W implies that $W(x,y,s) = 0$, $s > 2(y-x)$

The relations between W and $V^\pm$ on p. 82 then give $V^\pm(x,y,s) = 0$, $s > 2(y-x)$

$$\boxed{\begin{aligned} W(x,y,s) &= 0 \\ V^\pm(x,y,s) &= 0 \end{aligned} \quad , \quad s > 2(y-x)}$$

Examples (A and B constants)





Notice that W, V^+ (and V^-) are identically zero for $s > 2$ (one round trip)

1.9 Extension of data

a. Transmission data

Compact support of W : $W(x,y,s) = 0 \qquad s > 2(y-x)$

Resolvent equation: $T + W + T * W = 0$

For $s > 2(y-x)$

$$T(x,y,s) + \int_{s-2(y-x)}^{s} W(x,y,s-s')\,T(x,y,s')\,ds' = 0 \quad , \quad s > 2(y-x)$$

Rewrite

$$\boxed{\begin{aligned} & T(x,y,s) + \int_{2(y-x)}^{s} W(x,y,s-s')\,T(x,y,s')\,ds' = G(x,y,s) \quad , \quad s > 2(y-x) \\ & G(x,y,s) = \begin{cases} -\displaystyle\int_{s-2(y-x)}^{2(y-x)} W(x,y,s-s')\,T(x,y,s')\,ds' & , \ 2(y-x) < s < 4(y-x) \\ 0 & , \ s > 4(y-x) \end{cases} \end{aligned}}$$

This is a Volterra equation of the second kind for $T(x,y,s)$, $s > 2(y-x)$.
Notice that if $T(x,y,s)$ is known for $0 < s < 2(y-x)$ and thus $W(x,y,s)$ in this interval, then $T(x,y,s)$ can be determined at times $s > 2(y-x)$ by the eq. above.

b. Reflection data

The same arguments can be used on
$V^{\pm} = R^{\pm} + R^{\pm} * W$ to extend $R^{\pm}$-data beyond
one round trip.

The results are:

$$\boxed{\begin{aligned} & R^{\pm}(x,y,s) + \int_{2(y-x)}^{s} W(x,y,s-s')\, R^{\pm}(x,y,s')\, ds' = \\ & = \begin{cases} -\int_{s-2(y-x)}^{2(y-x)} W(x,y,s-s')\, R^{\pm}(x,y,s')\, ds' & , 2(y-x) < s < 4(y-x) \\ 0 & , s > 4(y-x) \end{cases} \end{aligned}}$$

Again we have a Volterra equation of the second kind for
$R^{\pm}(x,y,s)$, $s > 2(y-x)$.
However, this time not only reflection data for one round trip
are needed, but transmission data as well (to get W)

Specifically, the concept of extension of reflection data can be used to calculate the jump in $R^{\pm}$ at one round trip.

$$\left[R^{\pm}(x,y,s)\right]_{s=2(y-x)^{+}}^{s=2(y-x)^{-}} = R^{\pm}(x,y,2(y-x)^{+}) - R^{\pm}(x,y,2(y-x)^{-}) =$$

$$= -\int_0^{2(y-x)} W(x,y,2(y-x)-s')R^{\pm}(x,y,s')\,ds' - R^{\pm}(x,y,2(y-x)^{-})$$

In particular, for $R^{-}(x,y,s)$ and $y=1$

$$\left[R^{-}(x,1,s)\right]_{s=2(1-x)^{-}}^{s=2(1-x)^{+}} = -\int_0^{2(1-x)} W(x,1,2(1-x)-s')R^{-}(x,1,s')\,ds' - R^{-}(x,1,2(1-x)^{-})$$

However, $R^{-}(x,1,s) = R^{-}(0,1,s)$, $s < 2(1-x)$ (see p. 74)

$$\boxed{\begin{aligned}\left[R^{-}(x,1,s)\right]_{s=2(1-x)^{-}}^{s=2(1-x)^{+}} &= -\int_0^{2(1-x)} W(x,1,2(1-x)-s')R^{-}(0,1,s')\,ds' \\ &\quad - R^{-}(0,1,2(1-x)^{-})\end{aligned}}$$

Note! The physical data $R^{-}(0,1,s)$ enter in this relation.

1.10 The inverse problem

We present an algorithm that simultaneously reconstructs $A(x)$ and $B(x)$ (or $Z(x)$, L, $\varepsilon(z)$ and $\sigma(z)$) from the following data.

$$\begin{cases} R^+(0,1,s) \ , \ 0<s<2 \\ R^-(0,1,s) \ , \ 0<s<2 \\ T(0,1,s) \ , \ 0<s<2 \\ g(1)= \exp\left(-\int_0^1 B(x)\,dx\right) \quad \text{attenuation factor, can be obtained from the transmitted field} \\ \ell \quad \text{related to the total time of measurement.} \\ \varepsilon(0) \ (\text{or } \varepsilon(L)) \end{cases}$$

Equations used (reconstruction from the left) $y = 1$

① $$R^+_x(x,1,s) = 2R^+_s(x,1,s) - B(x)\,R^+(x,1,s) -$$

$$- \tfrac{1}{2}\{A(x)+B(x)]\int_0^s R^+(x,1,s-s')\,R^+(x,1,s')\,ds' \ , \quad s>0$$

② $$W_x(x,1,s) = \tfrac{1}{2}[A(x)+B(x)][\,R^+(x,1,s) +$$

$$+ \int_0^s W(x,1,s-s')\,R^+(x,1,s')\,ds'\,] \ , \quad s>0$$

③ $$R^+(x,1,0^+) = -\tfrac{1}{4}[A(x) - B(x)]$$

④ $$-\int_0^{2(1-x)} W(x,1,2(1-x)-s')\,R^-(0,1,s')\,ds' - R^-(0,1,2(1-x)^-)\,) = [R^-] =$$

$$= -\tfrac{1}{4}\{A(x)+B(x)]\,\exp\int_x^1 B(x')\,dx'$$

Algorithm

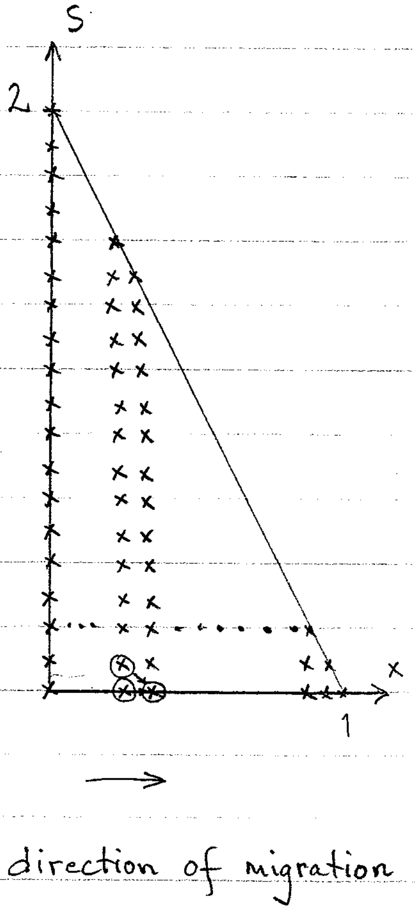
1. Use Eq. ② to explicitly step W forward in the x-direction to the next set of x grid points.

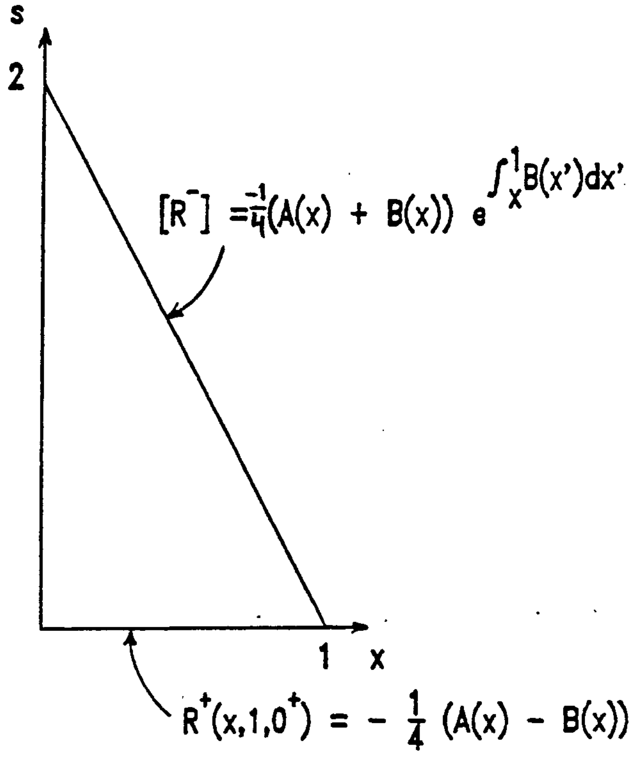
2. Use Eq. ① to implicitly step R^+ forward in the x-direction to the next x grid point at s=0.

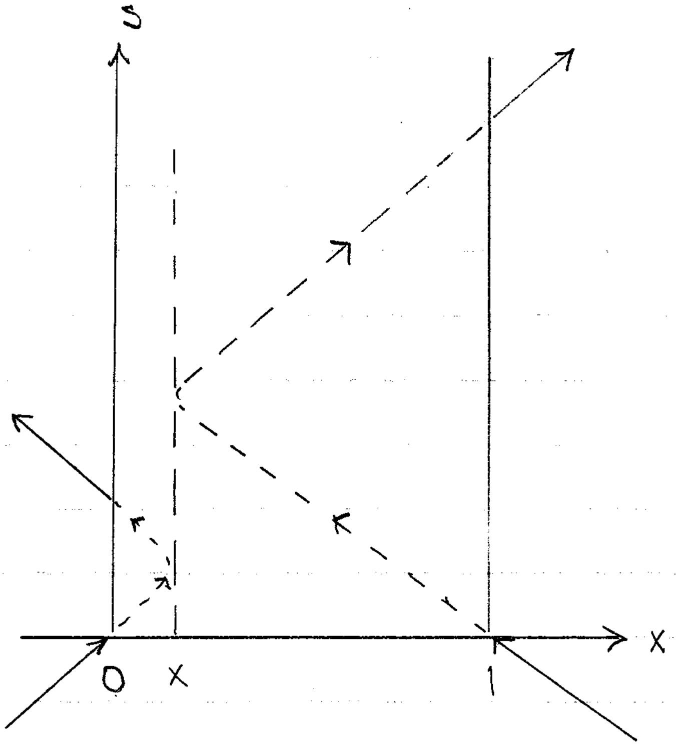
3. Eqs. ③ & ④ are used to find A(x) and B(x) at this new x grid point. (non-linear in B(x))

4. Use Eq. ① to implicitly step R^+ forward in the x-direction at the remaining x grid points.

5. Repeat 1. to 4. to move one step deeper into the medium.

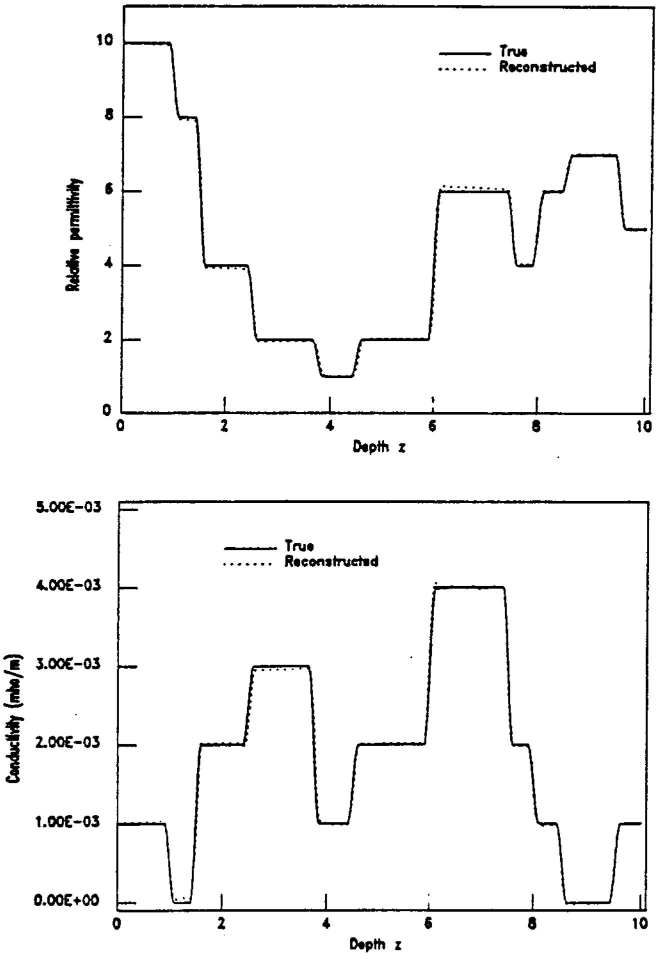




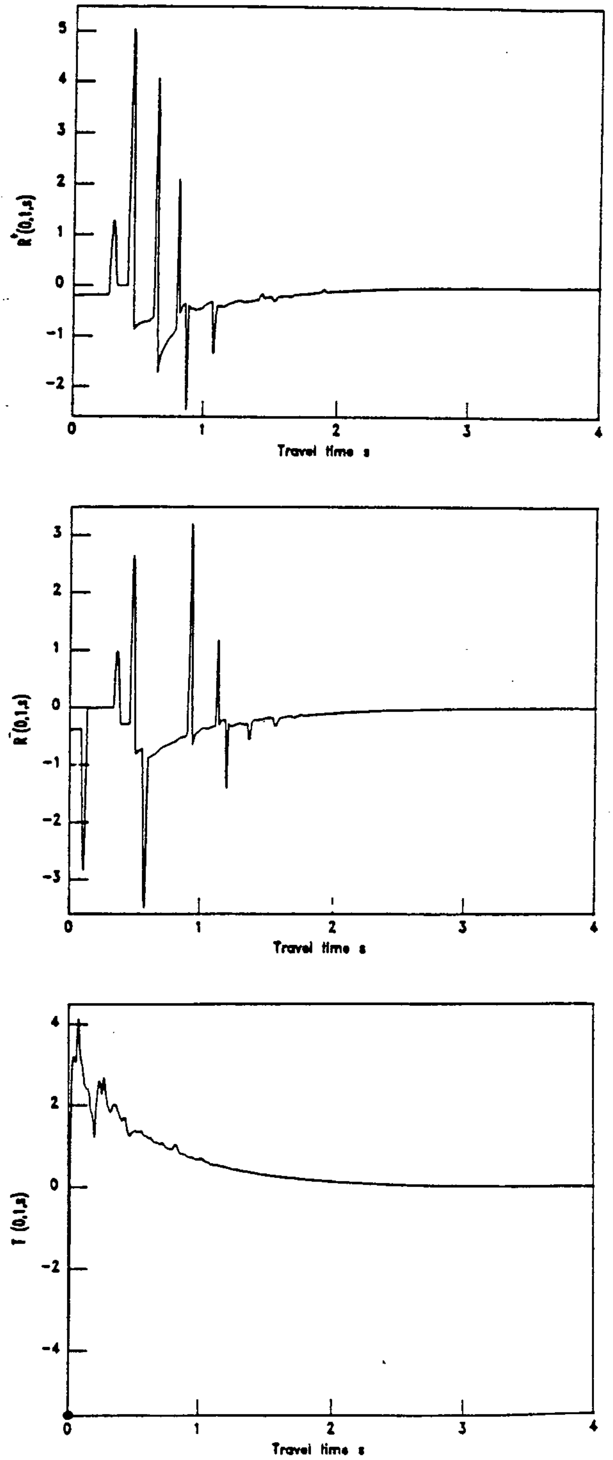


Early time behaviour of R^+ is combined with late time behaviour of R^-.
Transmission data (or rather its resolvent W) decipher the R^- data such that the right combinations can be made.

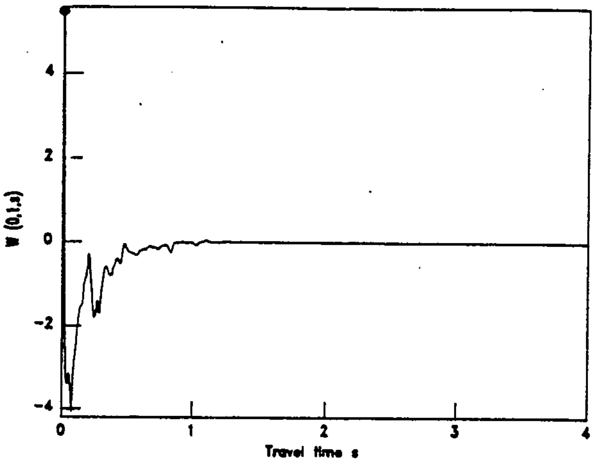
1.11 Examples of numerical reconstructions



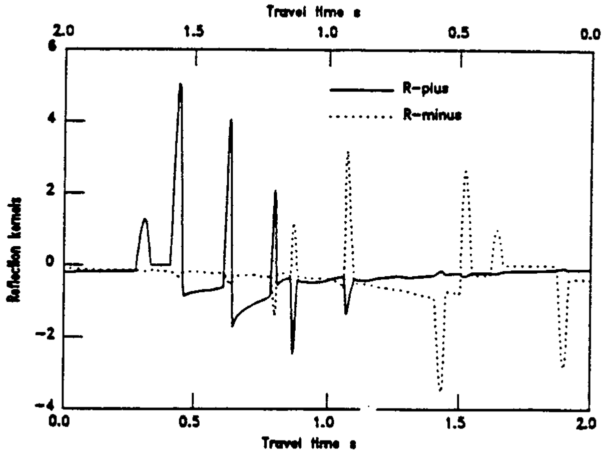
Data



Propagator kernel



Reflection kernels

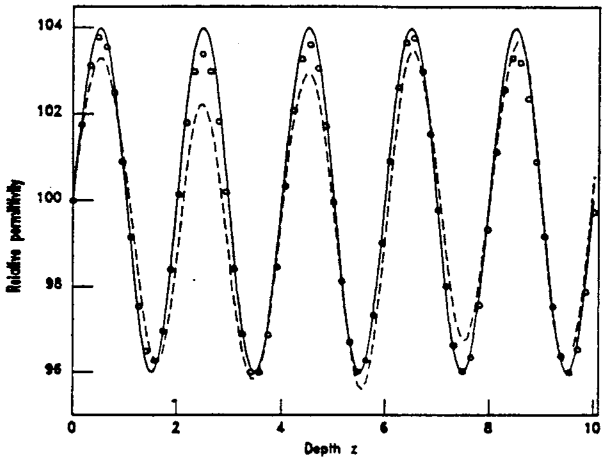


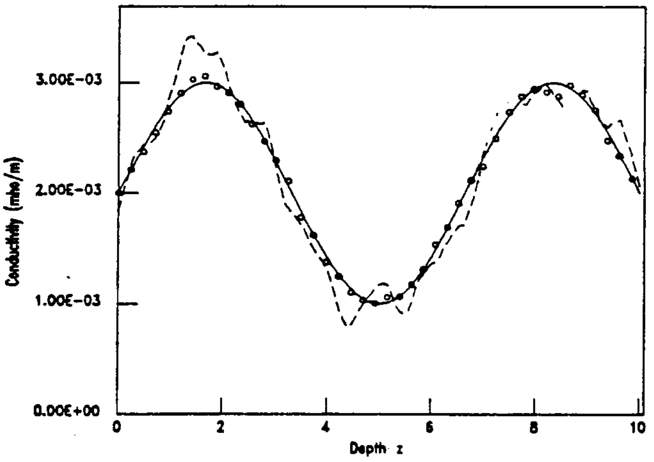
Reconstructions with noisy data

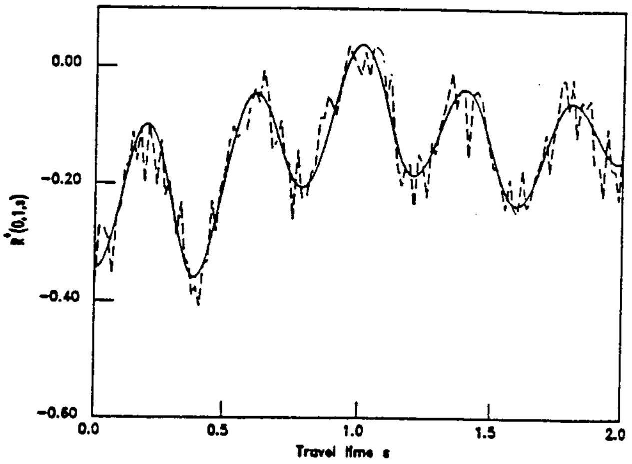
Data

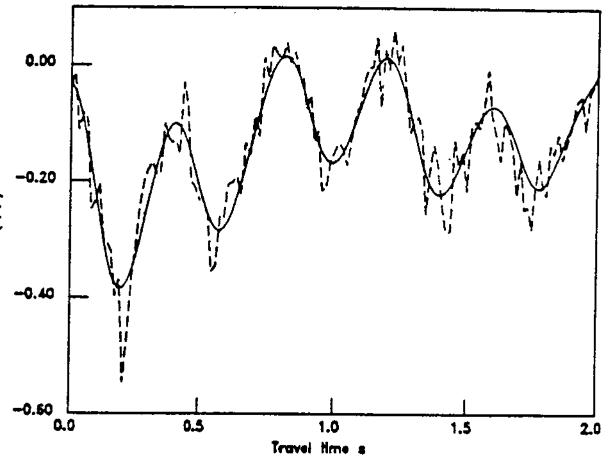
Reconstructions

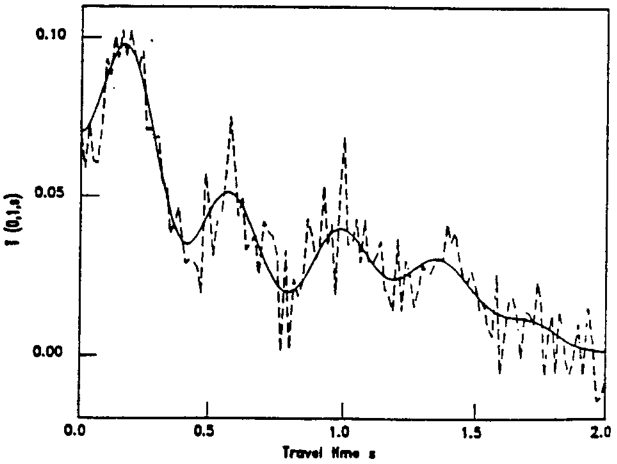
(smoothed twice)











——— clean data

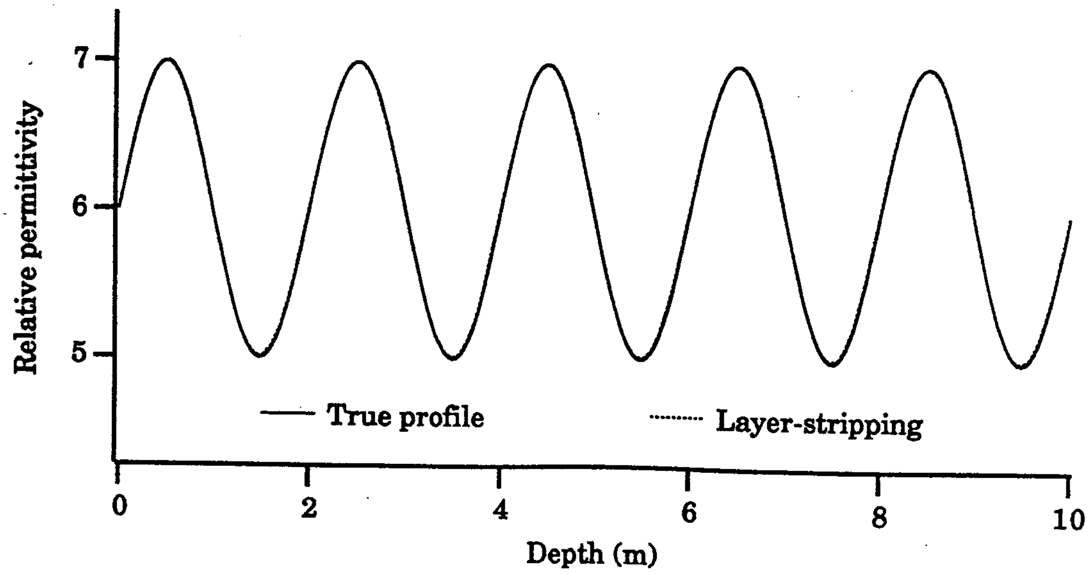
— — — noise s/n = 1.8

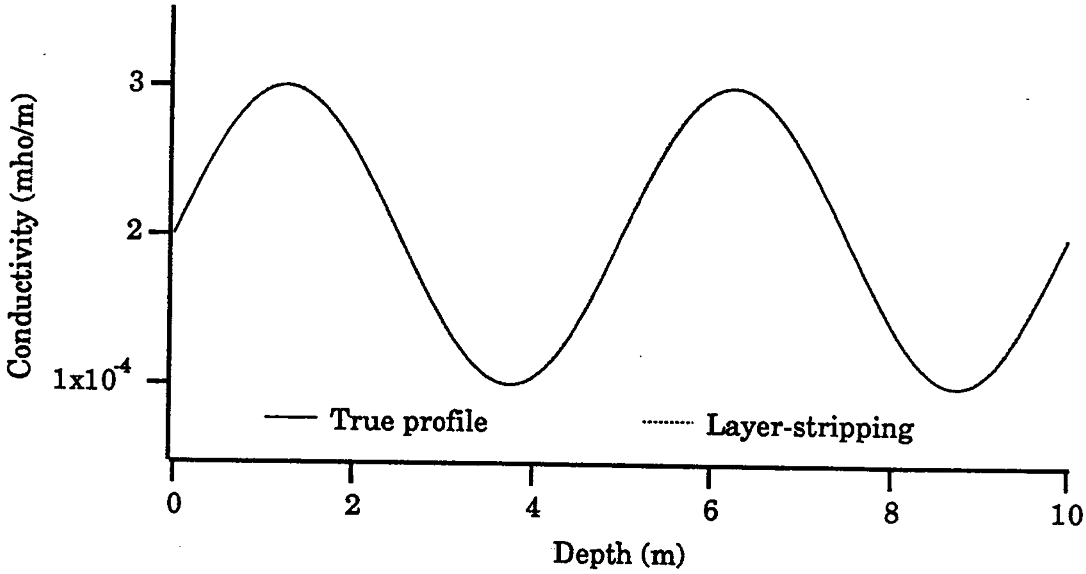
o o o o o noise s/n = 6.8

$$s/n = \sqrt{\int_0^2 (K(s) - \bar{K})^2 ds} \,/\, 2\sigma \qquad \left(\bar{K} = \int_0^2 K(s)\, ds / 2 \right)$$

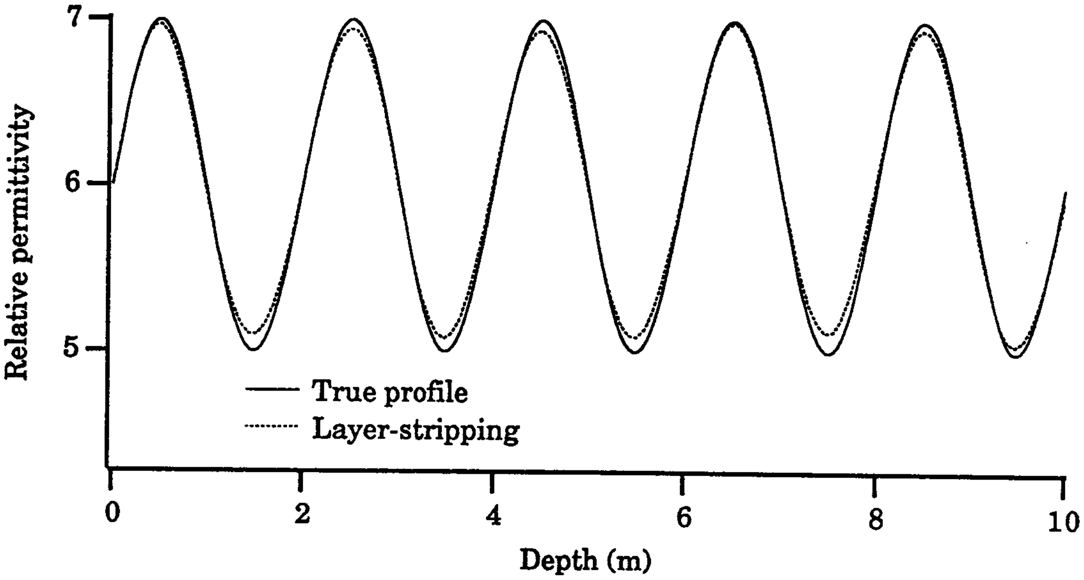
One more example

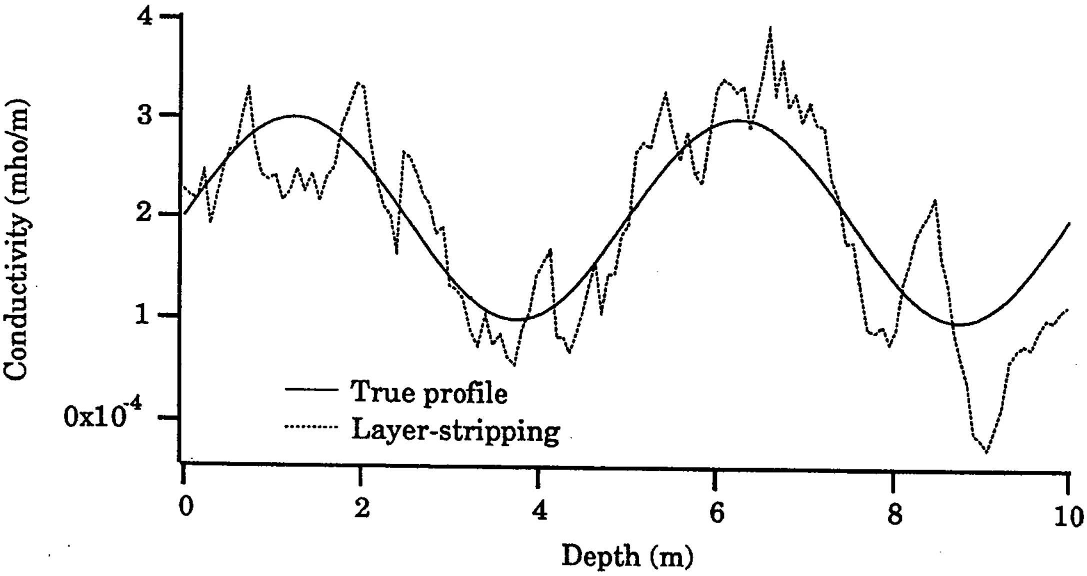
Reconstructions without noise





Reconstruction with noisy data





Gaussian noise STD= 0.05

The inverse problem revisited

In the previous algorithm complete data were used to simultaneously determine $A(x)$ and $B(x)$.

Under certain conditions it suffices just to use a subset of these data.

$$\begin{cases} R^{+}(0,1,s) & , \ 0 < s < 2 \\ R^{-}(0,1,s) & , \ 0 < s < 2 \\ \ell & \\ \varepsilon(0) \ (\text{or } \varepsilon(L)) & \end{cases}$$

Note! No transmission data $T(0,1,s)$ or attenuation factor G.

This inversion algorithm is based upon iteration techiques.

The basic equation is again the imbedding equation, written in integrated form.

$$R^+_x = 2R^+_s - BR^+ - \tfrac{1}{2}(A+B)\,R^+ * R^+$$

or (let $y=1$)

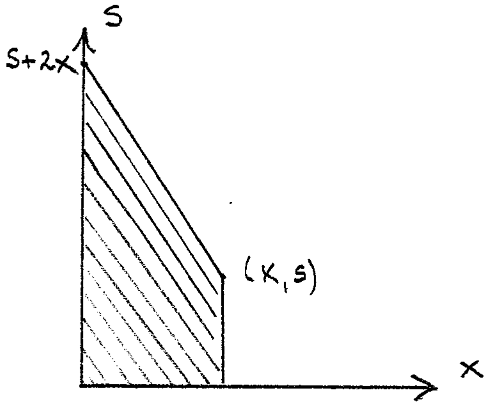
$$\frac{d}{dx'}\,R^+(x',1,s+2(x-x')) = -\,B(x')\,R^+(x',1,s+2(x-x'))$$

$$-\tfrac{1}{2}\left[A(x')+B(x')\right](R^+ * R^+)(x',1,s+2(x-x'))$$

Integrate x' from 0 to x

$$R^+(x,1,s) - R^+(0,1,s+2x) = -\int_0^x \Big\{ B(x')\,R^+(x',1,s+2(x-x'))$$

$$+\tfrac{1}{2}\left[A(x')+B(x')\right](R^+ * R^+)(x',1,s+2(x-x'))\Big\}\,dx'$$



Similarly, for the $R^-(x,y,s)$ kernel

$$R^-(0,y,s) - R^-(0,1,s+2(1-y)) = -\int_y^1 \Big\{ B(y')R^-(0,y',s+2(y'-y))$$

$$-\tfrac{1}{2}[A(y')-B(y')]\,(R^- * R^-)(0,y',s+2(y'-y))\Big\}\,dy'$$

Boundary values:

$$\begin{cases} R^+(x,1,0^+) = -\frac{1}{4}(A(x)-B(x)) \\ R^-(0,y,0^+) = \frac{1}{4}(A(y)+B(y)) \end{cases}$$

Solve for $A(x)$ and $B(x)$

$$\begin{cases} A(x) = 2[R^-(0,x,0^+) - R^+(x,1,0^+)] \\ B(x) = 2[R^-(0,x,0^+) + R^+(x,1,0^+)] \end{cases}$$

Data: $F^\pm(s) = R^\pm(0,1,s)$

Define <u>the iteration algorithm</u> $n = 0, 1, 2, \dots$

$$\left\{ \begin{aligned}
& R_0^+(x,1,s) = 0 \quad \longleftarrow \quad \text{other start values also possible} \\
& R_0^-(0,y,s) = 0 \\
& R_{n+1}^+(x,1,s) = F^+(s+2x) - \int_0^x \Big\{ B_n(x') R_n^+(x',1,s+2(x-x')) \\
& \qquad + \tfrac{1}{2}\left[A_n(x') + B_n(x')\right] (R_n^+ * R_n^+)(x',1,s+2(x-x')) \Big\} dx' \\
& R_{n+1}^-(0,y,s) = F^-(s+2(1-y)) - \int_y^1 \Big\{ B_n(y') R_n^-(0,y',s+2(y'-y)) \\
& \qquad - \tfrac{1}{2}\left[A_n(y') - B_n(y')\right] (R_n^- * R_n^-)(0,y',s+2(y'-y)) \Big\} dy' \\
& \text{where} \\
& A_n(x) = 2\left[R_n^-(0,x,0^+) - R_n^+(x,1,0^+) \right] \\
& B_n(x) = 2\left[R_n^-(0,x,0^+) + R_n^+(x,1,0^+) \right]
\end{aligned} \right.$$

Under certain specific conditions on the data $F^{\pm}(s)$ this iteration scheme converges and

$R_n^{\pm} \to R^{\pm}$ is unique and the result depends continuously on data F.

Furthermore, $A_n, B_n \to A, B$.

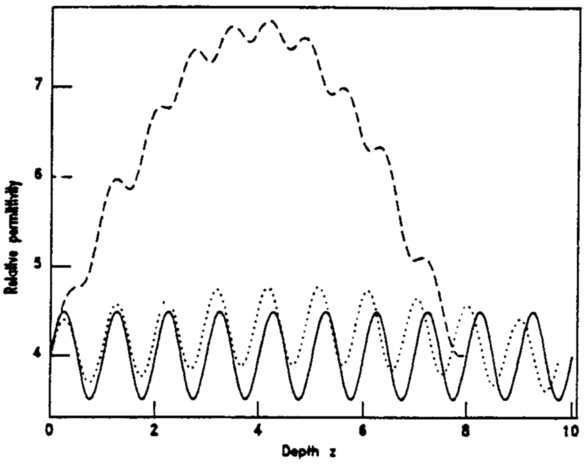
The proof is based upon contraction mapping arguments in the $\|\cdot\|_{\infty}$-norm.

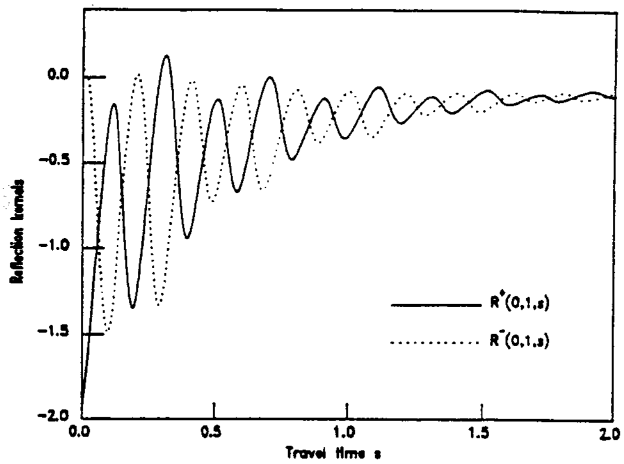
Sufficient conditions on $F^{\pm}$: $|F^{\pm}(s)| < f$, $0 < s < 2$

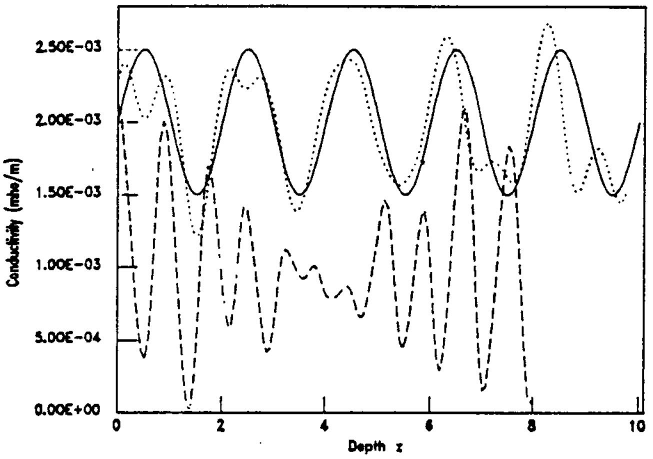
where $f = (11\sqrt{22} - 50)/27 \approx 0.05906$.

Numerical example

Data







Note $\sup |F^{\pm}| > f$

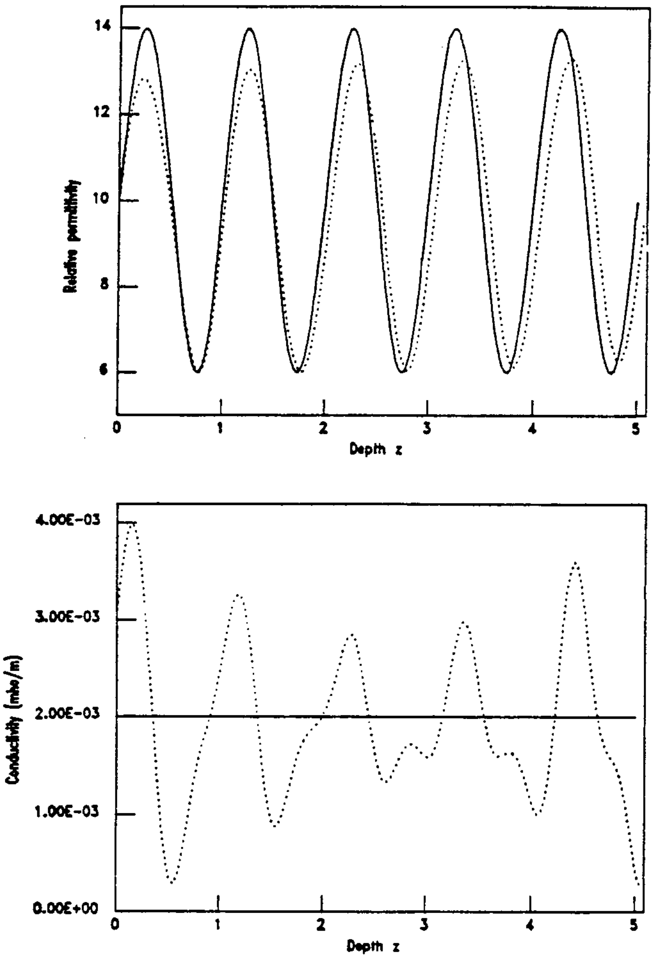
——— Original profile = after 60 iterations

– – – – After 1 iteration

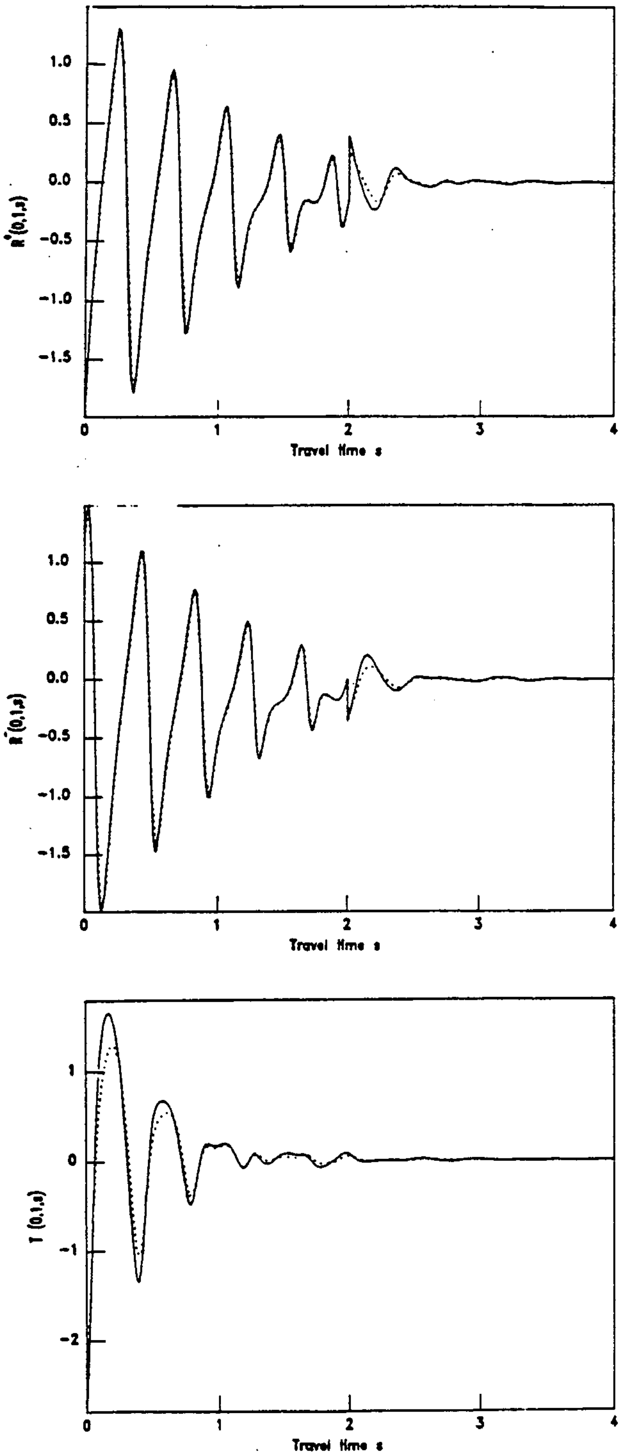
.... After 20 iterations

The importance of transmission data
(Numerical example)

Two different profiles



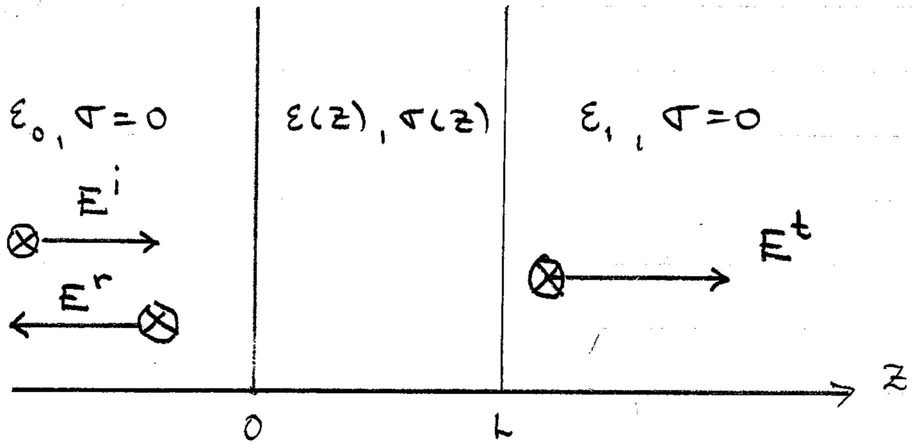
Data

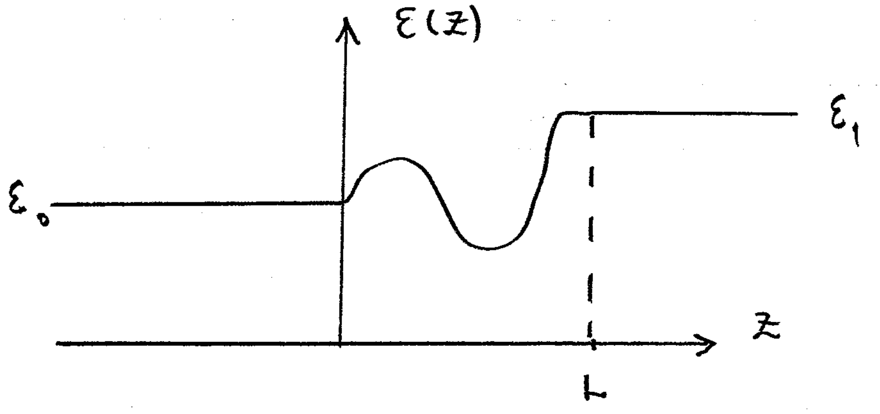


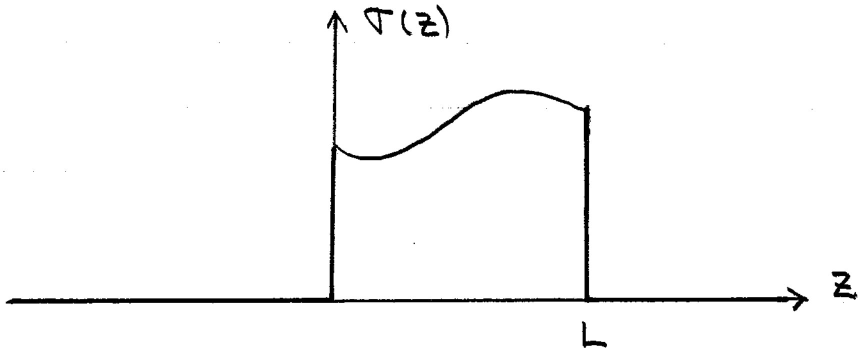
2. Green functions approach (direct scattering)

2.1 Basic equations

The same assumptions as in Section B.1 are made.







The same basic equation holds in travel time coordinates (see p. 63)

$$\boxed{\begin{aligned}&\left[\partial_x^2 - \partial_s^2 + A(x)\partial_x + B(x)\partial_s\right] u(x,s) = 0\\ &A(x) = -\partial_x\left[\ln c(z(x))\right] = \tfrac{1}{2}\partial_x\left[\ln \varepsilon(z(x))\right]\\ &B(x) = -\varrho\sigma(z(x))/\varepsilon(z(x))\end{aligned}}$$

Wave splitting

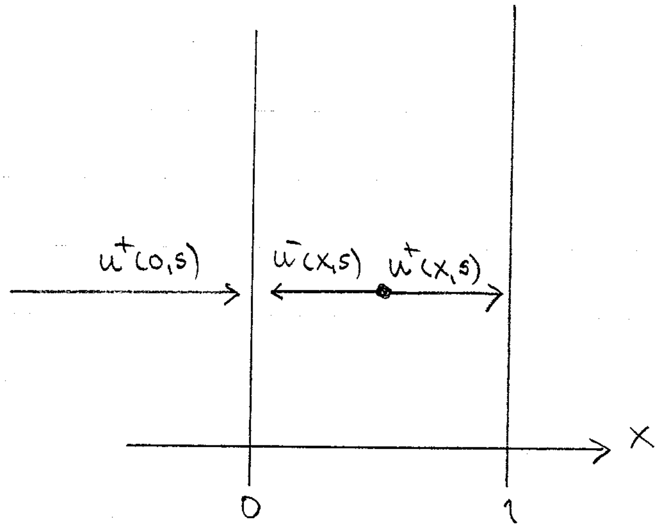
$$u^{\pm}(x,s) = \tfrac{1}{2}\left(u(x,s) \mp \partial_s^{-1} u_x(x,s)\right)$$

implies the PDE (cf. p. 65)

$$\boxed{\begin{aligned}&\partial_x\begin{pmatrix}u^+\\u^-\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}u^+\\u^-\end{pmatrix}\\[2em] &\left\{\begin{aligned}\alpha &= -\partial_s - \tfrac{1}{2}(A-B)\\ \beta &= \tfrac{1}{2}(A+B)\\ \gamma &= \tfrac{1}{2}(A-B)\\ \delta &= \partial_s - \tfrac{1}{2}(A+B)\end{aligned}\right.\end{aligned}}$$

2.2 Relation between $u^{\pm}$

The Green functions $G^{\pm}(x,s)$ map the excitation $u^+(0,s)$ on the left hand side of the slab to the field inside.



Excitations on the right $u^-(1,s)$ can also be considered. However, the presentation in this section uses only excitation from the left.

Similar analysis as in Section A.2 gives (cf. p. 39 – 42 and 43)

$$\boxed{\begin{aligned} u^+(x,x+s) &= t^-(0,x)\left\{ u^+(0,s) + \int_{-\infty}^{s} G^+(x,s-s')\,u^+(0,s')\,ds' \right] \\ u^-(x,x+s) &= \left(t^+(0,x)\right)^{-1} \int_{-\infty}^{s} G^-(x,s-s')\,u^+(0,s')\,ds' \end{aligned}}$$

$$t^{\pm}(0,x) = \exp\left\{ \pm \frac{1}{2} \int_0^x \left[A(x') \pm B(x')\right] dx' \right\}$$

(For $B=0$ these are identical to the ones on p. 43)

The boundary conditions are the same as in Section A.2. (cf. p. 44)

$$\boxed{\begin{array}{ll} G^+(0,s)=0 & G^-(1,s)=0 \\ G^-(0,s)=\overset{+}{R}(s) & G^+(1,s)=T(s) \end{array}}$$

$\overset{+}{R}(s) = \overset{+}{R}(0,s) = \overset{+}{R}(0,1,s)$ The physical reflection kernel

$T(s) = T(0,s) = T(0,1,s)$ — " — transmission kernel

2.3 Differential equation for $G^{\pm}(x,s)$

The system of linear PDE for $G^{\pm}$ on p. 50 is modified due to losses.

The result is

$$\boxed{\begin{aligned} G^+_x(x,s) &= \tfrac{1}{2}\big(A(x)+B(x)\big)\, e^{-\int_0^x B(x')dx'}\, G^-(x,s) \\ G^-_x(x,s) - 2\, G^-_s(x,s) &= \tfrac{1}{2}\big(A(x)-B(x)\big)\, e^{\int_0^x B(x')dx'}\, G^+(x,s) \end{aligned}}$$

Initial conditions

$$\boxed{\begin{aligned} G^+(x,0) &= -\tfrac{1}{8}\int_0^x \big(A^2(x') - B^2(x')\big)\, dx' \\ G^-(x,0) &= -\tfrac{1}{4}\big(A(x)-B(x)\big)\exp\Big[\int_0^x B(x')\,dx'\Big] \end{aligned}}$$

Finite jump discontinuities at $s = 2-2x$

$$\boxed{\big[G^-(x,2(1-x))\big] = \tfrac{1}{4}\big(A(1)-B(1)\big)\exp\Big[\int_0^1 B(x)\,dx\Big]}$$

2.4 Extension of data with Green functions

The transformations between $u^+(x,s)$ and $u^+(x, s+1-x)$ are (cf. 77 with $y=1$)

$$\left\{\begin{aligned} u^+(1, s+1-x) &= t^-(x,1)\left\{u^+(x,s) + \int_{-\infty}^{s} T(x,s-s')\,u^+(x,s')\,ds'\right\} \\ u^+(x,s) &= [t^-(x,1)]^{-1}\left\{u^+(1,s+1-x) + \int_{-\infty}^{s} W(x,s-s')\,u^+(1,s'+1-x)\,ds'\right\}\end{aligned}\right.$$

where $T(x,s) = T(x,1,s)$ and $W(x,s) = W(x,1,s)$

Let $x=0$ in the first equation and denote

$T(0,s) = T(s) =$ physical transmission kernel, and

evaluate the second at $s+x$

$$u^+(1,s+1) = t^-(0,1)\left\{u^+(0,s) + \int_{-\infty}^{s} T(s-s')\,u^+(0,s')\,ds'\right\}$$

$$\begin{aligned} u^+(x,s+x) &= [t^-(x,1)]^{-1}\left\{u^+(1,s+1) + \int_{-\infty}^{s+x} W(x,s+x-s')\,u^+(1,s'+1-x)\,ds'\right\} \\ &= [t^-(x,1)]^{-1}\left\{u^+(1,s+1) + \int_{-\infty}^{s} W(x,s-s'')\,u^+(1,s''+1)\,ds''\right\}\end{aligned}$$

Combine!

$$\begin{aligned} u^+(x,s+x) = [t^-(x,1)]^{-1}\,t^-(0,1)\Big\{ & u^+(0,s) + \int_{-\infty}^{s} T(s-s')\,u^+(0,s')\,ds' \\ & + \int_{-\infty}^{s} W(x,s-s')\Big[u^+(0,s') + \int_{-\infty}^{s'} T(s'-s'')\,u^+(0,s'')\,ds''\Big]\,ds'\Big\}\end{aligned}$$

Compare this expression with the representation of $G^+(x,s)$ on p. 100. Since $[t^-(x,1)]^{-1}t^-(0,1) = t^-(0,x)$ the result is

$$T(s) + W(x,s) + \int_0^s W(x,s-s')\,T(s')\,ds' = G^+(x,s)$$

Convolve with $W(s) = w(0,1,s)$

$$T*W + W(x,\cdot)*W + w(x,\cdot)*T*W = G^+(x,\cdot)*W$$

Use the resolvent equation $T + W + T*W = 0$

$$-T - W + \cancelto{①}{w(x,\cdot)}*W - w(x,\cdot)*[T + \cancelto{①}{W}] = G^+(x,\cdot)*W$$

which becomes

$$w(x,s) - G^+(x,s) - W(s) = \int_0^s G^+(x,s-s')\,W(s')\,ds'$$

For $x=1$ this is the resolvent equation, since

$w(1,s)=0$ and $G^+(1,s) = T(s)$.

However, $W(x,s)=0 \quad s>2(1-x)$ (cf. p 84)

Thus

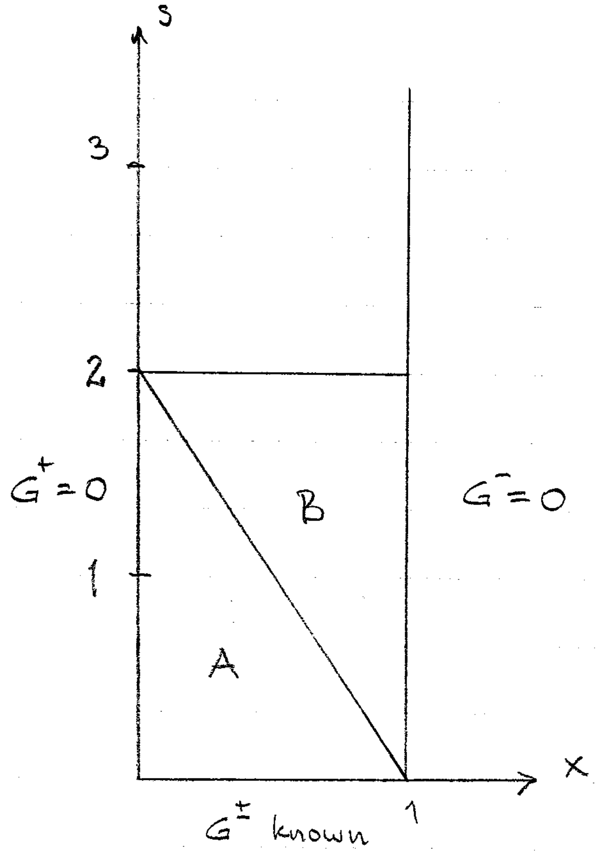
$$\boxed{G^+(x,s)+w(s)+\int_0^s w(s-s')\,G^+(x,s')\,ds'=0, \quad s>2(1-x)}$$

In the same way as transmission is extended beyond one round trip (cf p. 86), this equation can be used to extend $G^+(x,s)$ beyond $s=2(1-x)$, once $w(s)$ is known (enough with $W(s)$, $0\leq s\leq 2$).

Similarly, for $G^-(x,s)$ (not derived)

$$\boxed{G^-(x,s)+\int_0^s w(s-s')\,G^-(x,s')\,ds'=0 \quad , \quad s>2(1-x)}$$

2.5 The direct problem

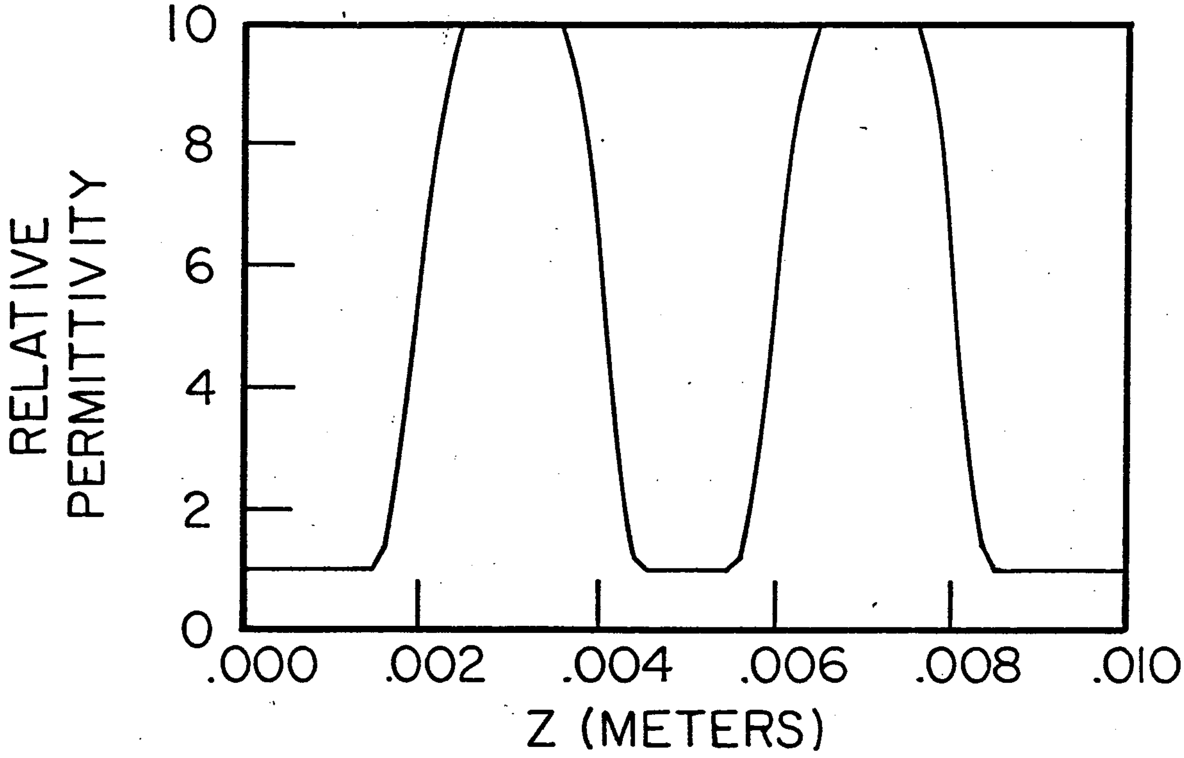


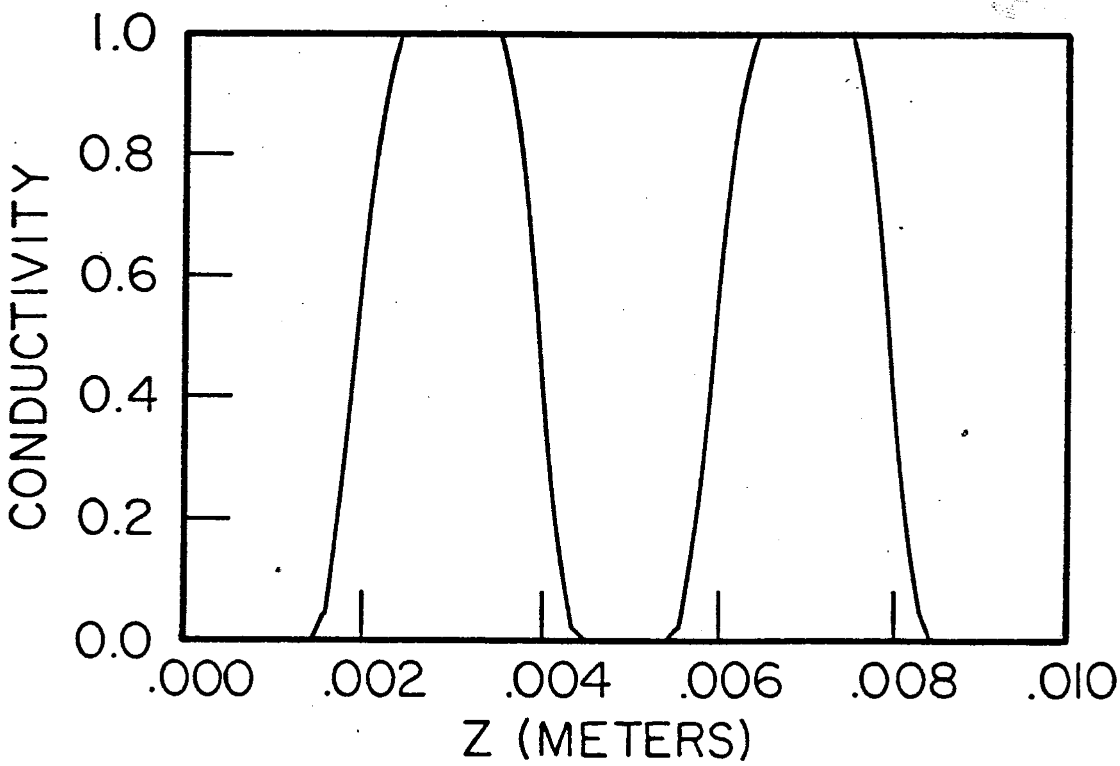
1. Solve for $G^\pm$ in the characteristic triangles A + B, and be careful with the discontinuity in $G^-(x,s)$ at $s = 2(1-x)$.

2. Compute the resolvent kernel $W(s)$, $0 \le s \le 2$, from the knowledge of $T(s) = G^+(1,s)$

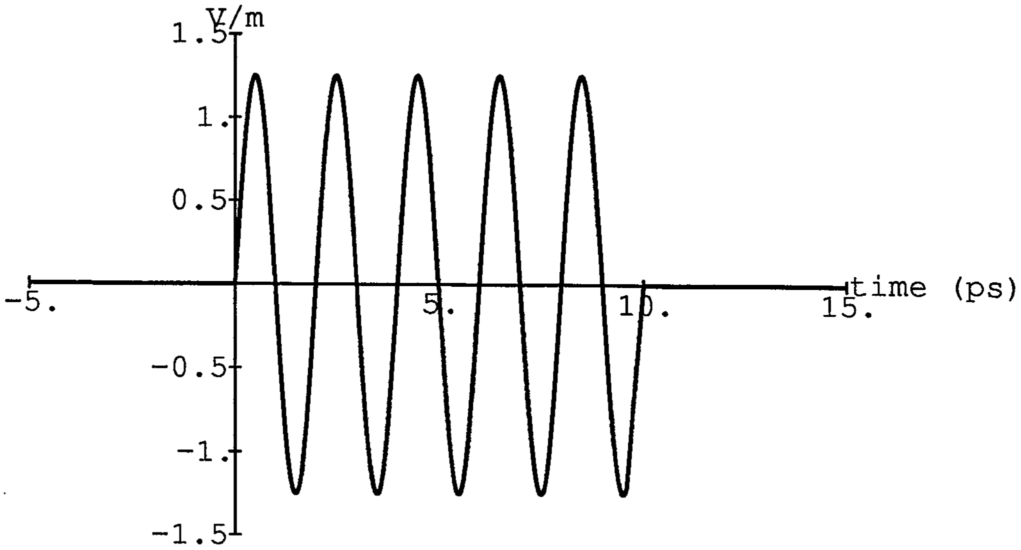
3. Use extension of data to generate $G^\pm$ for arbitrary time s.

2.6 Numerical example



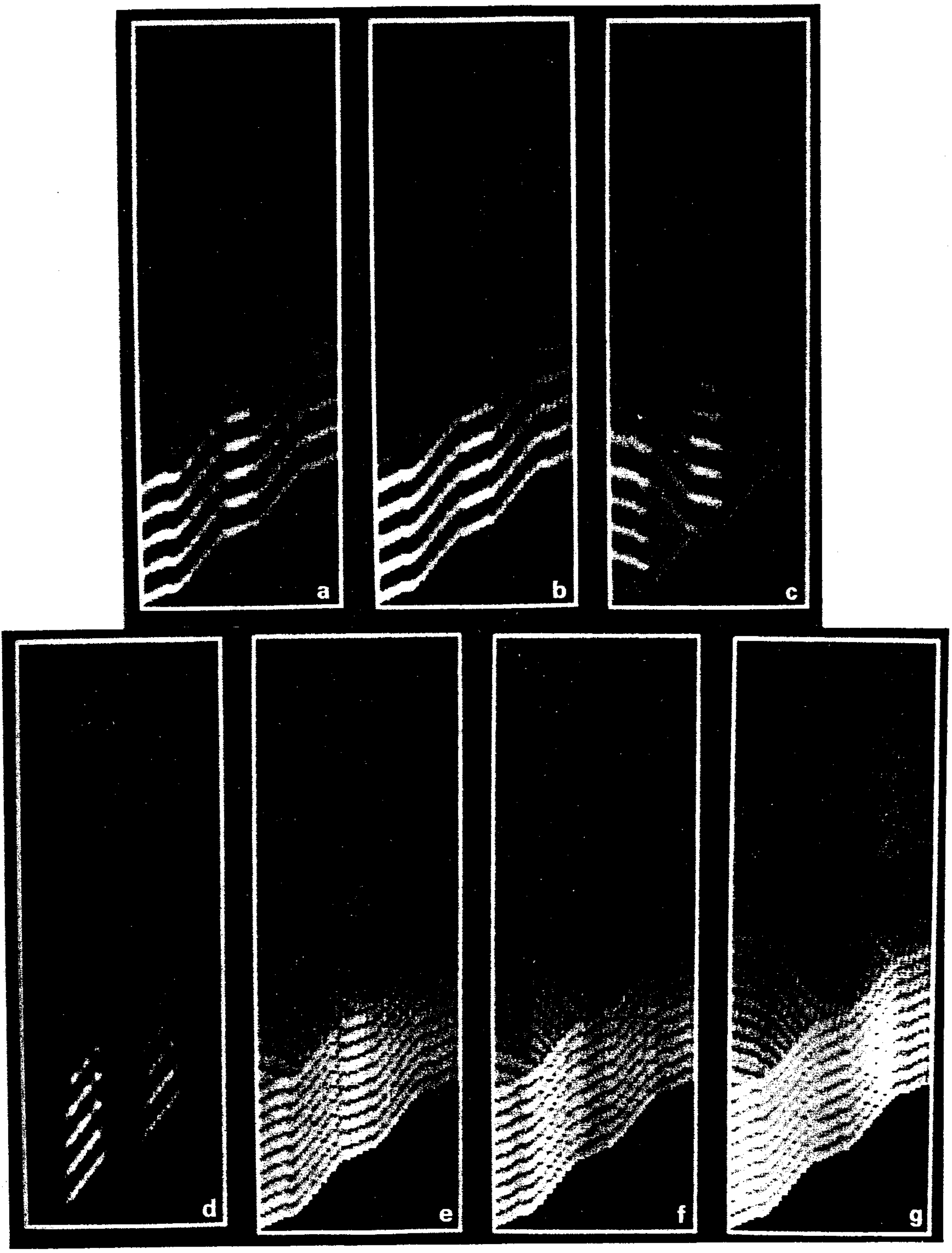


The incoming field

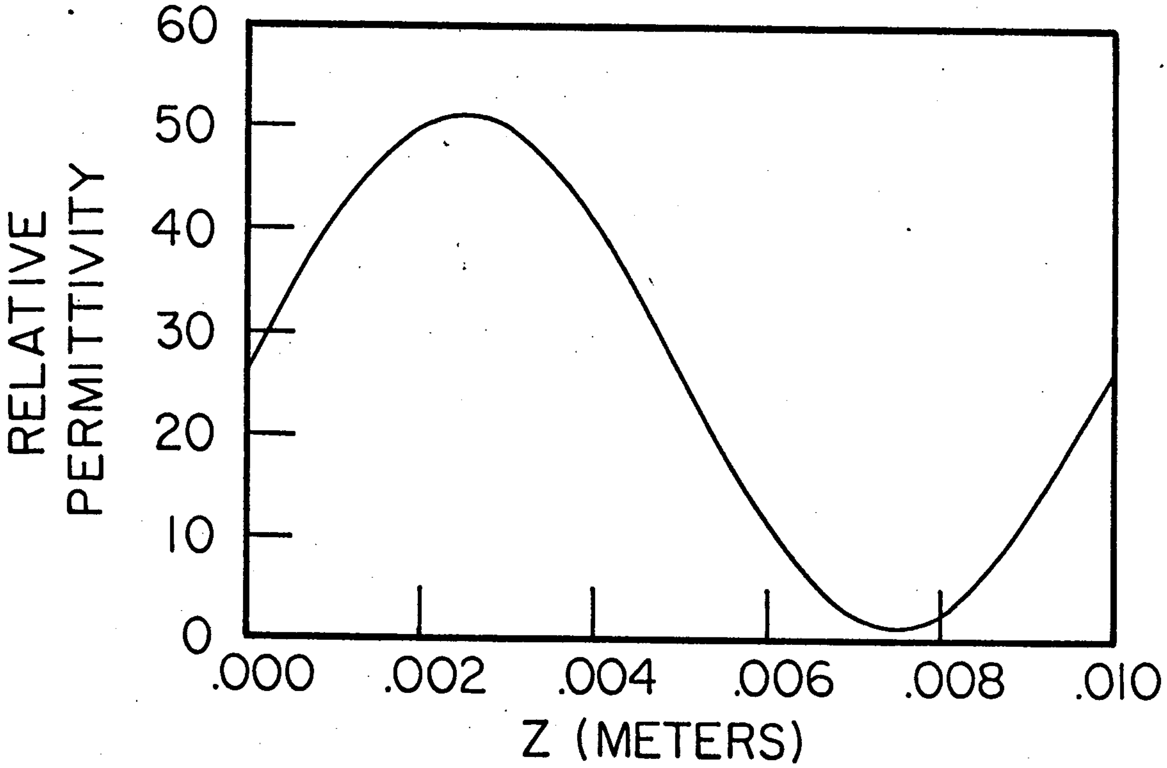


$$E^{i}(t) = \begin{cases} A \sin(2\pi f t) & , \quad 0 \le t \le 10\,\text{ps.} \\ 0 & , \quad \text{elsewhere} \end{cases}$$

$$\begin{cases} A = 1.25 \text{ V/m} \\ f = 500 \text{ GHz} \end{cases}$$



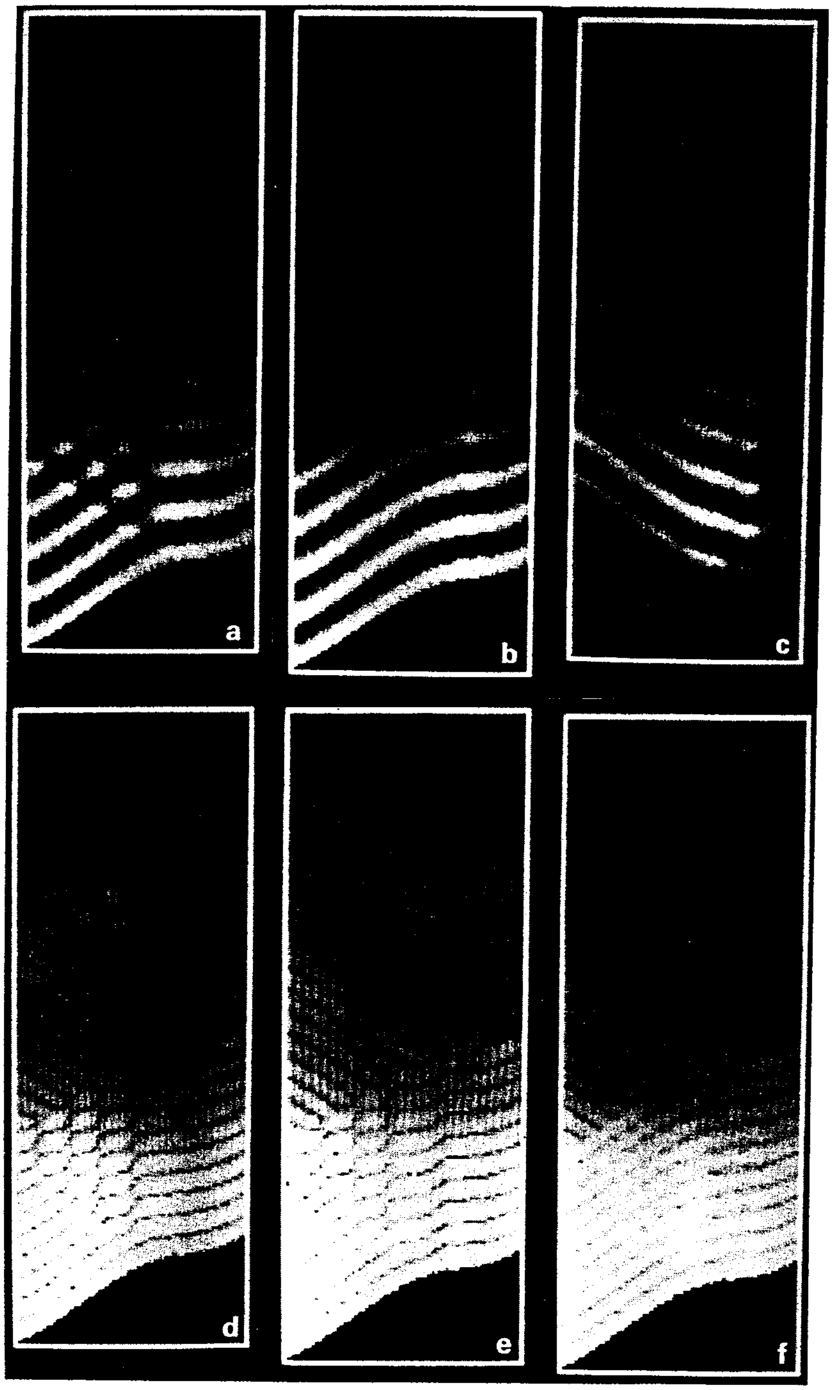
Gray-scale plots of the internal fields. (a) $E(z,t)$, (b) $E^{+}(z,t)$, (c) $E^{-}(z,t)$, (d) $J(z,t)$, (e) $U_E(z,t)$, (f) $U_B(z,t)$, (g) $U(z,t)=U_E(z,t)+U_B(z,t)$.



$\sigma = 0$ everywhere , ε continuous at the edges

$$E^i(t) = \begin{cases} A \sin(2\pi f t), & 0 \le t \le 10 \text{ ps.} \\ 0 & \text{elsewhere} \end{cases}$$

$$\begin{cases} A = 10 \text{ V/m} \\ f = 100 \text{ GHz} \end{cases}$$

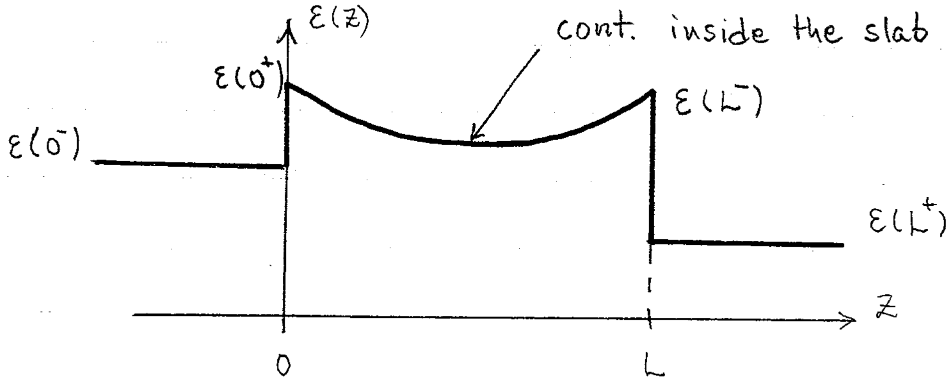


Gray-scale plots of the internal fields. (a) $E(z,t)$, (b) $E^{+}(z,t)$, (c) $E^{-}(z,t)$, (d) $U_E(z,t)$, (e) $U_B(z,t)$, (f) $U(z,t)= U_E(z,t)+U_B(z,t)$.

C. Scattering in non-dispersive lossy media
Discontinuous permittivity profile

1. Imbedding approach
1.1. Basic equations

The permittivity profile is now discontinuous at $z=0, L$



Same PDE as before $0 < z < L$

$$\begin{cases} \partial_z^2 E(z,t) - c^{-2}(z)\partial_t^2 E(z,t) - b(z)\partial_t E(z,t) = 0 \\ c^{-2}(z) = \varepsilon(z)\mu_0 \qquad b(z) = \sigma(z)\mu_0 \end{cases}$$

Boundary conditions at $z=0, L$ (E and H continuous)

$$\begin{cases} E(0^-,t) = E(0^+,t) \\ E_z(0^-,t) = E_z(0^+,t) \end{cases}$$

$$\begin{cases} E(L^-,t) = E(L^+,t) \\ E_z(L^-,t) = E_z(L^+,t) \end{cases}$$

Reduce to normalized travel time coordinates

$$\begin{cases} x = x(z) = \int_0^z \ell^{-1} c^{-1}(z')\,dz' & \left(\ell = \int_0^L c^{-1}(z)\,dz \right) \\ s = t/\ell \\ u(x,s) = E(z,t) \end{cases}$$

$$\begin{cases} u_{xx}(x,s) - u_{ss}(x,s) + A(x)\,u_x(x,s) + B(x)\,u_s(x,s) = 0 \\ A(x) = -\frac{d}{dx}\left\{ \ln\left(c(z(x))\right)\right\} \\ B(x) = -\ell\, b(z(x))\, c^2(z(x)) \end{cases}$$

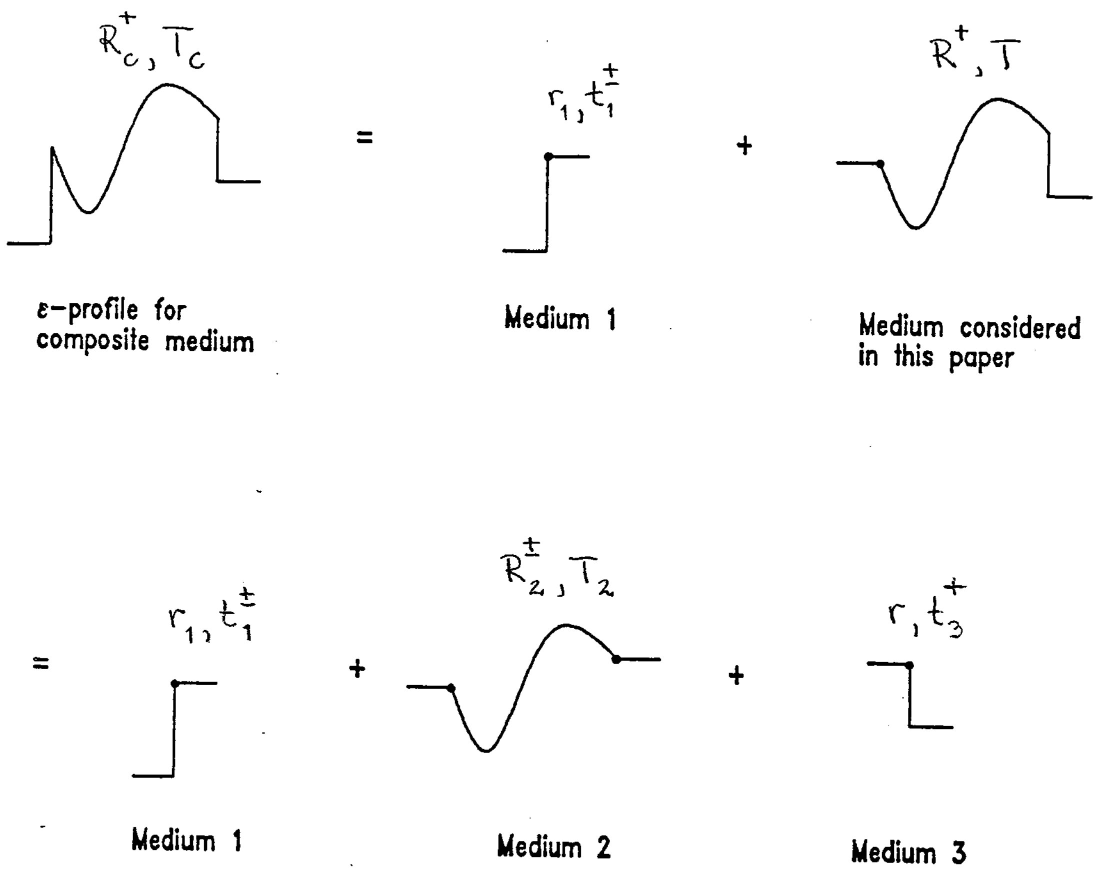
Boundary conditions imply

$$\begin{cases} u(0^-,s) = u(0^+,s) \\ c_0\, u_x(0^-,s) = u_x(0^+,s) \end{cases}$$

$$\begin{cases} u(1^-,s) = u(1^+,s) \\ c_1\, u_x(1^+,s) = u_x(1^-,s) \end{cases}$$

$$\begin{cases} c_0 = \sqrt{\varepsilon(0^-)/\varepsilon(0^+)} & (\text{The cont. case} \quad c_0 = 1) \\ c_1 = \sqrt{\varepsilon(L^+)/\varepsilon(L^-)} & (\quad - " - \quad c_1 = 1) \end{cases}$$

1.2 "Reduction" of the problem



The effects of the front end jump discontinuity can be removed with a suitable transformation.

Similarly the effects of the back end jump discontinuity can be removed. This is advantageous for the direct problem. For the inverse problem we like to retain the back end jump discontinuity.

The results are (physical kernels R_c^+, T_c, R^+, T)

$$\boxed{\begin{aligned} & R^+(0,s) - R_c^+(s) - H(s-2)\, r_1 \rho(0) \left[R^+(0,s-2) + (t_1^+ t_1^-)^{-1} R_c^+(s-2) \right] \\ & + H(s-4) (r_1 \rho(0))^2 R^+(0,s-4) - \\ & - r_1 (t_1^+ t_1^-)^{-1} \int_0^s R^+(0,s') R_c^+(s-s') \, ds' = 0 \quad , \quad 0 < s < 6 \end{aligned}}$$

$$\begin{cases} r_1 = (c_0 - 1)/(c_0 + 1) \quad ; \quad \rho(x) = \dfrac{1-c_1}{1+c_1} \exp\left[\int_x^1 B(x') \, dx' \right] \\ t_1^+ = 2c_0/(c_0+1) \quad ; \quad t_1^- = 2/(c_0+1) \end{cases}$$

$$\boxed{\begin{aligned} T(0,s) = {} & T_c(s) + r_1 R^+(0,s) + H(s-2)\, r_1 \rho(0) \left[T_c(s-2) - r_1 R^+(0,s-2) \right] \\ & + H(s-4)\, r_1^3 (\rho(0))^2 R^+(0,s-4) \\ & + r_1 \int_0^s T_c(s') R^+(0,s-s') \, ds' \quad , \quad 0 < s < 6 \end{aligned}}$$

$R_c^+, r_1, t_1^\pm, \rho(0) \Rightarrow R^+$ (delayed Volterra equation)

$R^+, r_1, t_1^\pm, \rho(0) \Rightarrow R_c^+$ (– " –)

$T, r_1, \rho(0), R^+ \Rightarrow T_c$ (– " –)

$T_c, r_1, \rho(0), R^+ \Rightarrow T$ (direct expression)

1.3. Scattering and propagator kernels

Without loss of generality we can assume $\varepsilon(z)$ at the front end continuous. $(c_0 = 1\,,\ c_1 \neq 1)$

The wave splitting is defined as before

$$u^{\pm}(x,s) = \frac{1}{2}\left(u(x,s) \mp \partial_s^{-1} u_x(x,s)\right)$$

However, the relations between the fields $u^{\pm}(x,s)$ and $u^{\pm}(1,s)$ now look differently.

$$\begin{cases} u^-(x,s) = \rho(x)\, u^+(x, s-2(1-x)) + \int_0^s R^+(x, s-s')\, u^+(x,s')\, ds' \\ u^+(1, s+1-x) = \tau(x)\left[u^+(x,s) + \int_0^s T(x, s-s')\, u^+(x,s')\, ds' \right] \end{cases}$$

$$\begin{cases} u^+(x,s) = \tau^{-1}(x)\left[u^+(1, s+1-x) + \int_0^s W(x, s-s')\, u^+(1, s'+1-x)\, ds' \right] \\ u^-(x,s) = v(x)\, u^+(1, s-1+x) + \tau^{-1}(x) \int_0^s V^+(x, s-s')\, u^+(1, s'+1-x)\, ds' \end{cases}$$

where

$$\begin{cases} \rho(x) = r \exp\left(\int_x^1 B(x')\, dx' \right) & r = (1-c_1)/(1+c_1) \\ \tau(x) = 2\, t^-(x,1)/(1+c_1) & t^{\pm}(x,1) = \exp\left\{ \pm\frac{1}{2} \int_x^1 [A(x') \pm B(x')]\, dx' \right\} \\ v(x) = (1-c_1)\, t^+(x,1)/2 \end{cases}$$

1.4 The imbedding equations for R^+, T, W

are also more complicated

$$R^+_x(x,s) = 2R^+_s(x,s) - B(x)R^+(x,s) -$$

$$- \frac{1}{2}[A(x)+B(x)]\int_0^s R^+(x,s-s')R^+(x,s')ds'$$

$$- H(s-2(1-x))\rho(x)[A(x)+B(x)]R^+(x,s-2(1-x)),$$

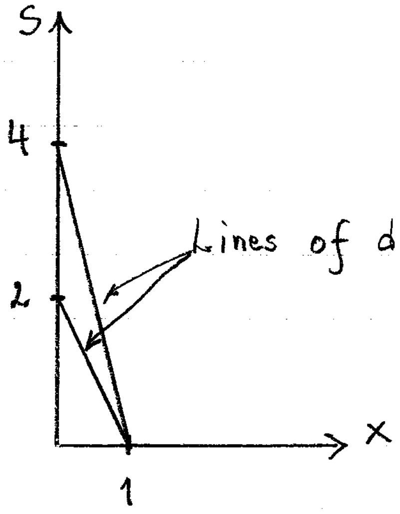
$$s>0,\ s \neq 4(1-x)$$

$$R^+(x,0^+) = -\frac{1}{4}[A(x)-B(x)], \quad 0<x<1$$

$$R^+(1,s) = 0, \quad s>0$$

R^+ has a jump discontinuity at $s=2(1-x)$ as before but also an additional one at $s=4(1-x)$ with jump

$$[R^+(x,s)]_{s=4(1-x)^-}^{s=4(1-x)^+} = -\frac{1}{4}\rho^2(x)[A(x)+B(x)]$$



(This disc. is forced on the solution and does not come from prop of sing.)

$$T_x(x,s) = -\frac{1}{2}\left[A(x)+B(x)\right]\Big\{R^+(x,s) + \int_0^s T(x,s-s')R^+(x,s')\,ds' + H(s-2(1-x))\,\varrho(x)\,T(x,s-2(1-x))\Big\} \quad , s>0 \; , \; s \neq 2(1-x)$$

$$T(1,s)=0 \quad , \; s>0$$

and a jump discontinuity at $s = 2(1-x)$

$$\left[T(x,s)\right]_{s=2(1-x)^-}^{s=2(1-x)^+} = -\frac{1}{4}\,\varrho(x)\left[A(x)+B(x)\right]$$

and for the resolvent kernel

$$T(x,s) + W(x,s) + \int_0^s T(x,s-s')\,W(x,s')\,ds' = 0 \quad , \; s>0$$

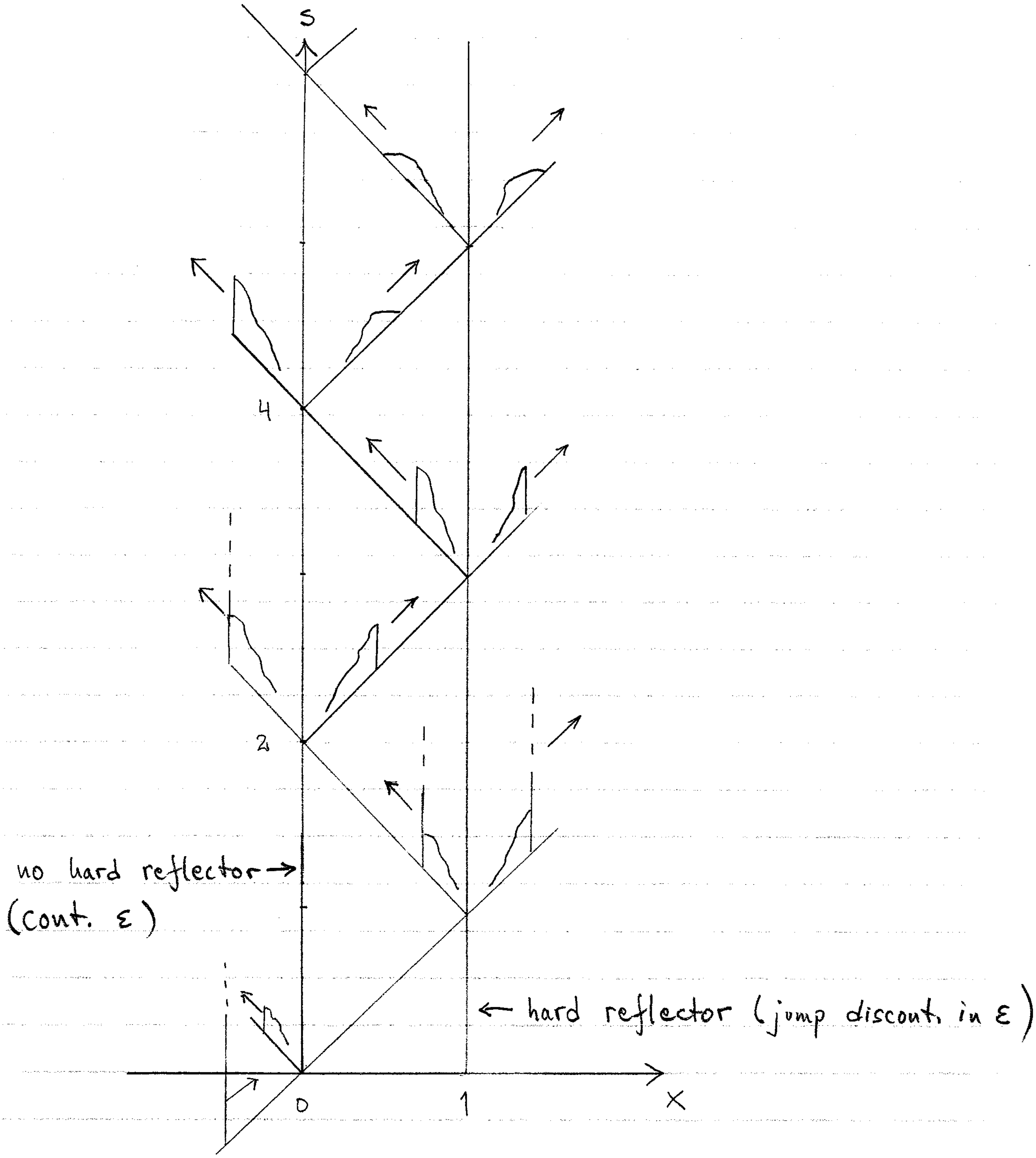
$$W_x(x,s) = \frac{1}{2}\left[A(x)+B(x)\right]\Big\{R^+(x,s) + \int_0^s W(x,s-s')R^+(x,s')\,ds' + H(s-2(1-x))\,\varrho(x)\,W(x,s-2(1-x))\Big\} \quad , \; s>0 \; , \; s \neq 2(1-x)$$

$$W(1,s) = 0 \quad , \; s>0$$

$$\left[W(x,s)\right]_{s=2(1-x)^-}^{s=2(1-x)^+} = -W(x,2(1-x)^-) = \frac{1}{4}\,\varrho(x)\left[A(x)+B(x)\right]$$

$W(x,s) = 0 \quad , \; s > 2(1-x)$ (Compact support)

1.5 Physical interpretation with jump in $\mathcal{E}(z)$



1.6 Numerical example

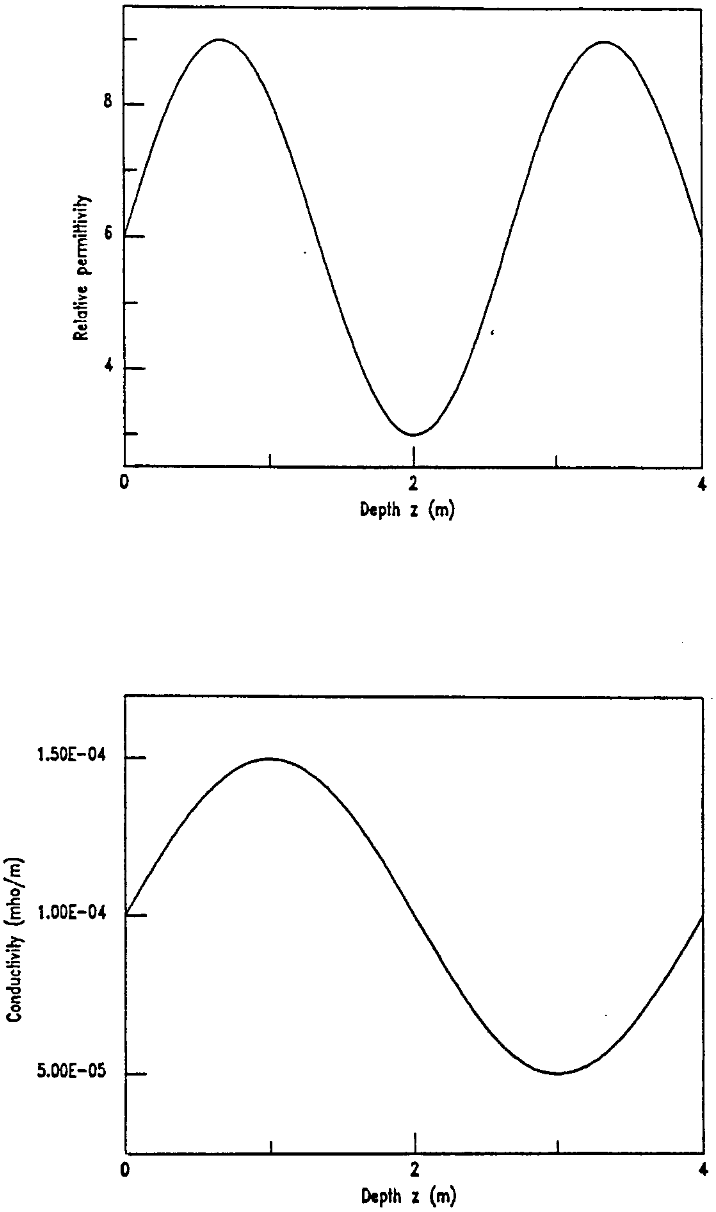


Figure 1

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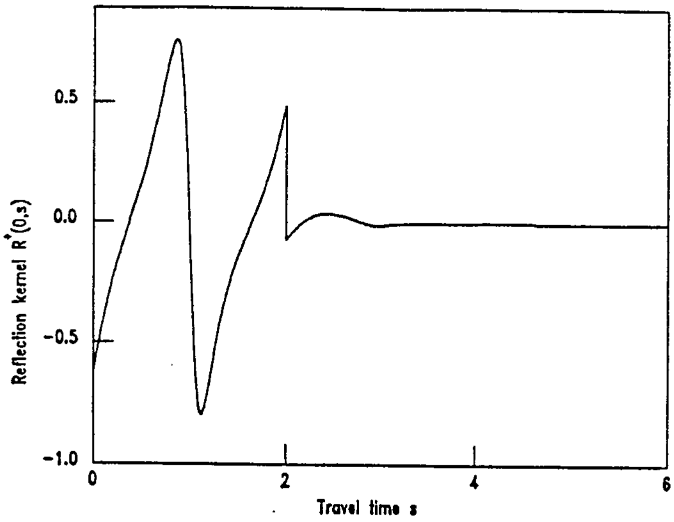
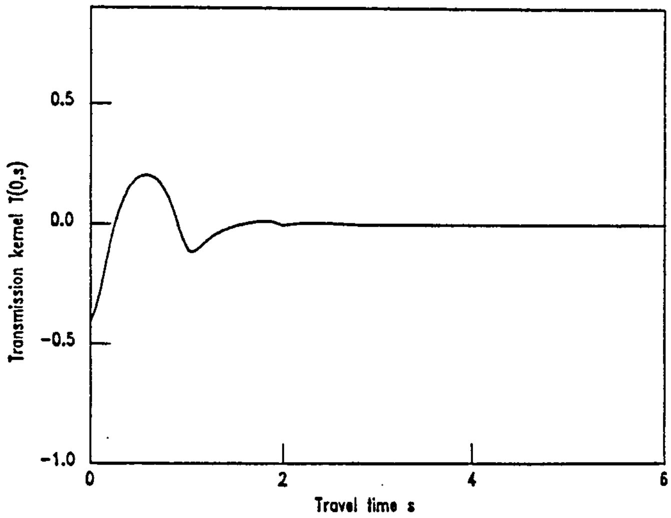
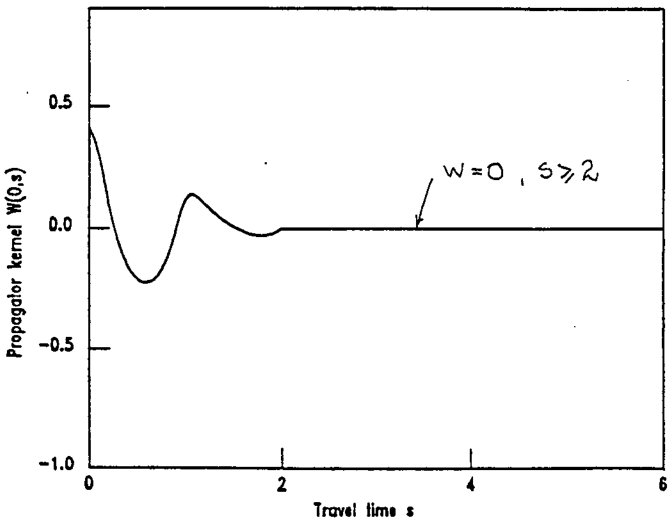


$C_1 = 1$ (continuous backing medium)





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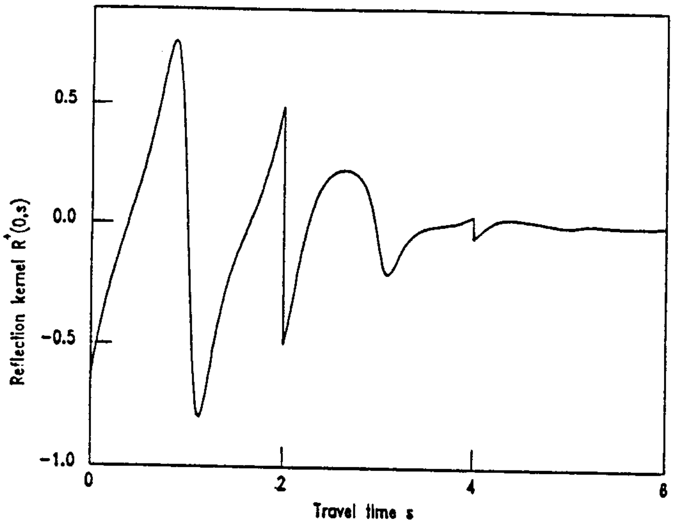
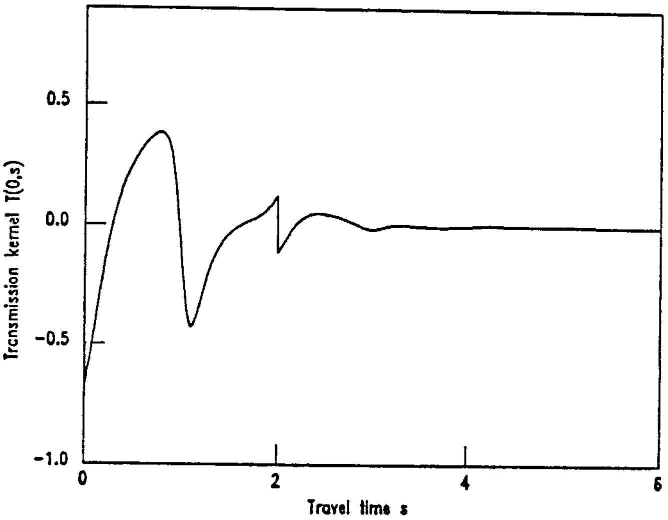
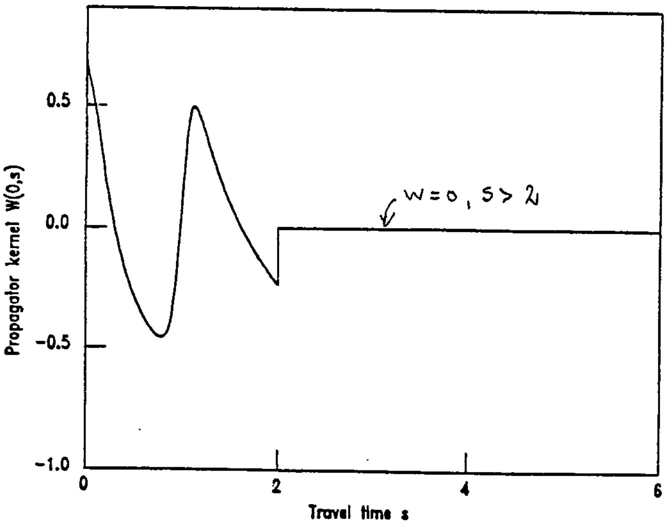
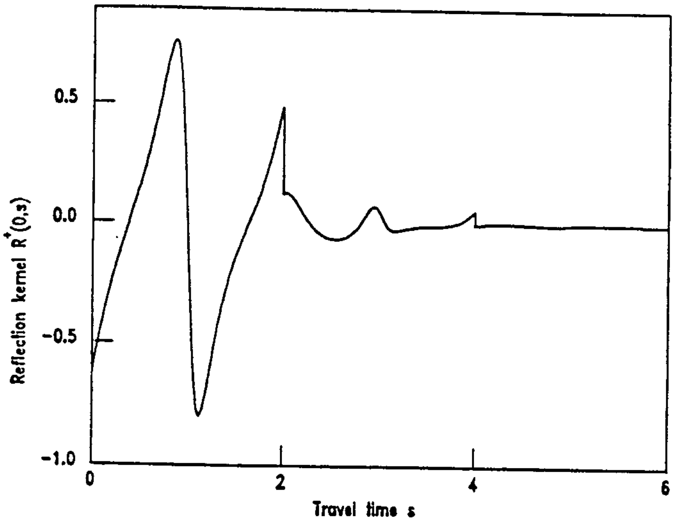


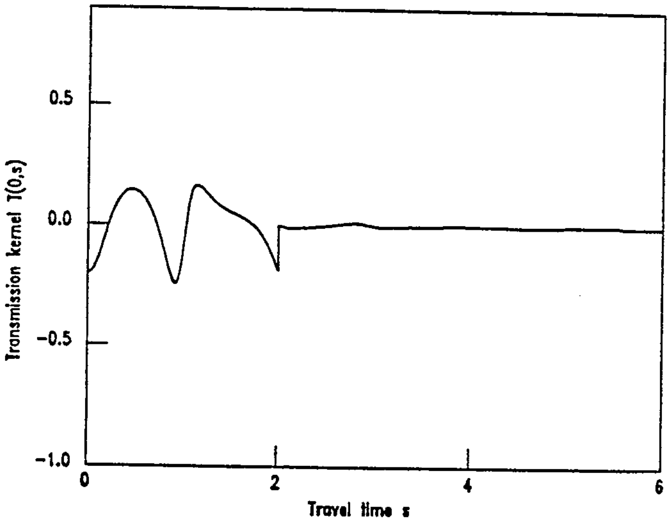
$C_1 = 1/\sqrt{6}$ (vacuum as backing medium)

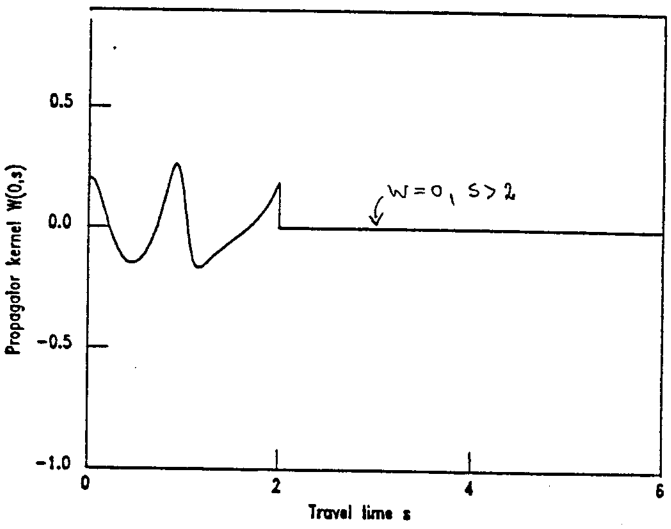


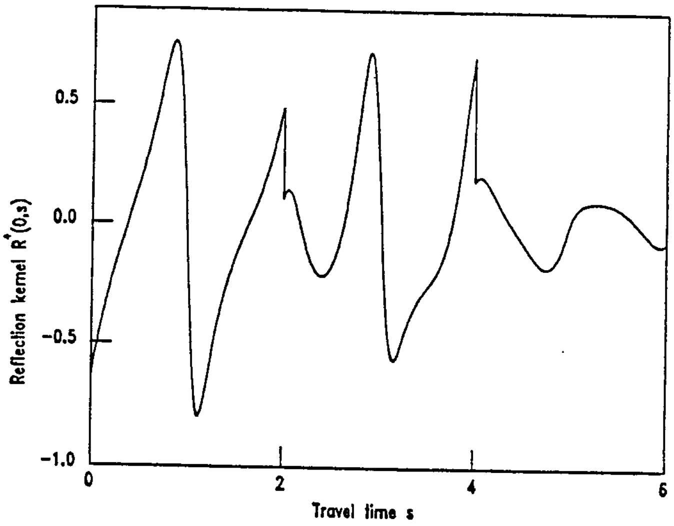



0.5
0.0
-0.5
-1.0
0
2
4
6
Reflection kernel $R^+(0,s)$
Travel time s

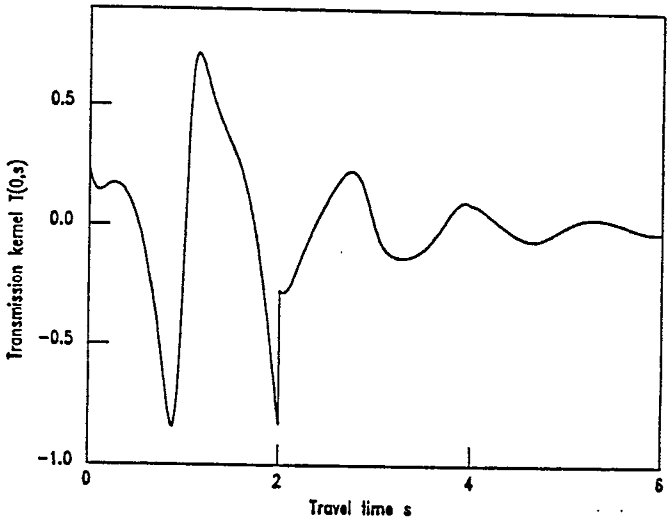
$C_1 = 2$

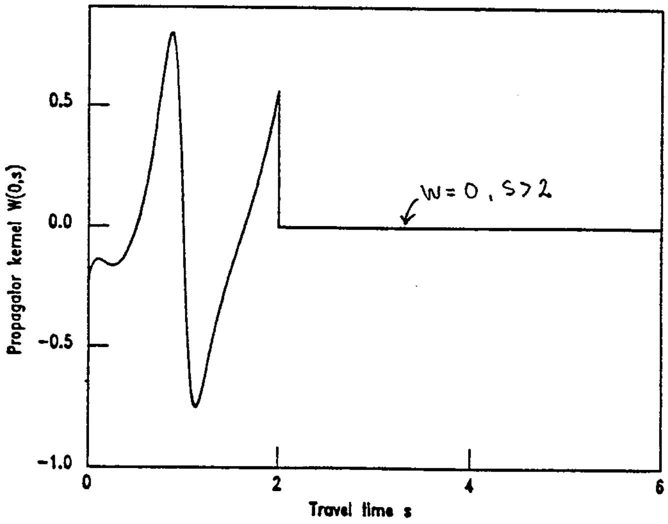

0.5
0.0
-0.5
-1.0
0
2
4
6
Transmission kernel $T(0,s)$
Travel time s


0.5
0.0
-0.5
-1.0
0
2
4
6
Propagator kernel $W(0,s)$
Travel time s
$w = 0, s > 2$



$C_1 \rightarrow \infty$ (perfectly conducting backing)





1.7 Extension of data

Works as in the continuous case

$$\boxed{T(x,s) + \int_{2(1-x)}^{s} W(x,s-s')\,T(x,s')\,ds' = \begin{cases} -\int_{s-2(1-x)}^{2(1-x)} W(x,s-s')\,T(x,s')\,ds' , & 2(1-x) < s < 4(1-x) \\ 0 , & s > 4(1-x) \end{cases}}$$

$T(x,s)$ for one round trip in $[x,1]$, i.e. $0 < s < 2(1-x)$

implies $T(x,s)$ for $s > 2(1-x)$

$$\boxed{R^+(x,s) + \int_{2(1-x)}^{s} W(x,s-s')\,R^+(x,s')\,ds' = \begin{cases} -\int_{s-2(1-x)}^{2(1-x)} W(x,s-s')\,R^+(x,s')\,ds' , & 2(1-x) < s < 4(1-x) \\ 0 , & s > 4(1-x) \end{cases}}$$

$R^+(x,s)$ <u>and</u> $T(x,s)$, $0 < s < 2(1-x)$

imply $R^+(x,s)$, $s > 2(1-x)$

1.8 The inverse problem (discontinuous back end)

Data requirements for a simultaneous reconstruction of $A(x)$ and $B(x)$ $(\Rightarrow Z(x), L, \varepsilon(z)$ and $\sigma(z))$

$$\begin{cases} R^+(0,s) \,, & 0 < s < 2 \\ W(0,s) \,, & 0 < s < 2 \\ \rho(0) & \\ \ell & \\ \varepsilon(0) & \end{cases}$$

We will later talk about how to obtain $W(0,s)$ from physical measurements.

The constants $\rho(0)$, ℓ and $\varepsilon(0)$ can be obtained from physical experiments.

$$u^r(s) = \underset{\uparrow}{\rho(0)}\, u^i(s-2) + \int_0^s R^+(0,s-s')\, u^i(s')\, ds'$$

echo from back jump discontinuity.

Equations used

① $$R^+_x(x,s) = 2R^+_s(x,s) - B(x)\,R^+(x,s) -$$
$$- \tfrac{1}{2}\left[A(x)+B(x)\right]\int_0^s R^+(x,s-s')\,R^+(x,s')\,ds'$$
$$- H(s-2(1-x))\,\rho(x)\left[A(x)+B(x)\right] R^+(x,s-2(1-x)) \quad , \quad s>0$$

② $$W_x(x,s) = \tfrac{1}{2}\left[A(x)+B(x)\right]\Big\{ R^+(x,s) + \int_0^s W(x,s-s')\,R^+(x,s')\,ds'$$
$$+ H(s-2(1-x))\,\rho(x)\,W(x,s-2(1-x))\Big\} \quad , \quad s>0$$

③ $$R^+(x,0^+) = -\tfrac{1}{4}\left[A(x)-B(x)\right]$$

④ $$W(x,2(1-x)^-) = -\tfrac{1}{4}\rho(x)\left[A(x)+B(x)\right]$$

and $$\rho(x) = \frac{1-c_1}{1+c_1}\exp\left[\int_x^1 B(x')\,dx'\right]$$

$$c_1 = \sqrt{\varepsilon(L^+)/\varepsilon(L^-)}$$

Algorithm

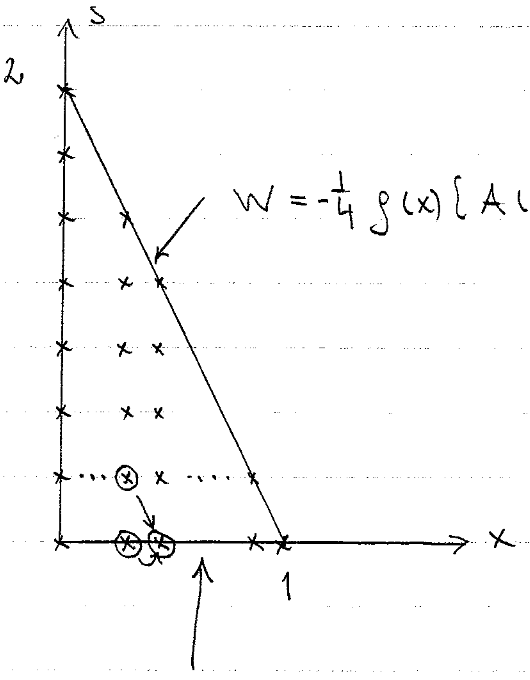
1. Use Eq. ② to explicitly step W forward in the x-direction to the next set of x grid points.

2. Use Eq. ① to implicitly step R^+ forward in the x-direction to the next grid point at $s=0$.

3. Eqs. ③ & ④ are used to find $A(x)$ & $B(x)$ at this new x grid point. (non-linear in $B(x)$).

4. Use Eq. ① to implicitly step R^+ forward in the x-direction at the remaining x grid points.

5. Repeat step 1. to 4. to move one step deeper into the slab.



$W = -\frac{1}{4}\rho(x)[A(x)+B(x)]$

$R^+ = -\frac{1}{4}[A(x)-B(x)]$

1.9 How to obtain $W(0,s)$, $0<s<2$

1/ Measure the transmitted field $T(0,s)$, $0<s<2$ and use the resolvent equation

$$T(0,s)+W(0,s)+\int_0^s T(0,s-s')\,W(0,s')\,ds'=0\ ,\quad s>0$$

to obtain $W(0,s)$, $0<s<2$.

This is a Volterra equation of the second kind and a well posed problem.

2/ It is interesting to notice that there is a second independent way of finding $W(0,s)$. For numerical purposes, however, this method seems to be less attractive.

Let $x=0$ in Eq. ② (on p. 127) and replace $s\to s+2$. Since $W(0,s)=0$, $s>2$ we have

$$R^+(0,s+2)+\rho(0)\,W(0,s)+\int_0^2 W(0,s')\,R^+(0,s+2-s')\,ds'=0$$

$$,\ s>0$$

Restrict the domain of validity to $0 \leq s \leq 2$ and assume we know $R^+(0,s)$, $0 \leq s \leq 4$ and $\rho(0)$

$$R^+(0,s+2) + \rho(0)\,W(0,s) + \int_0^2 R^+(0,s+2-s')\,W(0,s')\,ds' = 0 \,; \qquad 0 \leq s \leq 2 \qquad (✡)$$

known   unknown

This is a <u>Fredholm equation of the second kind</u> for W.

<u>Q.</u> Is this equation (✡) uniquely solvable?

<u>A.</u> Yes!

<u>Proof:</u> The question of existence of a solution is clear from the fact that it is an identity in R^+ and W.

(from p. 130 – 137)

Is the solution unique?

Study the homogeneous equation

$$\rho(0)\,y(s) + \int_0^2 R^+(0,s+2-s')\,y(s')\,ds' = 0 \qquad ; \; 0 \leq s \leq 2 \qquad (*)$$

We want to show that this equation has only the zero solution, $y(s) \equiv 0$.

Define $$f(s) = \begin{cases} 0 & , s < 0 \\ y(s) & , 0 \leq s \leq 2 \\ 0 & , s > 2 \end{cases}$$

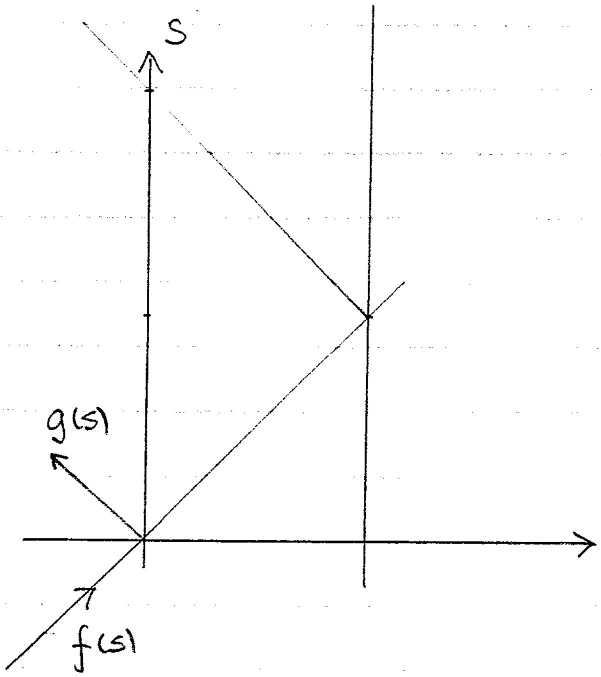
where $y(s)$ is a solution to $(*)$

Also define

$$g(s) = g(0)\, f(s-2) + \int_0^s f(s') R^+(0, s-s')\, ds' \quad , \quad s > 0$$

Physical interpretation:

$$\begin{cases} f(s) & \text{incident field} \\ g(s) & \text{reflected field} \end{cases}$$



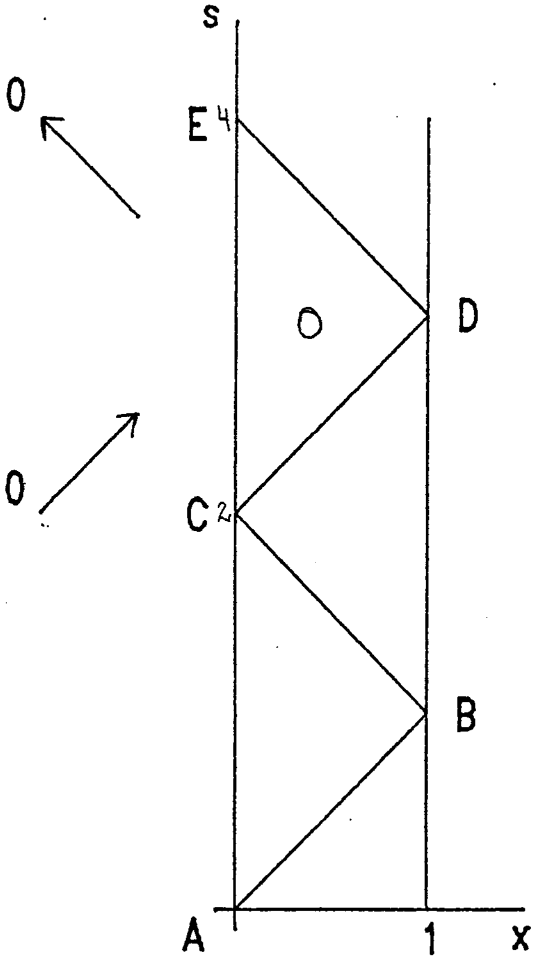
By construction $g(s) = 0$; $2 \leq s \leq 4$, since

$$g(s) = \rho(0) f(s-2) + \int_0^2 f(s') R^+(0, s-s') \, ds' =$$

$$= \rho(0) f(s-2) + \int_0^2 R^+(0, s-2+2-s') f(s') \, ds' =$$

$$= (\text{by } *) = 0 \quad , \quad 2 \leq s \leq 4$$

Thus $u(0,s) = u_x(0,s) = 0$, $2 < s < 4$ $\Rightarrow$ $u = 0$ in ΔCDE



This does <u>not</u> imply that $u = 0$ along CD

u could be discontinuous along the characteristic line CD, but this could only be if $f(2^-) \neq 0$!

So now prove that $f(2^-) = 0$!

Fredholm alternative theorem says that if $(*)$ has N linearly independent solutions

$y_1, y_2, \ldots, y_N$

then so does the adjoint equation

$$\rho(0)\, v(s) + \int_0^2 R^+(0, s'+2-s)\, v(s')\, ds' = 0 \quad ; \quad 0 \le s \le 2 \quad (**)$$

Call these solutions $v_1, v_2, \ldots, v_N$

Since (☆) has a solution $W(0,s)$ it follows from Fredholms alternative that the inhomogeneous term $R^+(0, s+2)$ in (☆) must be orthogonal to all v_k's.

$$\int_0^2 v_i(s')\, R(0, s'+2)\, ds' = 0 \quad , \quad i = 1, 2, \ldots, N$$

Evaluate $(**)$ at $s=0^+$. This implies

$$v_i(0^+) = 0 \quad , i = 1,2,\dots,N$$

Change variables in $(**)$ $\quad s' \to 2-s'$, $s \to 2-s$

$$\rho(0)\, v_i(2-s) + \int_0^2 R^+(0, s+2-s')\, v_i(2-s')\, ds' = 0 \quad ; \quad 0 \le s \le 2$$
$$i = 1,2,\dots,N$$

But this equation is identical to $(*)$ (the original homogeneous equation). Thus

$$y_j(s) = \sum_{i=1}^{N} a_{ij}\, v_i(2-s) \quad , \quad 0 \le s \le 2 \quad , \quad j = 1,2,\dots,N$$

Valuate at $s = 2^-$ and use $v_i(0^+) = 0$, $i = 1,2,\dots,N$

$$y_j(2^-) = 0 \quad ; \quad j = 1,2,\dots,N$$

and thus $f(2^-) = 0$!

Conclusion: u is continuous and zero along CD

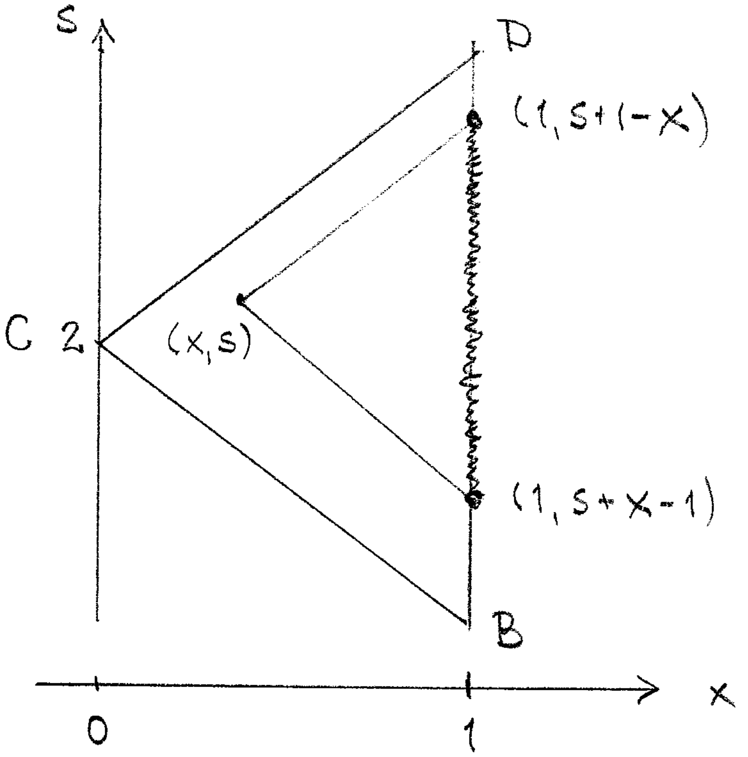
We proceed by proving that the transmitted field vanishes along BD.

In the triangle BCD, it is possible to write the solution $u(x,s)$ in terms of the transmitted field $u(1,s)$ as:

$$\boxed{\begin{aligned} 2\,u(x,s)\,t^{+}(x,1) &= (c_1+1)\,u(1,\,s+1-x) \\ &\quad - (c_1-1)\,t^{+}(x,1)\,t^{-}(x,1)\,u(1,\,s+x-1) \\ &\quad + \int_{s+x-1}^{s-x+1} N(x,\,s-s')\,u(1,s')\,ds' \quad , \quad 0<x<1 \end{aligned}}$$

The kernel $N(x,s)$ is a function related to the Riemann function of the original PDE.

Cf. domain of dependence!



Let (x,s) be a point on the line CD. $(s = x+2)$. The left hand side as well as the first term on the right hand side then vanish.

$$(c_1 - 1)\, t^+(x,1)\, t^-(x,1)\, u(1, 2x+1)$$

$$= \int_{2x+1}^{3} N(x, x+2-s')\, u(1,s')\, ds' \quad , \quad 0 \le x \le 1$$

Change of variables!

$$(c_1 - 1)\, u(1,s) + \int_s^3 K(s,s')\, u(1,s')\, ds' = 0 \quad , \quad 1 \le s \le 3$$

Volterra equation of the second kind $\Rightarrow$

$$u(1,s) = 0 \quad , \quad 1 \le s \le 3$$

However, (note $u(1,s) = u^+(1,s)$) (p. 70)

$$u(1, s+1) = \tau(0)\left[u^+(0,s) + \int_0^s T(0, s-s')\, u^+(0,s')\, ds' \right] \quad , \quad s \ge 0$$

Thus $u^+(0,s) = 0 \quad , \quad 0 \le s \le 2$

We then have

$$f(s) = 0 \quad , \quad 0 \leq s \leq 2$$

and thus

$$y(s) = 0 \quad , \quad 0 \leq s \leq 2$$

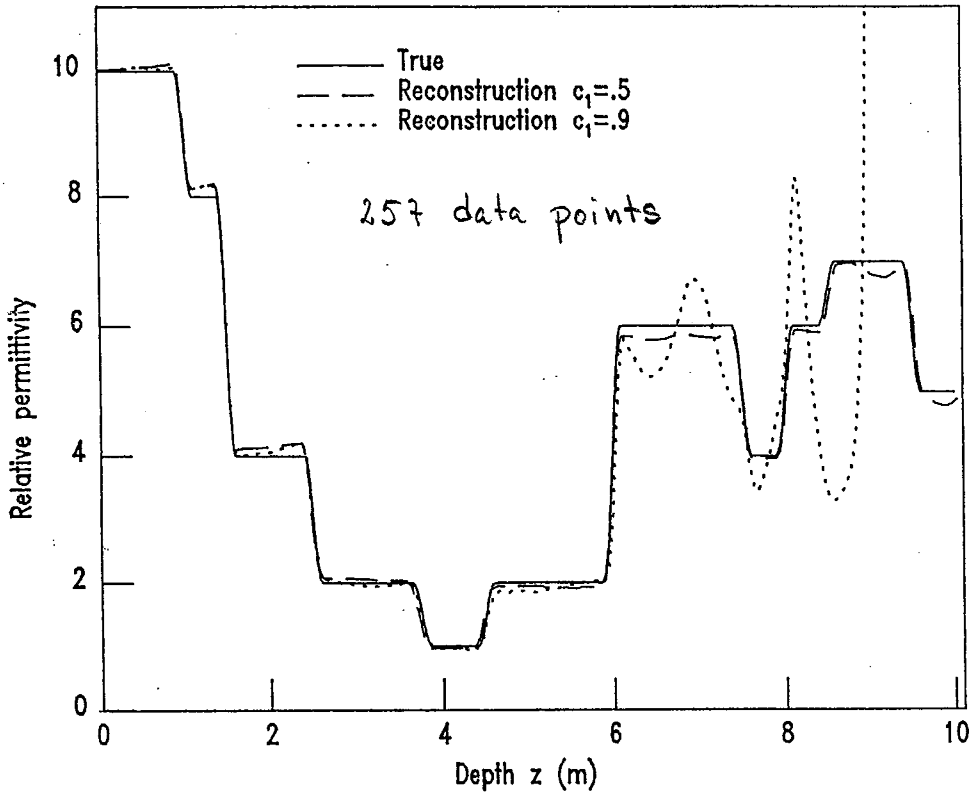
and

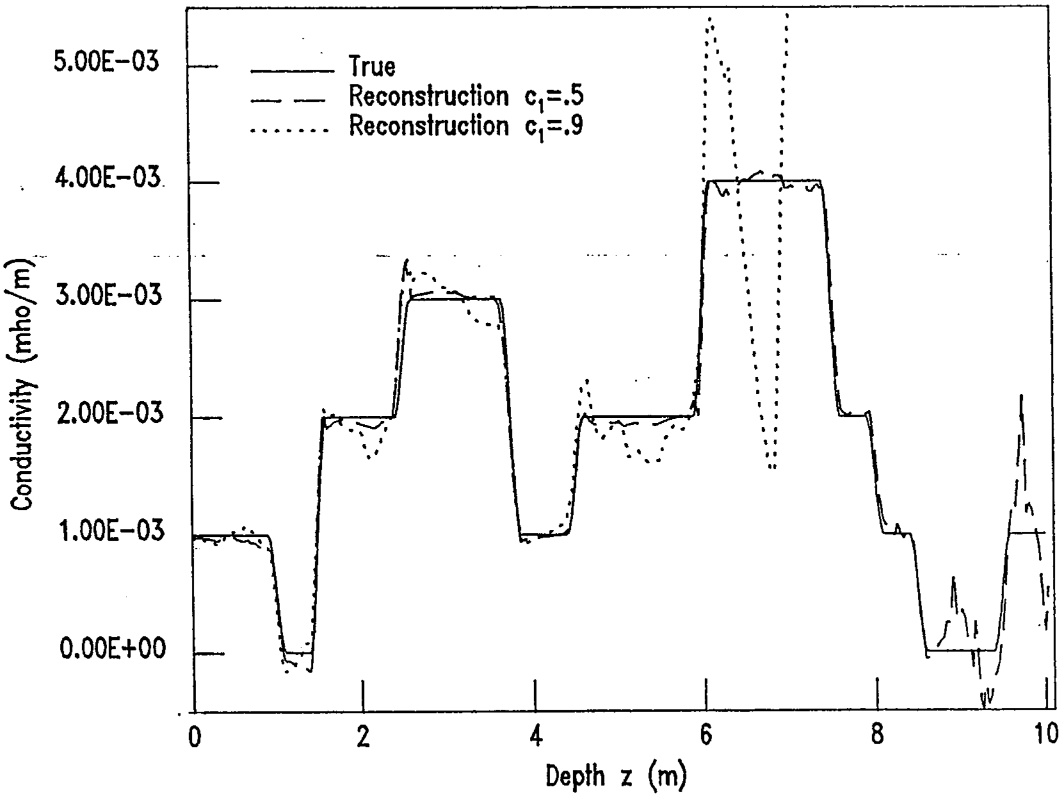
$$\overset{+}{R}(0, s+2) + \rho(0)\, W(0, s) + \int_0^2 \overset{+}{R}(0, s+2-s')\, W(0, s')\, ds' = 0$$

$$0 \leq s \leq 2$$

is uniquely solvable.

1.10 Numerical example





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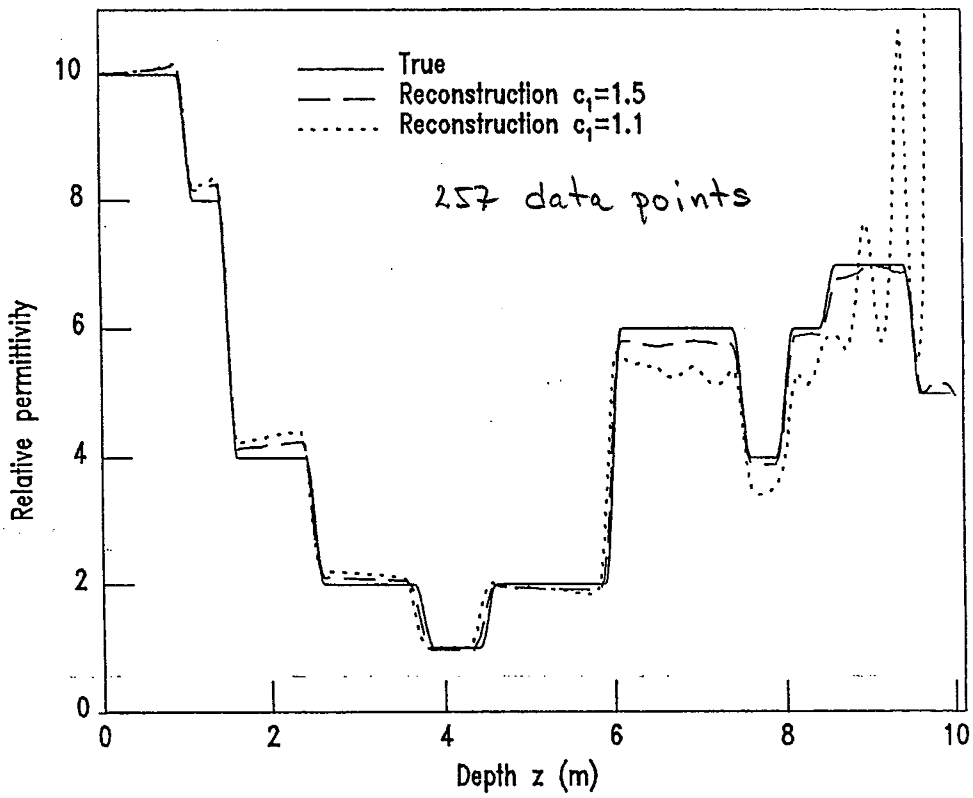
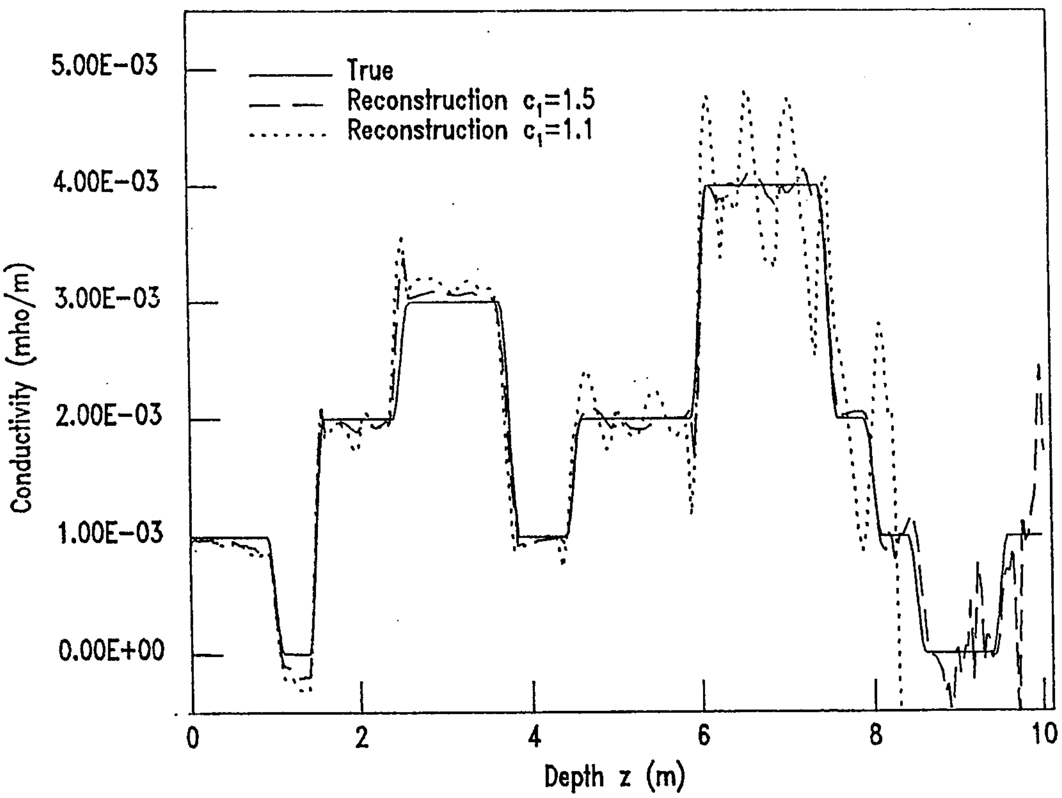
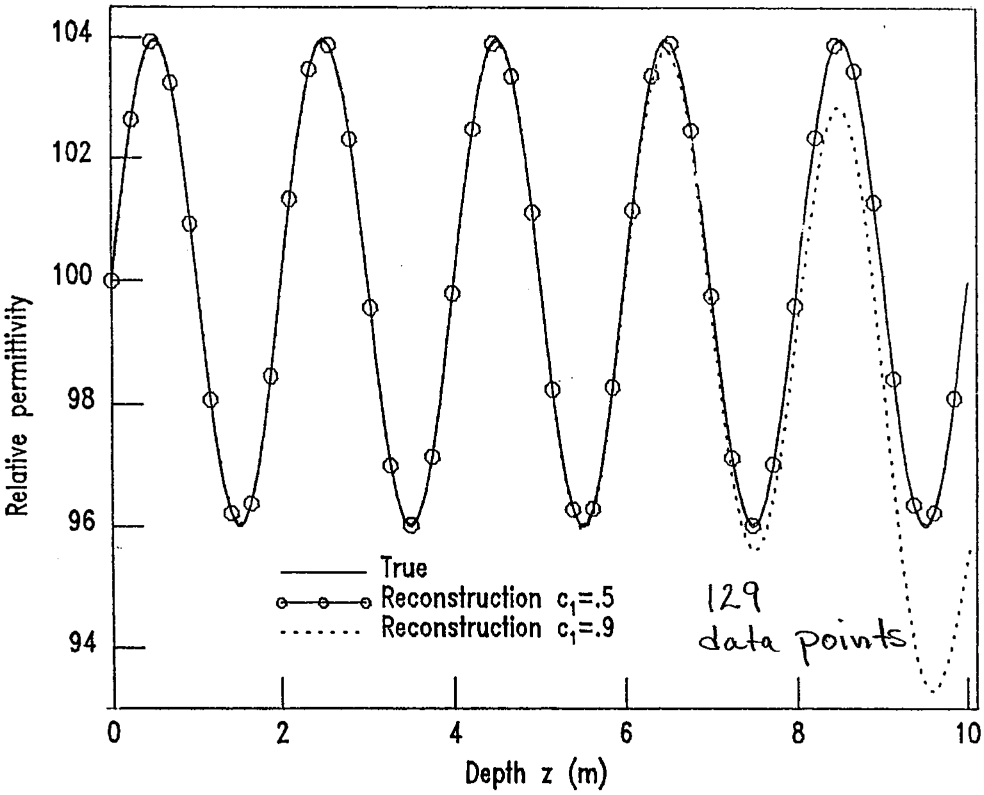
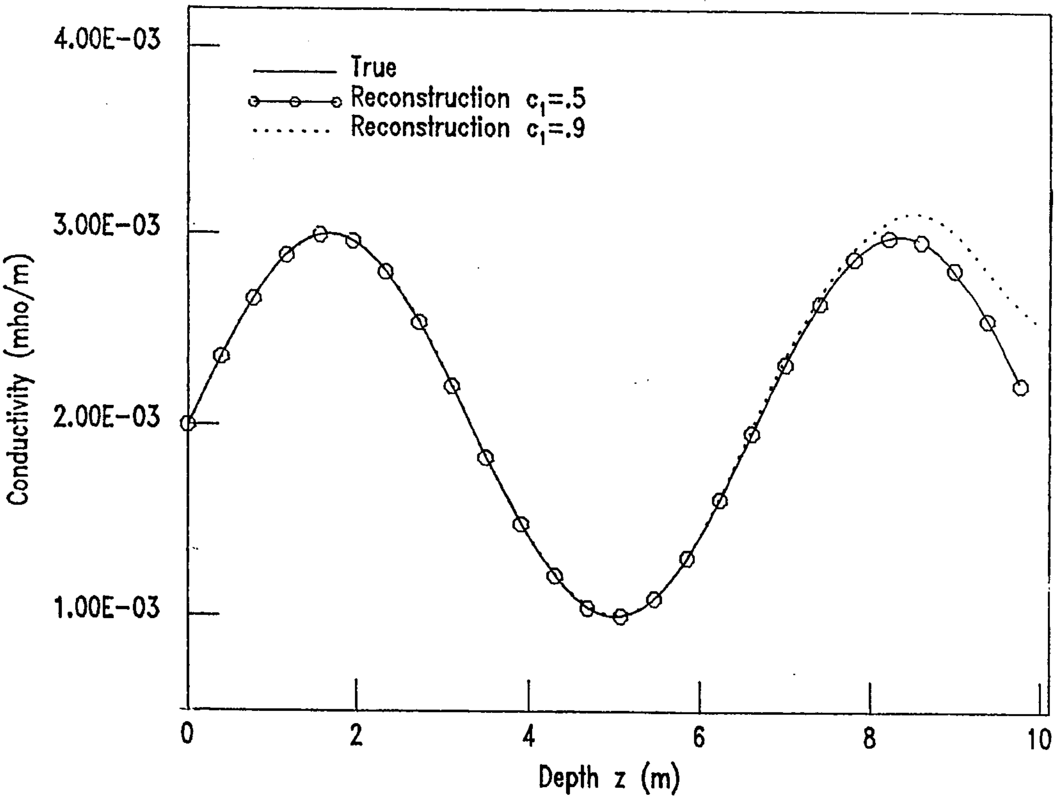

True
Reconstruction c_1=1.5
Reconstruction c_1=1.1
257 data points
Relative permittivity
Depth z (m)
0
2
4
6
8
10
0
2
4
6
8
10

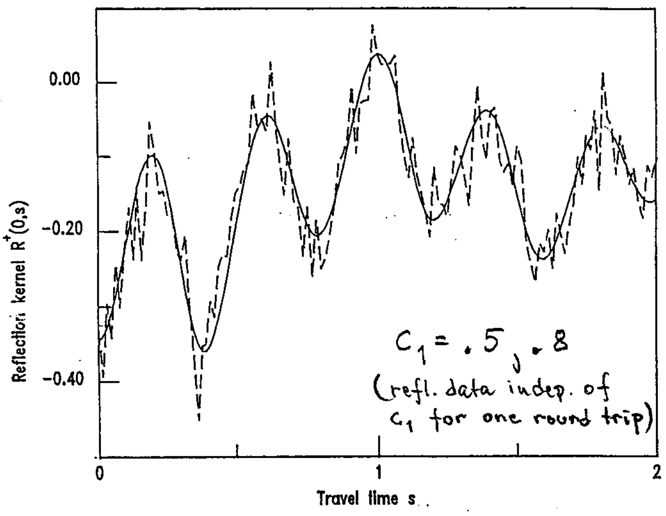

True
Reconstruction c_1=1.5
Reconstruction c_1=1.1
Conductivity (mho/m)
5.00E-03
4.00E-03
3.00E-03
2.00E-03
1.00E-03
0.00E+00
Depth z (m)
0
2
4
6
8
10



104
102
100
98
96
94
Relative permittivity
0
2
4
6
8
10
Depth z (m)
True
Reconstruction c1=.5
Reconstruction c1=.9
129 data points

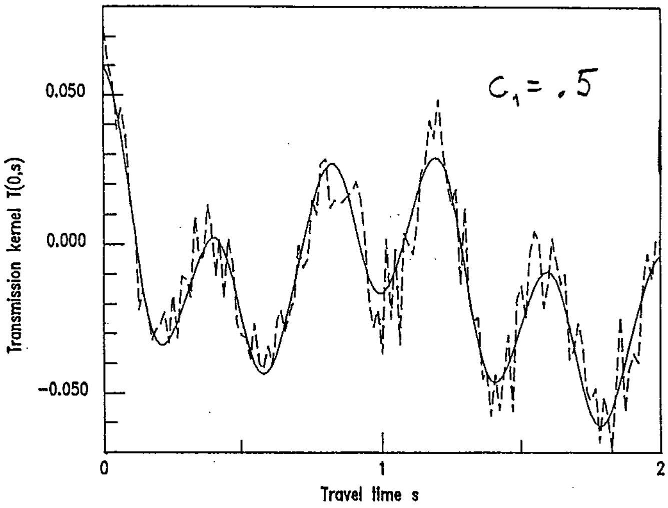


4.00E-03
3.00E-03
2.00E-03
1.00E-03
Conductivity (mho/m)
0
2
4
6
8
10
Depth z (m)
True
Reconstruction c1=.5
Reconstruction c1=.9

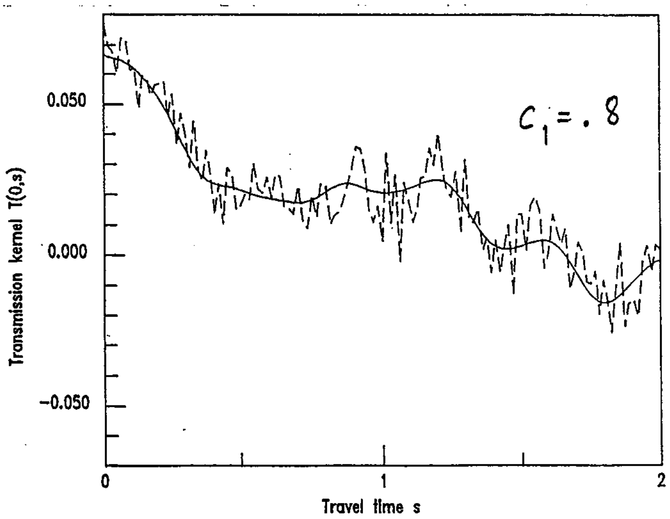


$C_1 = .5, .8$
(refl. data indep. of C_1 for one round trip)

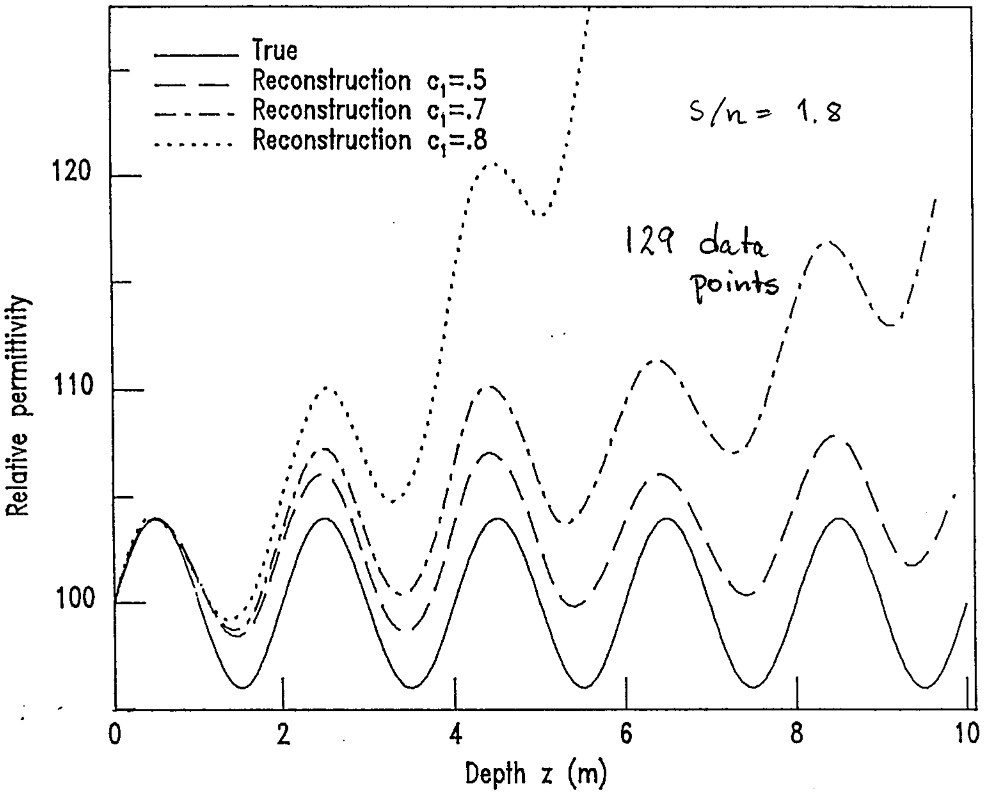
$s/n = 1.8$



$C_1 = .5$

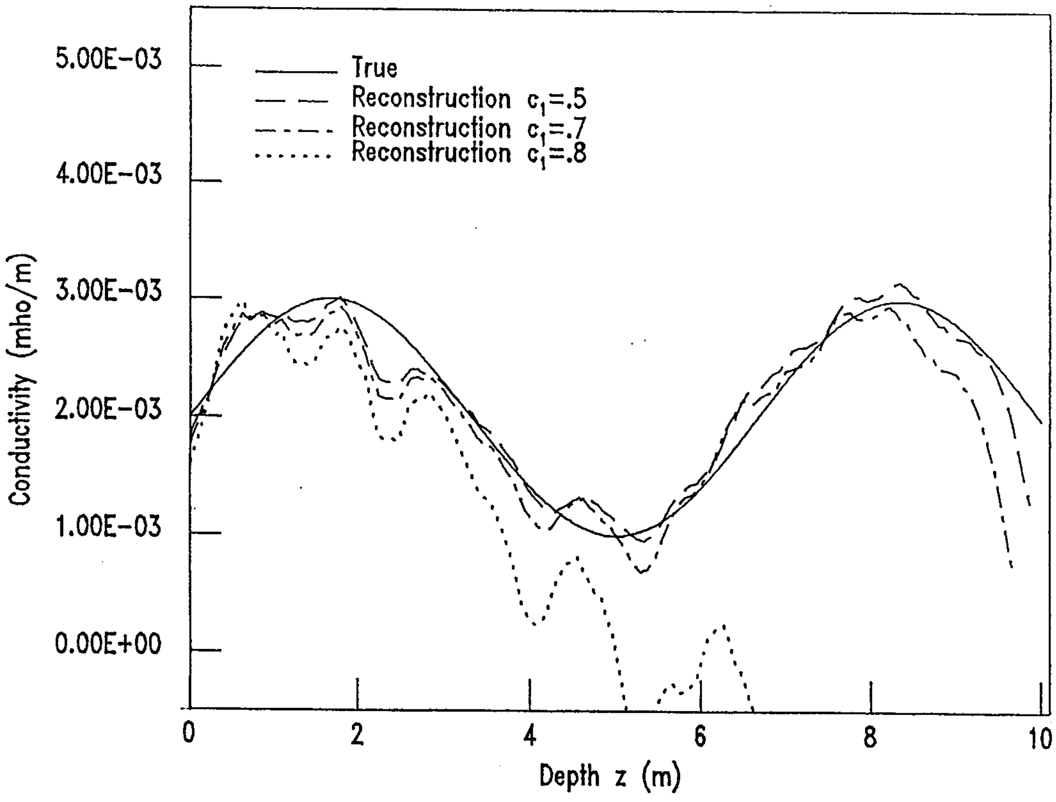


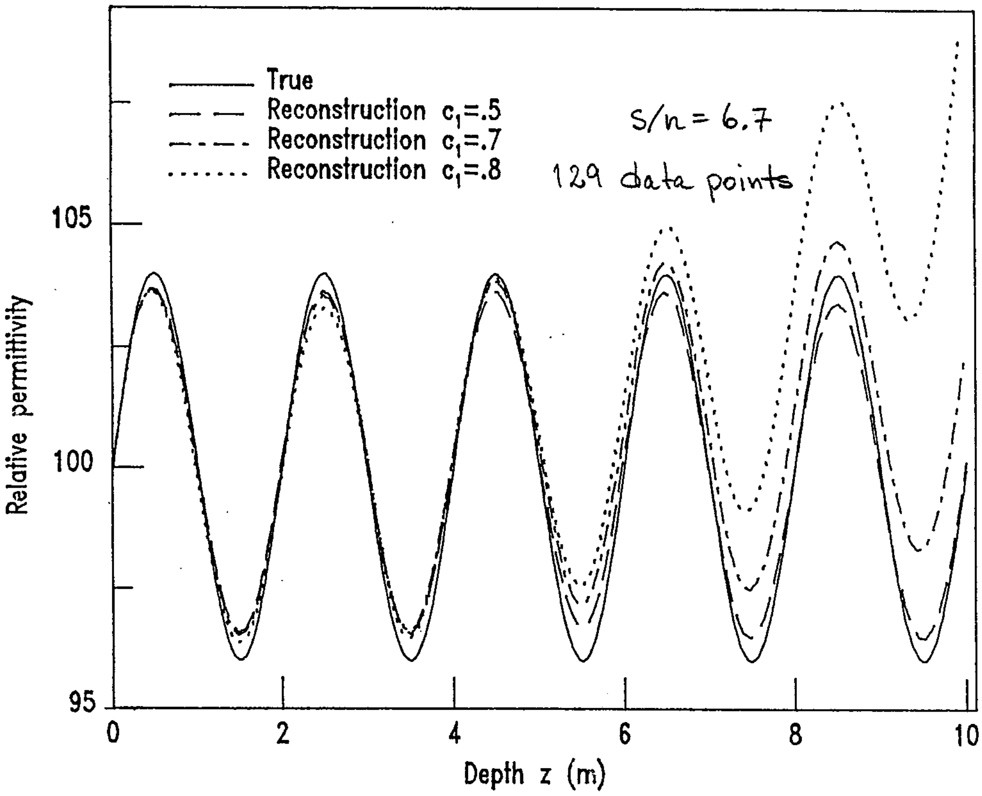
$C_1 = .8$

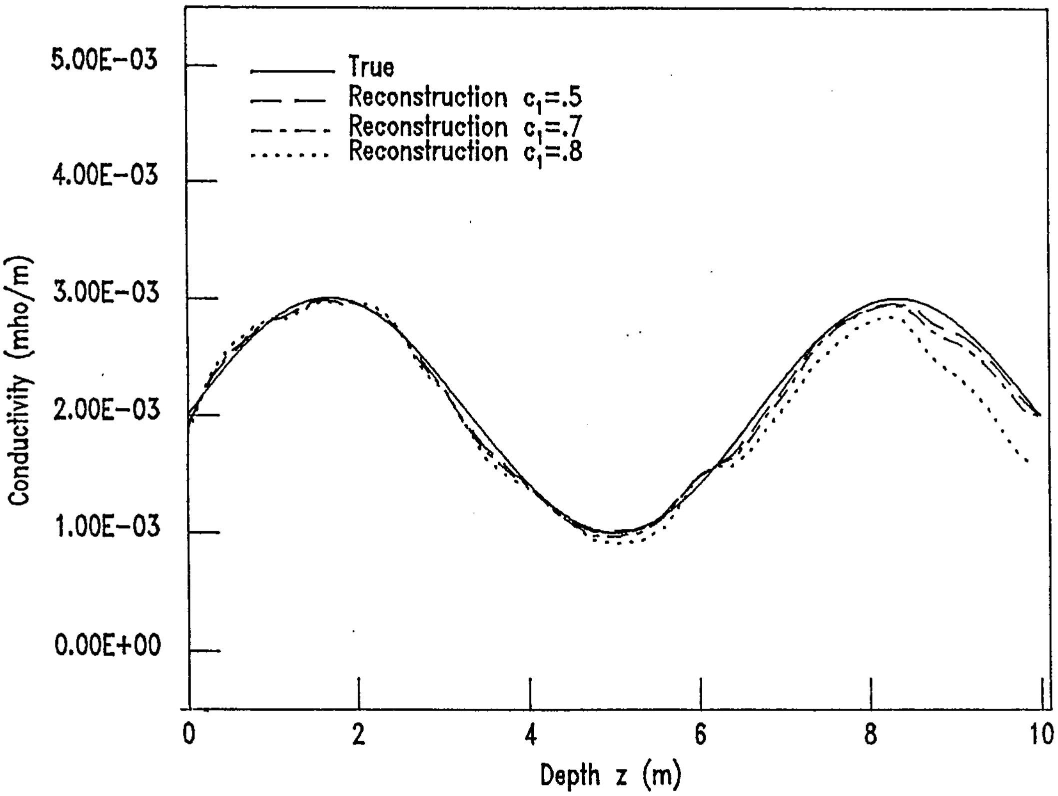


S/n = 1.8

129 data points




True
Reconstruction c_1=.5
Reconstruction c_1=.7
Reconstruction c_1=.8
S/n = 6.7
129 data points
105
100
95
Relative permittivity
0
2
4
6
8
10
Depth z (m)


True
Reconstruction c_1=.5
Reconstruction c_1=.7
Reconstruction c_1=.8
5.00E-03
4.00E-03
3.00E-03
2.00E-03
1.00E-03
0.00E+00
Conductivity (mho/m)
0
2
4
6
8
10
Depth z (m)

1.11 Summary of data sets for the inverse problem

Lossless medium

Data set for inversion
$R^+(s), \quad 0 < s < 2$
l
$\epsilon(0)$

Lossy media

Continuous profile

Data set for inversion
$R^+(s), \quad 0 < s < 2$
$R^-(s), \quad 0 < s < 2$
$T(s), \quad 0 < s < 2$
$g(1) = \exp\left\{\int_0^1 B(x')\, dx'\right\}$
l
$\epsilon(0)$

Discontinuous profile

Data set for inversion
$R^+(s), \quad 0 < s < 2$
$T(s), \quad 0 < s < 2$
l
$\epsilon(0)$

or

Data set for inversion
$R^+(s), \quad 0 < s < 4$
$\rho(0)$
l
$\epsilon(0)$

The inverse problem with iteration

Just as in the continuous profile case

the imbedding equation

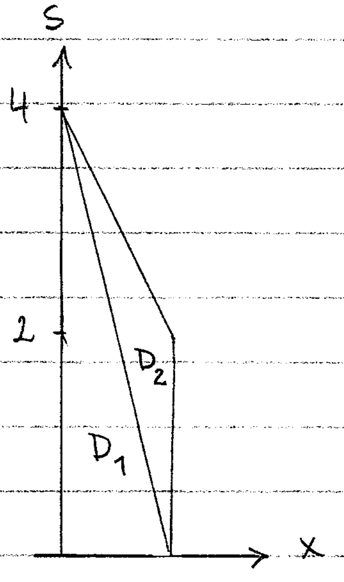
$$R^+_x(x,s) = 2R^+_s(x,s) - B(x)R^+(x,s)$$
$$- \tfrac{1}{2}[A(x)+B(x)](R^+ * R^+)(x,s)$$
$$- H(s-2(1-x))\,\rho(x)[A(x)+B(x)]\,R^+(x,s-2(1-x))$$

can be turned into an integral equation.

Analogous derivation gives

$$\begin{cases} R^+(x,s) = R(0,s+2x) + \int_0^x M(x,y,s)\,dy & ;\ (x,s) \in D_1 \\ R^+(x,s) = -\int_x^1 M(x,y,s)\,dy & ;\ (x,s) \in D_2 \end{cases}$$

$$M(x,y,s) = -B(y)R^+(y,s+2(x-y))$$
$$-\tfrac{1}{2}[A(y)+B(y)]\,(R^+ * R^+)(y,s+2(x-y))$$
$$-H(s-2(1-x))\,\rho(y)[A(y)+B(y)]\,R^+(y,s-2(1-x))$$



Assume the following data

1) $F(s) = R^{+}(0,s)$, $0 < s < 4$ (two round trips)

2) ℓ

3) $\varepsilon(0)$

Use the integral equation and the following identities

$$R^{+}(x,0^{+}) = -\tfrac{1}{4}\left[A(x) - B(x)\right]$$

$$\left[R^{+}(x,s)\right]_{s=4(1-x)^{-}}^{s=4(1-x)^{+}} = -\tfrac{1}{4}\rho^{2}(x)\left[A(x) + B(x)\right]$$

$$\rho(x) = r\exp\left[\int_x^1 B(x')\,dx'\right] \quad ; \quad r = \frac{1-c_1}{1+c_1}$$

to construct the iteration scheme

$$\left\{ \begin{array}{l}
R_0^+(x,s) = 0 \qquad \text{in } D_1 \cup D_2 \\[2ex]
R_{n+1}^+(x,s) = \left\{ \begin{array}{ll} F(s+2x) + \int_0^x M_n(x,y,s)\,dy & \text{in } D_1 \\ -\int_x^1 M_n(x,y,s)\,dy & \text{in } D_2 \end{array} \right. \\[3ex]
\rho_{n+1}(x) = r \exp\left[\int_x^1 B_n(y)\,dy \right] \\[3ex]
A_{n+1}(x) = -\dfrac{2}{\rho_{n+1}^2(x)} \left[R_{n+1}^+(x,s) \right]_{s=4(1-x)^-}^{s=4(1-x)^+} - 2R_{n+1}^+(x,0^+) \\[3ex]
B_{n+1}(x) = -\dfrac{2}{\rho_{n+1}^2(x)} \left[R_{n+1}^+(x,s) \right]_{s=4(1-x)^-}^{s=4(1-x)^+} + 2R_{n+1}^+(x,0^+) \\[3ex]
n = 0, 1, 2, 3, \ldots
\end{array} \right.$$

$M_n(x,y,s)$ has subscript n on all its entries.

First iteration

$$\left\{ \begin{array}{l} R_1^+(x,s) = \begin{cases} F(s+2x) & \text{in } D_1 \\ 0 & \text{in } D_2 \end{cases} \\ \\ \rho_1(x) = r \\ \\ A_1(x) = \frac{2}{r^2} F(4-2x) - 2F(2x) \\ \\ B_1(x) = \frac{2}{r^2} F(4-2x) + 2F(2x) \end{array} \right.$$

Under specific conditions on the data $F(s)$ the iteration scheme converges (contraction mapping in $\| \cdot \|_\infty$ norm)

$$R_n^+ \to R$$

$$A_n, B_n \to A, B$$

Sufficient conditions on $F(s)$:

$$|F(s)| \leq \begin{cases} f_1 & ; 0 \leq s < 2 \\ f_2 & ; 2 < s \leq 4. \end{cases}$$

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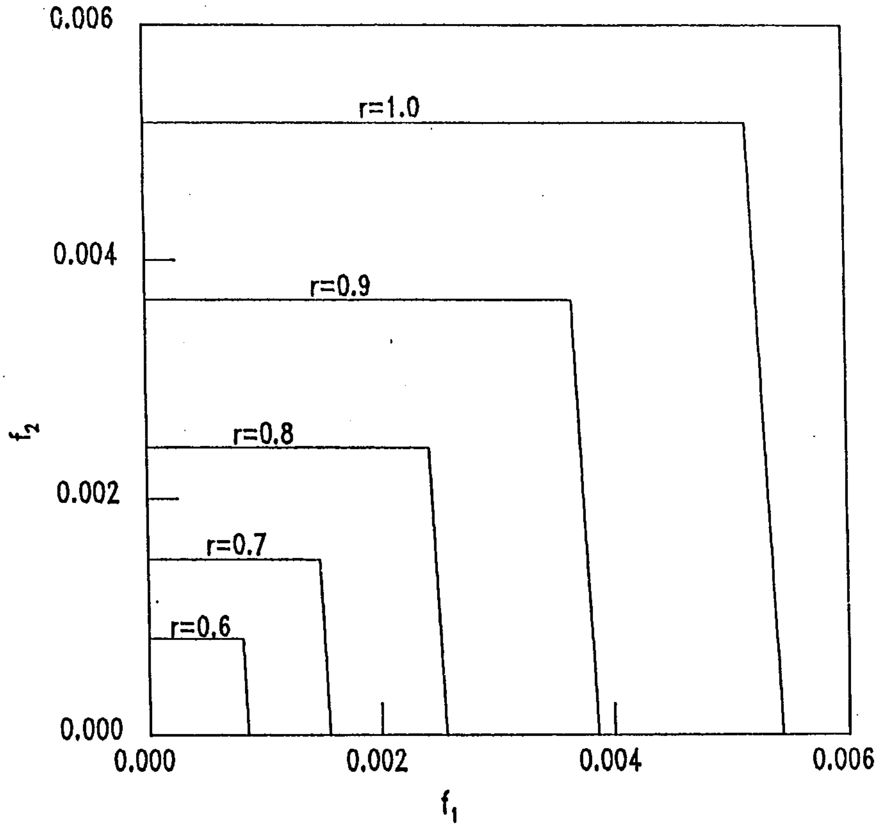
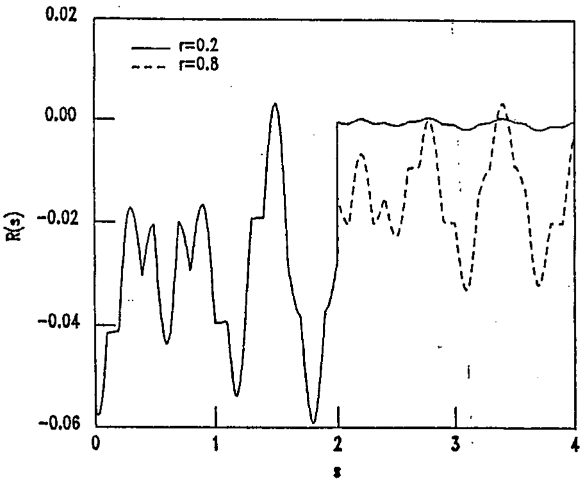
Domain of convergence

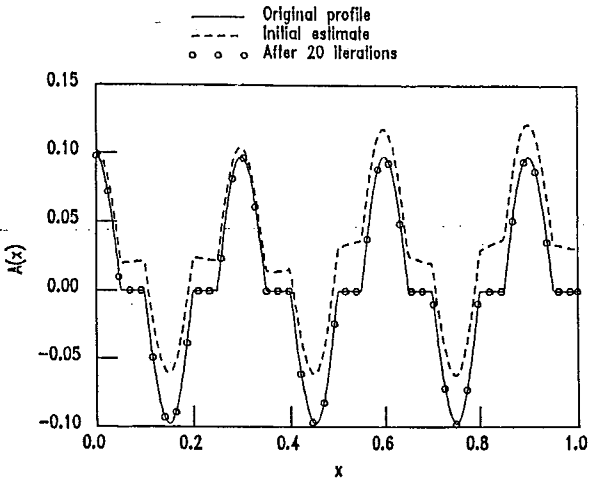


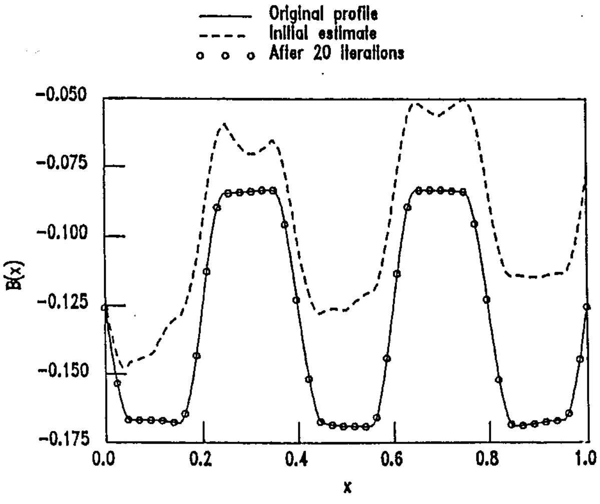
Numerical example



$$r = \frac{1-C_1}{1+C_1}$$

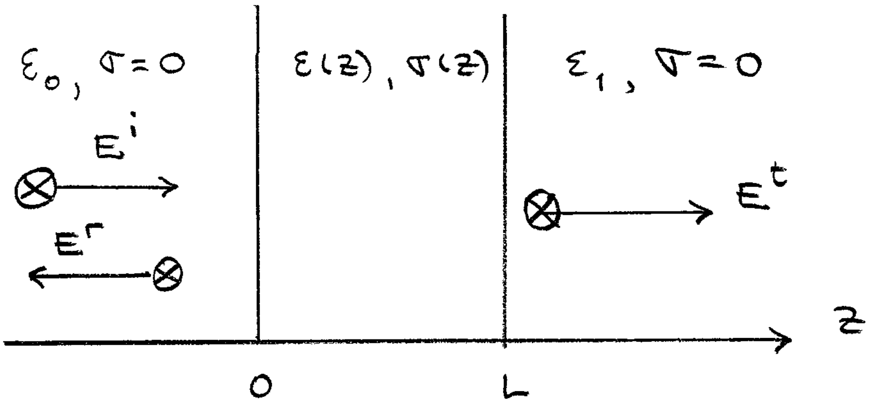
$$C_1 = \sqrt{\mathcal{E}(L^+)/\mathcal{E}(L^-)}$$



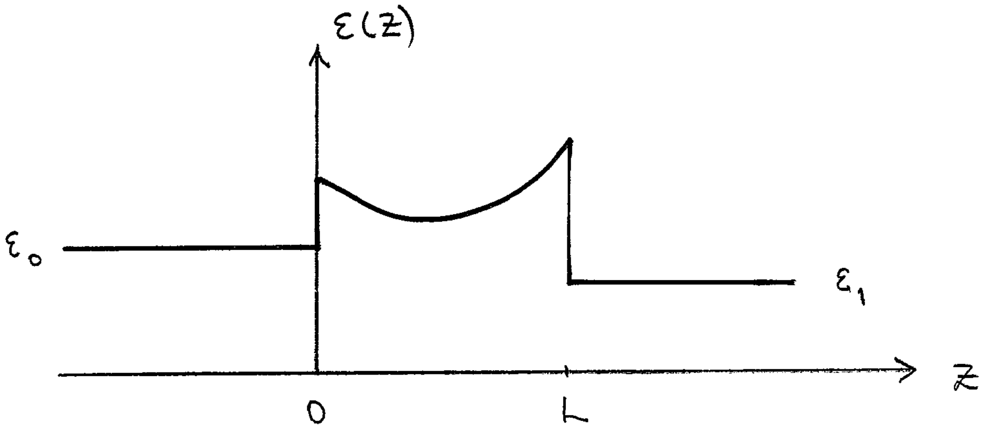


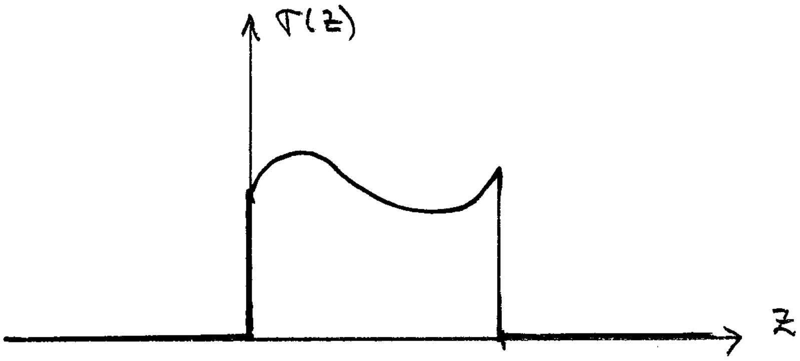
2. Green functions approach

2.1 Repetions of basic equations



The permittivity and conductivity profiles are discontinuous.





The Maxwell equations imply as before (cf. p. 113) in normalized travel time coordinates

$$\left[\partial_x^2 - \partial_s^2 + A(x)\,\partial_x + B(x)\,\partial_s\right] u(x,s) = 0$$

$$A(x) = -\partial_x \ln c(z(x)) = \tfrac{1}{2}\partial_x \ln \varepsilon(z(x))$$

$$B(x) = -\ell\,\sigma(z(x))/\varepsilon(z(x))$$

Boundary conditions at $x = 0, 1$ are

$$\begin{cases} u(0^-,s) = u(0^+,s) \\ c_0\, u_x(0^-,s) = u_x(0^+,s) \end{cases}$$

$$\begin{cases} u(1^-,s) = u(1^+,s) \\ c_1\, u_x(1^+,s) = u_x(1^-,s) \end{cases}$$

$$\begin{cases} c_0 = \sqrt{\varepsilon(0^-)/\varepsilon(0^+)} \\ \\ c_1 = \sqrt{\varepsilon(L^+)/\varepsilon(L^-)} \end{cases}$$

As before the front edge discontinuity can be removed, cf. pp. 114 - 115

Without loss of generality we can therefore assume $c_0 = 1$.

$$\boxed{c_0 = 1, \quad c_1 \neq 1.}$$

Wave splitting as before

$$u^{\pm}(x,s) = \frac{1}{2}\left[u(x,s) \mp \partial_s^{-1} u_x(x,s)\right]$$

implies the PDE for $u^{\pm}$ (cf. p. 99)

$$\boxed{\begin{gathered} \partial_x \begin{pmatrix} u^+ \\ u^- \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} u^+ \\ u^- \end{pmatrix} \\ \left\{ \begin{aligned} \alpha &= -\partial_s - \tfrac{1}{2}(A-B) \\ \beta &= \tfrac{1}{2}(A+B) \\ \gamma &= \tfrac{1}{2}(A-B) \\ \delta &= \partial_s - \tfrac{1}{2}(A+B) \end{aligned} \right. \end{gathered}}$$

2.2 The relation between $u^\pm$

The relations on p. 100 are modified due to the presence of the hard back wall ($c_1 \neq 1$). We get, $x \in (0,1)$ (inside the slab)

$$\boxed{\begin{aligned} u^+(x,x+s) &= t^-(0,x)\left[u^+(0,s) + \int_{-\infty}^{s} G^+(x,s-s')\,u^+(0,s')\,ds'\right] \\ u^-(x,x+s) &= \left[t^+(0,x)\right]^{-1}\Big[\rho(0)\,u^+(0,s-2(1-x)) \\ &\qquad + \int_{-\infty}^{s} G^-(x,s-s')\,u^+(0,s')\,ds'\Big] \end{aligned}}$$

$$\begin{cases} t^\pm(x,y) = \exp\left\{\pm\frac{1}{2}\int_x^y [A(x') \pm B(x')]\,dx'\right\} & (\text{cf. p. } 116) \\ \rho(x) = r\exp\left\{\int_x^1 B(x')\,dx'\right\} & (\text{—"—}) \\ r = (1-c_1)/(1+c_1) & (\text{—"—}) \end{cases}$$

It is immediately clear that

$$G^+(0,s) = 0$$

As before, it is possible to compare with the scattering kernels on p. 116. The result is (cf. p. 101)

$$\begin{cases} G^-(0,s) = R^+(s) = \text{Physical reflection kernel} \\ G^+(1,s) = T(s) = \text{Transmission reflection kernel} \end{cases}$$

Notice that the field $u^+(1, s+1-x)$ on p. 116 is the field at $x = 1^+$, i.e., outside the slab, and differs from the field $u^+(1^-, s+1-x)$ by a factor $2/(1+c_1)$. This is due to the back wall ($c_1 \neq 1$). Moreover, due to wave splitting

$$\begin{cases} u = u^+ + u^- \\ u_x = u_s^- - u_s^+ \end{cases} \qquad (\text{cf. p } 36)$$

and, since there is no sources at $x > 1$, $u^-(1^+, s) = 0$

$$\begin{cases} u(1^-, s) = u^+(1^-, s) + u^-(1^-, s) = u(1^+, s) = u^+(1^+, s) \\ u_x(1^-, s) = u_s^-(1^-, s) - u_s^+(1^-, s) = c_1 u_x(1^+, s) = -c_1 u_s^+(1^+, s) \end{cases}$$

Integration wrt s gives

$$\begin{cases} u^+(1^-, s) + u^-(1^-, s) = u^+(1^+, s) \\ u^-(1^-, s) - u^+(1^-, s) = -c_1 u^+(1^+, s) \end{cases} \Rightarrow \begin{cases} u^-(1^-, s) = r\, u^+(1^-, s) \\ u^+(1^+, s) = \frac{2}{1+c_1} u^+(1^-, s) \end{cases}$$

The representations on p. 148 therefore at $x = 1^-$ are related by

$$G^-(1,s) = \rho(0)\, G^+(1,s)$$

since $t^-(0,1)\, t^+(0,1)\, r = \rho(0)$

In summary, the boundary conditions for $G^\pm$ at $x=0,1$ are

$$\boxed{\begin{array}{ll} G^+(0,s) = 0 & G^-(1,s) = \rho(0)\, G^+(1,s) \\ G^-(0,s) = R^+(s) & G^+(1,s) = T(s) \end{array}}$$

The limit values at $x=0,1$ are taken from the inside

2.3. Differential equations for $G^{\pm}$

The PDEs for $G^{\pm}$ are identical to the ones without back wall on p. 102. The initial conditions are also the same.

$$\boxed{\begin{aligned} G^{+}_{x}(x,s) &= \tfrac{1}{2}\left(A(x)+B(x)\right) e^{-\int_0^x B(x')dx'} \; G^{-}(x,s) \\ G^{-}_{x}(x,s) - 2G^{-}_{s}(x,s) &= \tfrac{1}{2}\left(A(x)-B(x)\right) e^{\int_0^x B(x')dx'} \; G^{+}(x,s) \end{aligned}}$$

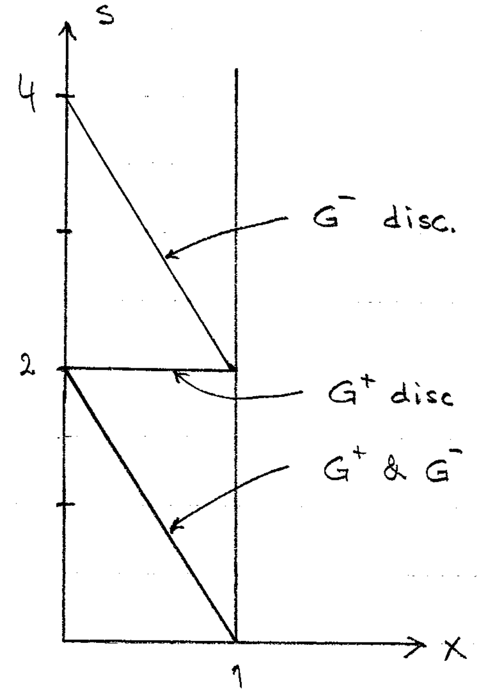
$$\boxed{\begin{aligned} G^{+}(x,0) &= -\tfrac{1}{8}\int_0^x \left(A^2(x') - B^2(x')\right) dx' \\ G^{-}(x,0) &= -\tfrac{1}{4}\left(A(x)-B(x)\right) \exp\left[\int_0^x B(x')dx'\right] \end{aligned}}$$

The boundary conditions are, however, not the same, see p. 150.

Any finite jump discontinuity in $G^{\pm}(x,s)$ can occur

$$\begin{cases} s = \text{constant} & \text{for } G^{+} \\ s = -2x + \text{constant} & \text{for } G^{-} \end{cases}$$

Due to the boundary condition at $x=1$, G^{+} has an additional finite jump discontinuity (follows from the derivation of the PDEs)



The finite jump discontinuities are

$$\boxed{\begin{aligned}
&\left[G^{+}(x,2(1-x))\right] = \tfrac{1}{4}\left(A(x)+B(x)\right)\rho(x) \\
&\left[G^{-}(x,2(1-x))\right] = \tfrac{1}{4}\left(A(1)-B(1)\right)\exp\left[\int_0^1 B(x)\,dx\right] \\
&\qquad + \tfrac{1}{8}\rho(0)\left\{-\int_0^1 (A^2(x)-B^2(x))\,dx - \int_x^1 (A^2(x')-B^2(x'))\,dx'\right\} \\
&\qquad + \tfrac{1}{4}r\rho(0)\left[A(1)+B(1)\right] \\
&\left[G^{+}(x,2)\right] = -\tfrac{1}{4}\left[A(0)+B(0)\right]\rho(0) \\
&\left[G^{-}(x,4-2x)\right] = -\tfrac{1}{4}\left[A(0)+B(0)\right]\rho^2(0)
\end{aligned}}$$

(The first disc. is forced on the solution)

2.4 Extension of data with $G^{\pm}$

Proceed as in the case without back wall, cf p. 103-105

The result is

$$\boxed{\begin{array}{ll} G^{+}(x,s) + W(s) + \int_0^s W(s-s')\, G^{+}(x,s')\, ds' = 0, & s > 2(1-x) \\[2ex] G^{-}(x,s) + \int_0^s W(s-s')\, G^{-}(x,s')\, ds' = 0, & s > 2(1-x) \end{array}}$$

Once $W(s)$, $0<s<2$, is known, the values of $G^{\pm}(x,s)$ below the line $s=2(1-x)$ can be used to compute the values of $G^{\pm}(x,s)$ above this line.

Specifically, the finite jump discontinuity in $G^{+}(x,s)$ at $s=2(1-x)$ can be written

$$[G^{+}(x,2(1-x))] = G^{+}(x,2(1-x)^{+}) - G^{+}(x,2(1-x)^{-}) =$$

$$-W(2(1-x)^{+}) - \int_0^{2(1-x)} W(2-2x-s')\, G^{+}(x,s')\, ds' - G^{+}(x,2(1-x)^{-})$$

From p. 152

$$[G^{+}(x,2(1-x))] = \frac{1}{4}\big(A(x)+B(x)\big)\, g(x)$$

2.5 Solution to the direct problem

The same grid of points (x_i, s_j) as before

Known: A_i , B_i ; $i = 0, 1, \ldots, N$

Sought:
$$\begin{cases} R^+(2jh) = G^-_{0,j} & ; \; j = 0,1,2,\ldots \\ T(2jh) = G^+_{N,j} & ; \; j = 0,1,2,\ldots \end{cases}$$

The direct procedure proceeds as in the case without back wall with special care taken to the jump discontinuities of G^+ at $s = 2$ and $s = 2(1-x)$ and G^- at $s = 2(1-x)$ and $s = 4-2x$.

2.6 Solution of the inverse problem

Data requirements for a simultaneous reconstruction of $A(x)$ and $B(x)$

$$\begin{cases} R^{+}(s), \quad 0 < s < 2 \\ W(s), \quad 0 < s < 2 \\ \rho(0) \\ \ell \\ \varepsilon(0) \end{cases}$$

How to get the $W(s)$ data from physical measurements is identical to the imbedding approach (cf C.1.9, p. 129 – 137).

The constants $\rho(0)$, ℓ and $\varepsilon(0)$ can also be obtained in exactly the same way as before, (cf. p. 126).

Equation used

① $$\partial_x G^+(x,s) =$$

$$= \frac{1}{2}\left[A(x) + B(x)\right] \exp\left[-\int_0^x B(x')\,dx'\right] G^-(x,s)$$

$$\partial_x G^-(x,s) - 2\partial_s G^-(x,s) =$$

$$= \frac{1}{2}\left[A(x) - B(x)\right] \exp\left[\int_0^x B(x')\,dx'\right] G^+(x,s)$$

② $$G^-(x,0) = -\frac{1}{4}\left[A(x) - B(x)\right] \exp\left[\int_0^x B(x')\,dx'\right]$$

③ $$W(2-2x) + \int_0^{2(1-x)} W(2-2x-s')\, G^+(x,s')\, ds'$$

$$+ G^+(x, 2(1-x)^-) = -\frac{1}{4}\left[A(x) + B(x)\right] \rho(x)$$

and $$\rho(x) = \frac{1-c_1}{1+c_1} \exp\left[\int_x^1 B(x')\,dx'\right]$$

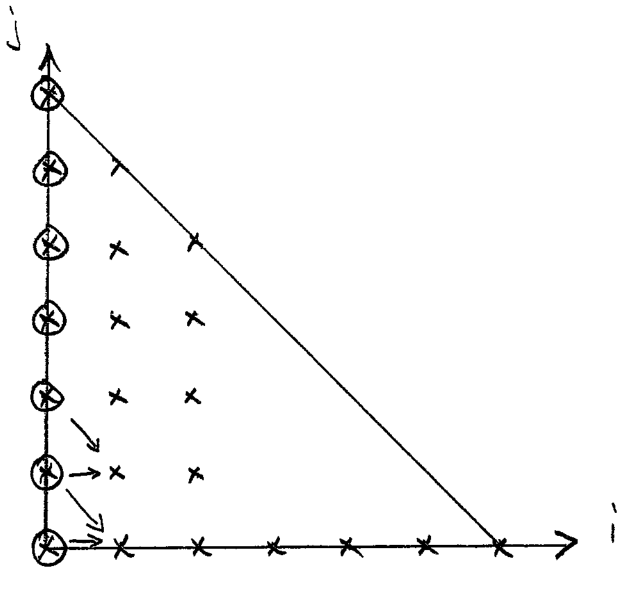
$$c_1 = \sqrt{\varepsilon(L^+)/\varepsilon(L^-)}$$

Algorithm

1.) Use eq. ① to step G^+ and G^- forward from the current grid line x_0 to the next grid line $x_1 = x_0 + h$

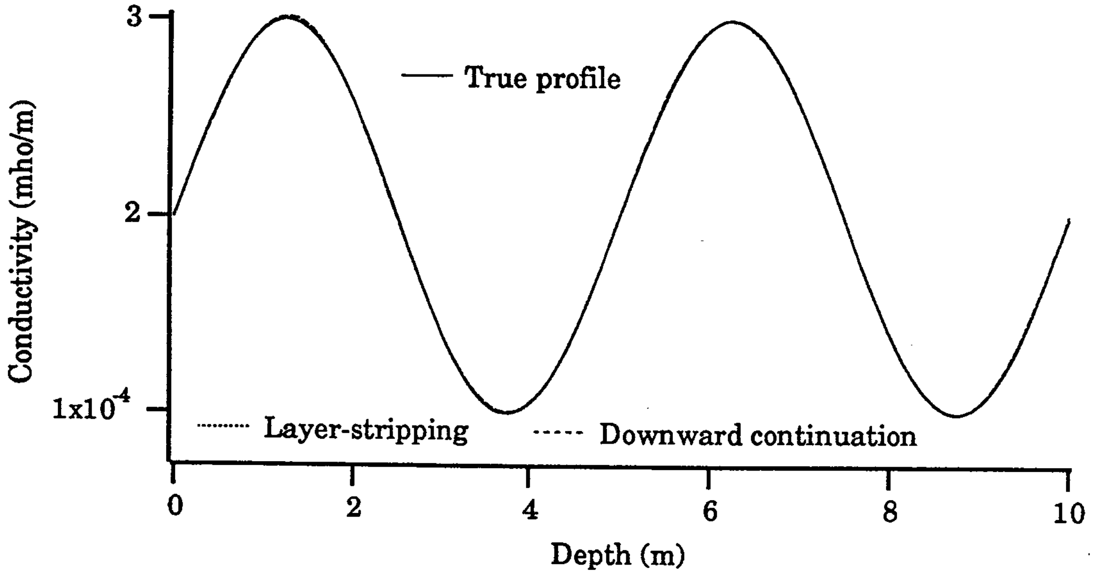
2.) At this new grid line solve for $A(x_1)$ and $B(x_1)$ using eqs. ② & ③.
Note this is a non-linear equation in B!

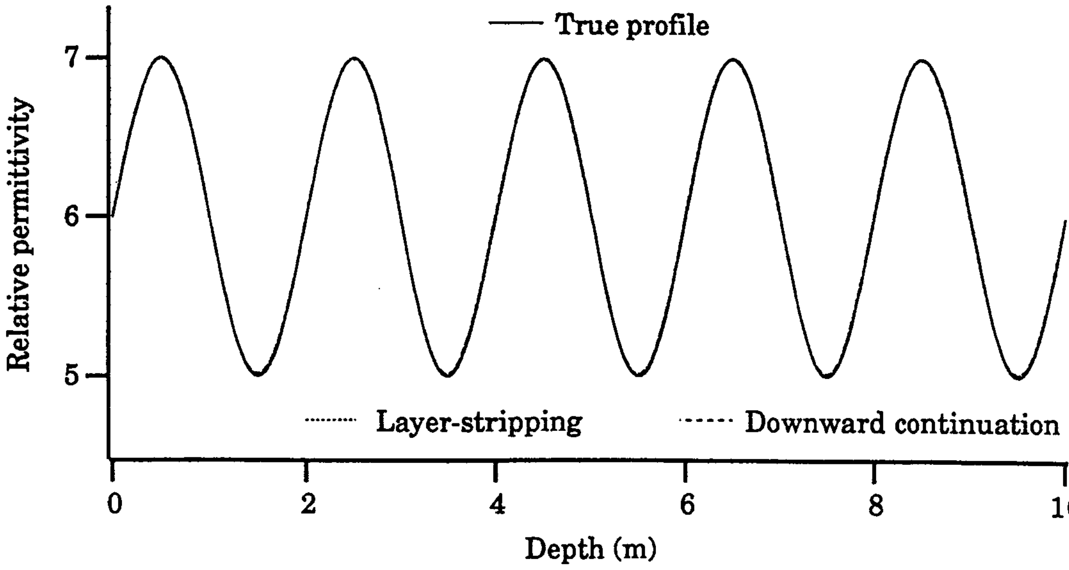
3) Repeat steps 1 & 2 to move one grid line deeper into the medium.



2.7 Numerical example

Reconstruction without noise $c_1 = 2$

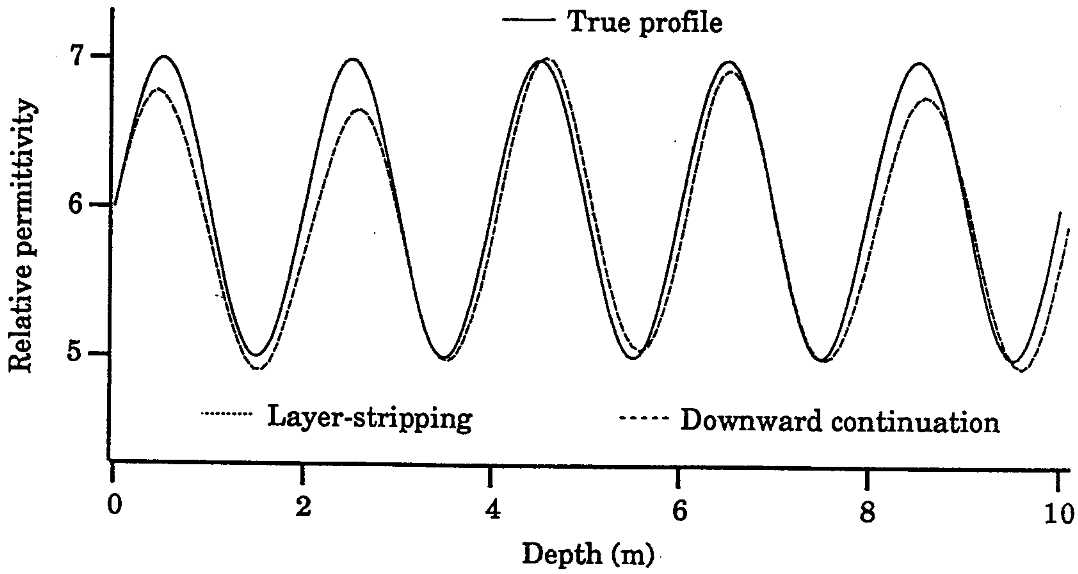


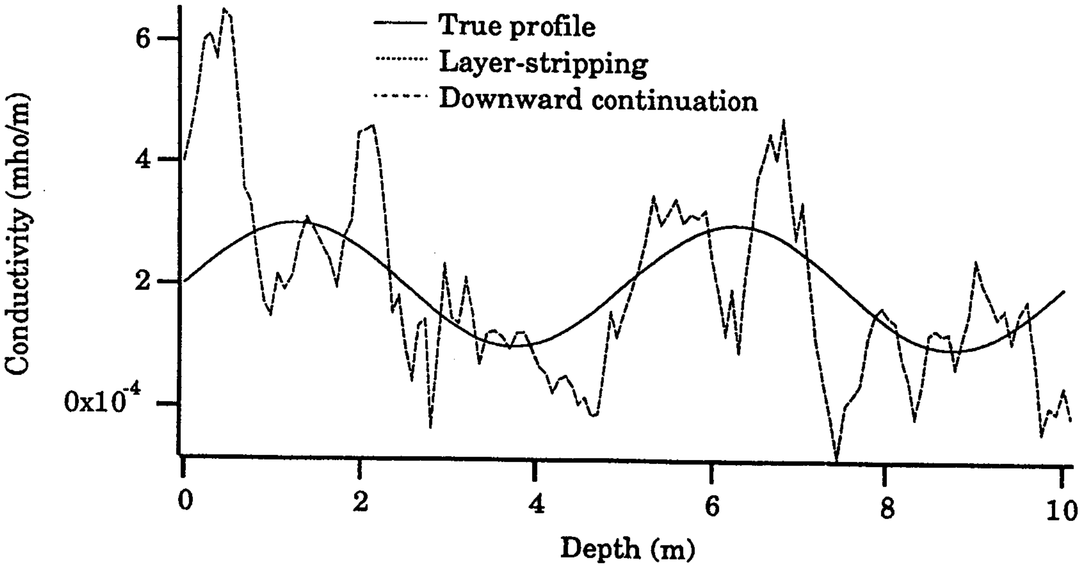


Layer-stripping = Imbedding approach

Downward continuation = Green function approach

Reconstruction with noisy data $c_1 = 2$





Gaussian noise STD = 0.05

D Scattering in dispersive media

1. Constitutive relations

1.1 Basic assumptions

In the previous chapters the losses were modeled with a DC conductivity $\sigma(z)$. That model ignored all frequency dependent effects.

In this chapter the constitutive relations between the displacement field $\bar{D}(\bar{r},t)$ and the electric field $\bar{E}(\bar{r},t)$ is assumed to be

$$\boxed{\bar{D}(\bar{r},t) = \epsilon_0 \left\{ \bar{E}(\bar{r},t) + \int_{-\infty}^{t} \chi(\bar{r},t-t')\, \bar{E}(\bar{r},t')\, dt' \right\}}$$

$\bar{D}$ depends on the previous time history of $\bar{E}$.

The function $\chi(\bar{r},t)$ is referred to as the <u>susceptibility kernel (function)</u> of the medium.

Some of the assuptions inherent in this model are now explained.

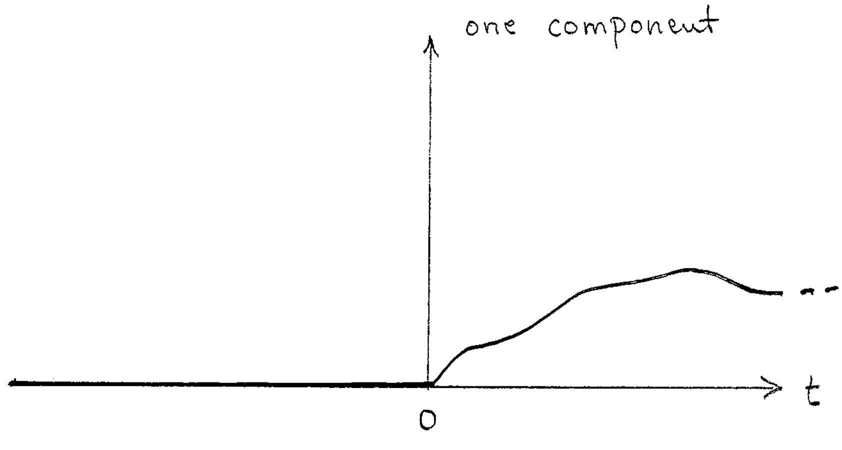
In general the constitutive relation is

$\bar{D} = \bar{D}(\bar{E})$ (a relation between $\bar{D}$ and $\bar{E}$-fields)

We will now make some assumptions about the functional behaviour in $\bar{D}(\bar{E})$ that are physically sound.

First a definition. Denote by

$C = \{$ all vector fields with continuous components as a function of time t, such that the components are zero on $(-\infty, 0]$ $\}$



Assume $\bar{D} = L(\bar{E})$ where the mapping L satisfies the following assumptions

① L is linear on C, i.e.

$$L(\alpha \bar{E} + \beta \bar{E}') = \alpha L(\bar{E}) + \beta L(\bar{E}'), \quad \forall \alpha, \beta \in R.$$

② L is invariant to time translations, i.e. $\forall \tau > 0$ the new field

$$\bar{E}'(t) = \bar{E}(t-\tau), \quad t \in (-\infty, \infty)$$

implies that

$$\bar{D}'(t) = L(\bar{E}')(t) = \bar{D}(t-\tau), \quad t \in (-\infty, \infty)$$

where $\bar{D}(t) = L(\bar{E})(t)$.

③ The transformation is causal, i.e. $\forall \tau$

$\bar{E}(t) = 0$ on $(-\infty, \tau]$ implies

$\bar{D}(t) = 0$ on $(-\infty, \tau]$.

④ The transformation is continuous, i.e. $\forall \tau$ and $\forall \epsilon > 0$ there exists a $\delta > 0$ such that

$$|\bar{E}(t)| < \delta \text{ on } (-\infty, \tau] \Rightarrow |\bar{D}(\tau)| < \epsilon$$

One can prove (proof not given here) that such a transformation maps C into C and that to every L there exists a unique tensor-valued function $G_{ij}(t)$ such that

$$\boxed{D_i(t) = \sum_{j=1}^{3} \int_{-\infty}^{t} E_j(t-t')\, d\, G_{ij}(t') \qquad , \quad i=1,2,3}$$

where G_{ij} has the following properties

1. $G_{ij}(t) = 0 \quad , \quad t < 0$ (implied by causality)

2. $G_{ij}(t)$ is of bounded variation on every closed subinterval of $(-\infty, \infty)$

3. $G_{ij}(t)$ is continuous on the right on $(-\infty,\infty)$, i.e.

$$G_{ij}(t) = \lim_{\epsilon \to 0^+} G_{ij}(t+\epsilon)$$

For an isotropic media, $G_{ij}(t) = \delta_{ij}\, G(t)$ and

$$\bar{D}(t) = \int_{-\infty}^{t} \bar{E}(t-t')\, d\,G(t')$$

Assume G continuously differentiable on $(0,\infty)$ and denote $G'(t) = \chi(t)$ and $a = G(0^+)$.

Then

$$\bar{D}(t) = a\,\bar{E}(t) + \int_{-\infty}^{t} \bar{E}(t-t')\,\chi(t')\,dt'$$

$$= a\bar{E}(t) + \int_{0}^{t} \chi(t-t')\,\bar{E}(t')\,dt'$$

$$= a\bar{E}(t) + (\chi * \bar{E})(t)$$

The constant a determines the high frequency behaviour and is usually taken as ϵ_0 (optical response)

These arguments justify the constitutive relation

$$\boxed{\bar{D}(t) = \epsilon_0\left[\bar{E}(t) + \int_{0}^{t} \chi(t-t')\,\bar{E}(t')\,dt'\right]}$$

(The lower limit in the integration can be replaced by $-\infty$)

1.2 Connection with frequency domain results

$$\begin{cases} \bar{D}(\bar{r},\omega) = \int_{-\infty}^{\infty} \bar{D}(\bar{r},t)\, e^{i\omega t}\, dt \\ \bar{D}(\bar{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{D}(\bar{r},\omega)\, e^{-i\omega t}\, d\omega \end{cases}$$

$\bar{D}(\bar{r},t)$ real valued for all times

$$\Rightarrow \quad \bar{D}(\bar{r},\omega) = \bar{D}^*(\bar{r},-\omega^*)$$

and similarly for $\bar{E}(\bar{r},t)$.

Constitutive relation in the frequency domain

$$\bar{D}(\bar{r},\omega) = \epsilon_0 \epsilon(\omega) \bar{E}(\bar{r},\omega) \qquad (\text{definition of } \epsilon(\bar{r},\omega))$$

$$\bar{D}(\bar{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \epsilon_0 \epsilon(\omega)\, \bar{E}(\bar{r},\omega)\, e^{-i\omega t}\, d\omega$$

$$= \frac{\epsilon_0}{2\pi}\int_{-\infty}^{\infty} d\omega\, \epsilon(\omega) \int_{-\infty}^{\infty} dt'\, \bar{E}(\bar{r},t')\, e^{i\omega(t'-t)}$$

$$= \epsilon_0 \Big\{ \bar{E}(\bar{r},t) + \int_{-\infty}^{\infty} dt' \Big[\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, (\epsilon(\omega)-1)\, e^{-i\omega(t-t')}\Big] \bar{E}(\bar{r},t')$$

$$= \epsilon_0 \Big\{ \bar{E}(\bar{r},t) + \int_{-\infty}^{\infty} \chi(t-t')\, \bar{E}(\bar{r},t')\, dt' \Big\}$$

$$= \epsilon_0 \Big\{ \bar{E}(\bar{r},t) + \int_{-\infty}^{\infty} \chi(t')\, \bar{E}(\bar{r},t-t')\, dt' \Big\}$$

$$\boxed{\chi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (\epsilon(\omega) - 1)\, e^{-i\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \chi_e(\omega)\, e^{-i\omega t}\, d\omega}$$

Causality implies that

$\chi(t) = 0 \quad t < 0$

$\Leftrightarrow$

$\epsilon(\omega)$ analytic in upper half plane of ω

$$\boxed{\epsilon(\omega) = 1 + \int_0^\infty \chi(t)\, e^{i\omega t}\, dt}$$

$\chi(t)$ realvalued function $\Rightarrow$ $\epsilon(\omega) = \epsilon^*(-\omega^*)$

Kramers - Kronigs relation

$$\begin{cases} \mathrm{Re}\,\epsilon(\omega) = 1 + \frac{2}{\pi} P \int_0^\infty \frac{\omega' \mathrm{Im}\,\epsilon(\omega')}{\omega'^2 - \omega^2}\, d\omega' \\ \mathrm{Im}\,\epsilon(\omega) = -\frac{2\omega}{\pi} P \int_0^\infty \frac{\mathrm{Re}\,\epsilon(\omega') - 1}{\omega'^2 - \omega^2}\, d\omega' \end{cases}$$

<u>Note</u> If $\chi(t) \sim \delta(t)$ $\Rightarrow$ ϵ independent of ω and no dispersion

Also conductivity effects can be included.

1.3 Explicit examples

a Resonance model

Each electron in the medium feels two forces in addition to the electric force $\overline{F}_e = (-e)\,\overline{E}$

a) a restoring force prop. to the displacement $\overline{r}$

$$\overline{F}_h = -k\,\overline{r} = -m\omega_0^2\,\overline{r}$$

b) a friction force prop. to the velocity $\dot{\overline{r}}$

$$F_f = -m\nu\dot{\overline{r}} = -m\nu\frac{d}{dt}\overline{r}$$

Newton II:

$$m(\ddot{\overline{r}} + \nu\dot{\overline{r}} + \omega_0^2\,\overline{r}) = (-e)\,\overline{E}(\overline{r},t)$$

Polarization vector $\overline{P} = (-e)\,\overline{r}\,N$

where N = electrons / unit volume

We get

$$\ddot{\overline{P}} + \nu\dot{\overline{P}} + \omega_0^2\,\overline{P} = \omega_p^2\,\epsilon_0\,\overline{E}$$

where $\omega_p^2 = \dfrac{Ne^2}{\epsilon_0 m}$ = <u>plasma frequency</u>

Insert the constitutive relation

$$\bar{P} = \bar{D} - \epsilon_0 \bar{E} = \epsilon_0 \int_{-\infty}^{t} \chi(t-t')\, \bar{E}(t')\, dt'$$

$$\begin{cases} \dot{\bar{P}} = \epsilon_0\, \chi(0)\, \bar{E}(t) + \epsilon_0 \int_{-\infty}^{t} \chi'(t-t')\, \bar{E}(t')\, dt' \\ \ddot{\bar{P}} = \epsilon_0\, \chi(0)\, \dot{\bar{E}}(t) + \epsilon_0\, \chi'(0)\, \bar{E}(t) + \epsilon_0 \int_{-\infty}^{t} \chi''(t-t')\, \bar{E}(t')\, dt' \end{cases}$$

We get

$$\chi(0)\, \dot{\bar{E}}(t) + \left(\chi'(0) + \nu\, \chi(0) - \omega_p^2\right) \bar{E}(t)$$
$$+ \int_0^t f(t-t')\, \bar{E}(t')\, dt' = 0$$

where $f(t) = \chi''(t) + \nu\, \chi'(t) + \omega_0^2\, \chi(t)$

The field $\bar{E}$ is arbitrary, so

$$\begin{cases} \chi(0) = 0 \\ \chi'(0) + \nu\, \chi(0) - \omega_p^2 = 0 \\ f(t) = \chi''(t) + \nu\, \chi'(t) + \omega_0^2\, \chi(t) = 0 \end{cases}$$

The unique solution is

$$\boxed{\begin{aligned} \chi(t) &= \omega_p^2 \, e^{-\nu t/2} \, \frac{\sin \nu_0 t}{\nu_0} \quad , \quad t > 0 \\ \nu_0^2 &= \omega_0^2 - \nu^2/4 \end{aligned}}$$

without friction $\nu = 0$

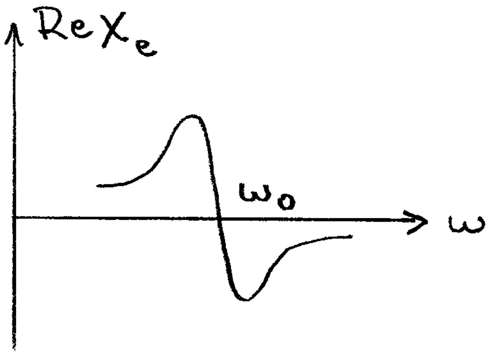
$$\chi(t) = \frac{\omega_p^2}{\omega_0} \sin \omega_0 t$$

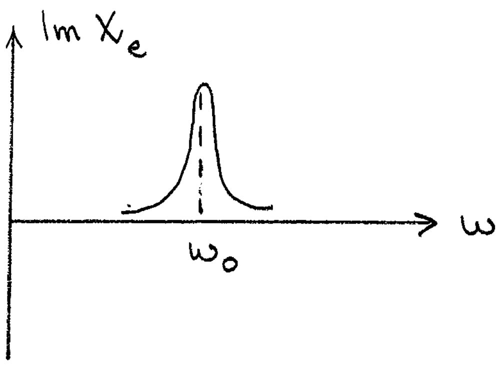
without restoring force $\omega_0^2 = 0$

$$\chi(t) = \frac{\omega_p^2}{\nu} (1 - e^{-\nu t})$$

The connection to fixed frequency p. 165-166 gives

$$\boxed{\chi_e(\omega) = \int_0^\infty \chi(t) e^{i\omega t} dt = \omega_p^2 (\omega_0^2 - i\omega\nu - \omega^2)^{-1}}$$





Resonance at $\omega = \omega_0$

We can formally compare this dispersion model with the previous DC conductivity model.

To do this comparison, use frequency domain results

① Conductivity model

$$\nabla \times \overline{H} = \overline{J} - i\omega \overline{D} = -i\omega \epsilon_0 \Big(\underset{\substack{\uparrow \\ \text{"normal" realvalued permittivity}}}{\epsilon} + i \frac{\sigma}{\omega \epsilon_0} \Big) \overline{E}$$

$$\approx -i\omega\epsilon_0 \left(i \frac{\sigma}{\omega\epsilon_0} \right) \overline{E} \quad \text{as} \quad \omega \to 0$$

② Dispersion model

$$\nabla \times \overline{H} = -i\omega \overline{D} = -i\omega\epsilon_0 \left(\overline{E} + \chi_e(\omega) \overline{E} \right)$$

$$= -i\omega\epsilon_0 \left(1 + \omega_p^2 (\omega_0^2 - i\omega\nu - \omega^2)^{-1} \right) \overline{E}$$

$$\approx -i\omega\epsilon_0 \left(\frac{i\omega_p^2}{\omega\nu} \right) \overline{E} \quad \text{as} \quad \omega \to 0 \quad (\omega_0 = 0)$$

Compare!

$$\boxed{\sigma = \frac{\omega_p^2 \epsilon_0}{\nu}} \qquad \text{(Drude)}$$

Conductor	lowest resonant frequency	zero
Insulator	— " —	non-zero

b Debye model

The resonance model is relevant in the optical regime for solids and for gyrotropic media in the microwave regime.

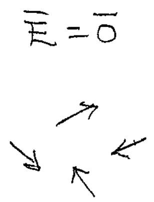
The Debye model is appropriate for polar liquids.

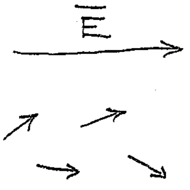
In a polar liquid, the molecules have permanent electric dipole moment.

Two competing processes for the polarization $\bar{P}$.

1. Alignment of $\bar{P}$ with the exterior electric field $\bar{E}$.

2. Disorder due to thermal motion.





$$\frac{d}{dt}\bar{P} = \underbrace{\epsilon_0 \alpha \bar{E}}_{\text{alignment}} - \underbrace{\frac{1}{\tau}\bar{P}}_{\text{thermal}}$$

α = alignment frequency > 0

τ = relaxation time > 0

With the constitutive relation

$$\bar{P} = \bar{D} - \epsilon_0 \bar{E} = \epsilon_0 \int_{-\infty}^{t} \chi(t-t')\,\bar{E}(t')\,dt'$$

$$\dot{\bar{P}}(t) = \epsilon_0\, \chi(0)\,\bar{E}(t) + \epsilon_0 \int_{-\infty}^{t} \chi'(t-t')\,\bar{E}(t')\,dt'$$

and insert in

$$\dot{\bar{P}} = \epsilon_0 \alpha \bar{E} - \frac{1}{\tau}\bar{P}$$

and balance terms

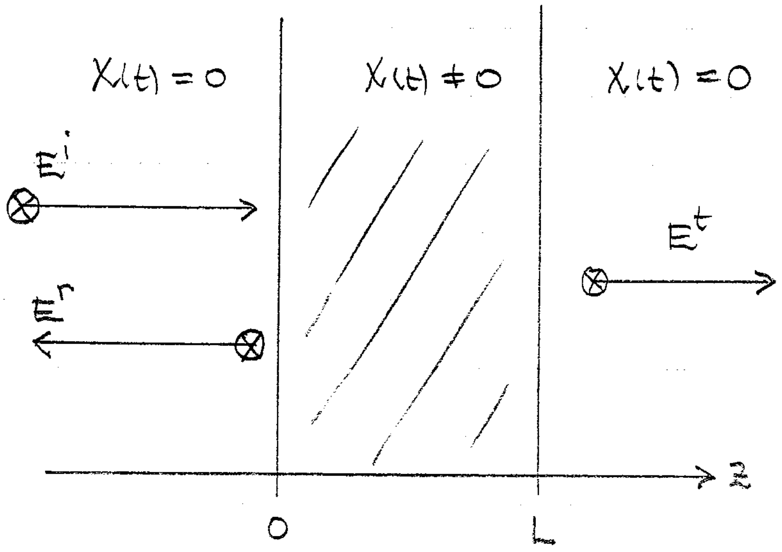
$$\begin{cases} \chi(0) = \alpha \\ \chi'(t) + \frac{1}{\tau}\,\chi(t) = 0 \end{cases}$$

with unique solution

$$\boxed{\chi(t) = \alpha\, e^{-t/\tau}}$$

2. Imbedding approach

2.1 Basic equations



The medium is assumed homogeneous (generalizations can be made for the direct scattering problem).

The Maxwell equations

$$\begin{cases} \nabla \times \bar{E} = -\partial_t \bar{B} \\ \nabla \times \bar{H} = \partial_t \bar{D} \end{cases}$$

Constitutive relations (non-magnetic dispersive medium)

$$\begin{cases} \bar{D}(\bar{r},t) = \epsilon_0 \left(\bar{E}(\bar{r},t) + \int_{-\infty}^{t} \chi(t-t') \bar{E}(\bar{r},t')\, dt' \right) \\ \qquad = \epsilon_0 (\bar{E} + \chi * \bar{E}) \\ \bar{B}(\bar{r},t) = \mu_0 \bar{H}(\bar{r},t) \end{cases}$$

Combine!

$$\nabla \times (\nabla \times \bar{E}) = -\partial_t(\nabla \times \bar{B}) = -\mu_0 \partial_t^2 \bar{D} = -\mu_0 \epsilon_0 \partial_t^2 (\bar{E} + \chi * \bar{E})$$

Denote $c = 1/\sqrt{\epsilon_0 \mu_0}$ = speed of light in vacuum.

$$\boxed{\nabla \times (\nabla \times \bar{E}) + \frac{1}{c^2}\left(\partial_t^2 \bar{E} + \chi * (\partial_t^2 \bar{E})\right) = \bar{0}}$$

since $\partial_t^2 \int_{-\infty}^{t} \chi(t-t') \bar{E}(t') dt' = \partial_t^2 \int_0^\infty \chi(t') \bar{E}(t-t') dt'$

$$= \int_0^\infty \chi(t') \partial_t^2 \bar{E}(t-t') dt' = \chi * (\partial_t^2 \bar{E})$$

Assume all fields transverse to $\hat{z}$ and dependent only on (z,t).

$$\nabla \times (\nabla \times \bar{E}) = -\nabla^2 \bar{E} + \nabla(\underbrace{\nabla \cdot \bar{E}}_{=0}) = -\nabla^2 \bar{E} = -\partial_z^2 \bar{E}$$

The wave equation for the transverse field

$$\boxed{E_{zz} - c^{-2}(E_{tt} + \chi * E_{tt}) = 0}$$

Rewrite the wave-equation in a system of first order equations

$$\partial_z \begin{pmatrix} E \\ E_z \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ c^{-2}(\partial_t^2 + \chi * \partial_t^2) & 0 \end{pmatrix} \begin{pmatrix} E \\ E_z \end{pmatrix}$$

$$= D \begin{pmatrix} E \\ E_z \end{pmatrix}$$

where $\chi * \partial_t^2 E = \int_{-\infty}^{t} \chi(t-t') \, E_{tt}(t') \, dt'$

If $\chi = 0$ $(z < 0 , z > L)$ the general solution is

$$E(z,t) = f(t - z/c) + g(t + z/c)$$

2.2 Wave splitting transformation

The same wave splitting as before

$$E^{\pm}(z,t) = \frac{1}{2}\left[E(z,t) \mp c \int_{-\infty}^{t} E_z(z,t')\,dt' \right]$$

In matrix notation

$$\boxed{\begin{pmatrix} E^+ \\ E^- \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -c\partial_t^{-1} \\ 1 & c\partial_t^{-1} \end{pmatrix}\begin{pmatrix} E \\ E_z \end{pmatrix} = P\begin{pmatrix} E \\ E_z \end{pmatrix}}$$

Note that c is here a constant $= 1/\sqrt{\mu_0\epsilon_0}$ everywhere

P has a formal inverse

$$P^{-1} = \begin{pmatrix} 1 & 1 \\ -c^{-1}\partial_t & c^{-1}\partial_t \end{pmatrix}$$

The wave splitting transformation has, in a region where $\chi = 0$, the effect of projecting the right and left-going parts of the wave, respectively. That is

$$E(z,t) = f(t - z/c) + g(t + z/c) \qquad \left(\begin{array}{l}\text{general}\\ \text{solution}\\ \text{when } \chi = 0\end{array}\right)$$

$$\begin{cases} E^+(z,t) = f(t - z/c) \\ E^-(z,t) = g(t + z/c) \end{cases}$$

2.3 Dynamics of $E^{\pm}$

"PDE" $$\partial_z \begin{pmatrix} E \\ E_z \end{pmatrix} = D \begin{pmatrix} E \\ E_z \end{pmatrix}$$

Wave splitting $$\begin{pmatrix} E^+ \\ E^- \end{pmatrix} = P \begin{pmatrix} E \\ E_z \end{pmatrix}$$

$$\partial_z \begin{pmatrix} E^+ \\ E^- \end{pmatrix} = \partial_z \left[P \begin{pmatrix} E \\ E_z \end{pmatrix} \right] = \Big(\underbrace{P_z}_{=0} P^{-1} + P D P^{-1} \Big) \begin{pmatrix} E^+ \\ E^- \end{pmatrix}$$

$$\boxed{\partial_z \begin{pmatrix} E^+ \\ E^- \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} E^+ \\ E^- \end{pmatrix}}$$

$$\begin{cases} \alpha = -\delta = -\left(\partial_t + \frac{1}{2} \chi * \partial_t \right) / c \\ \beta = -\gamma = -\frac{1}{2} \chi * \partial_t / c \end{cases}$$

2.4 Relation between E^+ and E^-

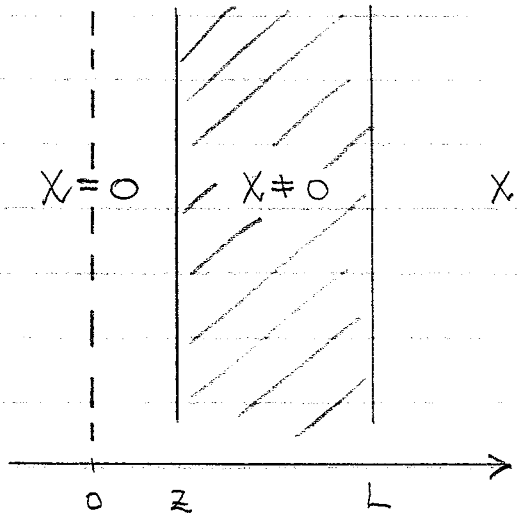
Duhamel's principle can be used to prove the general relation

$$\boxed{E^-(z,t) = \int_{-\infty}^{t} R(z,t-t')\, E^+(z,t')\, dt'}$$

As before

$R(0,t)$ is the physical reflection kernel
i.e. the reflection kernel for the slab $[0,L]$

$R(z,t)$ is the reflection kernel for the slab $[z,L]$



This can be interpreted as an invariant imbedding of the full scattering problem in a family of scattering problems as the left hand side z of the slab varies.

2.5 The imbedding equation

Differentiate $E^- = R * E^+$ wrt z and use

$$\partial_z \begin{pmatrix} E^+ \\ E^- \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} E^+ \\ E^- \end{pmatrix}$$

and assume a finite jump discontinuity in $R(z,t)$ at $t = d(z)$. Denote the jump with $[R]$

$$\partial_z E^- = R_z * E^+ - d'[R]\, E^+(t-d) + R * E^+_z =$$

$$= R_z * E^+ - d'[R]\, E^+(t-d) - \tfrac{1}{c} R * \left(\partial_t E^+ + \tfrac{1}{2} \chi * \partial_t E^+\right)$$

$$- \tfrac{1}{2c} R * \left(\chi * \partial_t E^-\right) =$$

$$= R_z * E^+ - d'[R]\, E^+(t-d) - \tfrac{1}{c}\left[R(0)\, E^+ + R_t * E^+ + [R]\, E^+(t-d)\right]$$

$$- \tfrac{1}{2c} R * \left[\chi(0)\, E^+ + \chi_t * E^+\right] - \tfrac{1}{2c} R * \left[\chi(0)\, E^- + \chi_t * E^-\right]$$

$$= R_z * E^+ - d'[R]\, E^+(t-d) - \tfrac{1}{c} R(0)\, E^+ - \tfrac{1}{c} R_t * E^+$$

$$- \tfrac{1}{c} [R]\, E^+(t-d) - \tfrac{1}{2c} \chi(0)\, R * E^+ - \tfrac{1}{2c} R * (\chi_t * E^+)$$

$$- \tfrac{1}{2c} \chi(0)\, R * (R * E^+) - \tfrac{1}{2c} R * (\chi_t * (R * E^+))$$

On the other hand

$$\partial_z E^- = \gamma E^+ + \delta E^- = \frac{1}{2c}\, \chi * E_t^+ + \frac{1}{c} E_t^- + \frac{1}{2c}\, \chi * E_t^-$$

$$= \frac{1}{2c}\, \chi(0) E^+ + \frac{1}{2c}\, \chi_t * E^+ + \frac{1}{c}\left(R_t * E^+ + [R]\, E^+(t-d) + R(0) E^+ \right)$$

$$+ \frac{1}{2c}\, \chi(0)\, R * E^+ + \frac{1}{2c}\, \chi_t * (R * E^+)$$

Balance terms!

$$\begin{cases} d'(z) = -\frac{2}{c} = \text{constant} \quad \Rightarrow \quad t = -\frac{2z}{c} + \text{constant} \\ R(0) = -\frac{1}{4}\chi(0) \\ cR_z = 2R_t + \chi(0) R + R * \chi_t + \frac{1}{2}\chi(0) R * R + \frac{1}{2} R * (\chi_t * R) + \frac{1}{2}\chi_t \end{cases}$$

Since $\partial_t (\chi * R) = \chi(0) R + \chi_t * R$

the imbedding equation becomes

$$\boxed{\begin{aligned} & cR_z = \partial_t \left\{ 2R + \tfrac{1}{2}\chi + \chi * R + \tfrac{1}{2}\, \chi * (R * R) \right\} \\ & R(z,0) = -\tfrac{1}{4}\chi(0) \\ & R(L,t) = 0 \end{aligned}}$$

Propagation of singularity gives that $R(z,t)$ is everywhere continous except along $t = 2(L-z)/c$ where

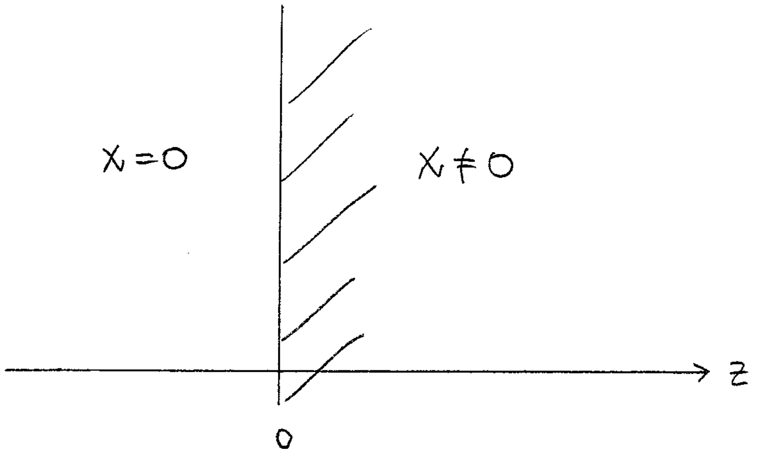
$$[R(z, 2(L-z)/c] = \frac{X(0)}{4} \exp\left(-X(0)(L-z)/c\right)$$

Thus if $X(0) = 0$ (like in resonance model) $R(z,t)$ is continuous everywhere.

This equation solves both the direct and inverse problems.

Problem	known (indata)	Sought
Direct	$X(t)$	$R(0,t)$
Inverse	$R(0,t)$	$X(t)$

2.6 Semi-infinite medium



The extent of the medium is $z > 0$ $(L \to \infty)$

This implies that the reflection kernel is independent of z.

$R(z,t) \to R(t)$ and $\partial_z R = 0$

The imbedding equation then implies

$$2R + \frac{1}{2}\chi + \chi * R + \frac{1}{2}\chi * (R * R) = 0$$

or

$$\boxed{4R(t) + \chi(t) + 2(\chi * R)(t) + (\chi * R * R)(t) = 0}$$

and the initial condition $R(0) = -\frac{1}{4}\chi(0)$ is automatically satisfied.

$$\boxed{4\,R(t) + X(t) + [X * (2R + R * R)](t) = 0 \quad , \quad t > 0}$$

Direct problem [$X(t)$ given] this is a nonlinear integral equation for $R(t)$

Inverse problem [$R(t)$ given] this is a <u>linear</u> integral equation (Volterra eq. of the second kind) for $X(t)$!

<u>Examples</u> (exact solutions)

1) $X(t) = \alpha\, e^{\beta t}$ $\qquad R(t) = -\, e^{(\beta - \frac{\alpha}{2})t}\, I_1(\alpha t/2)/t$

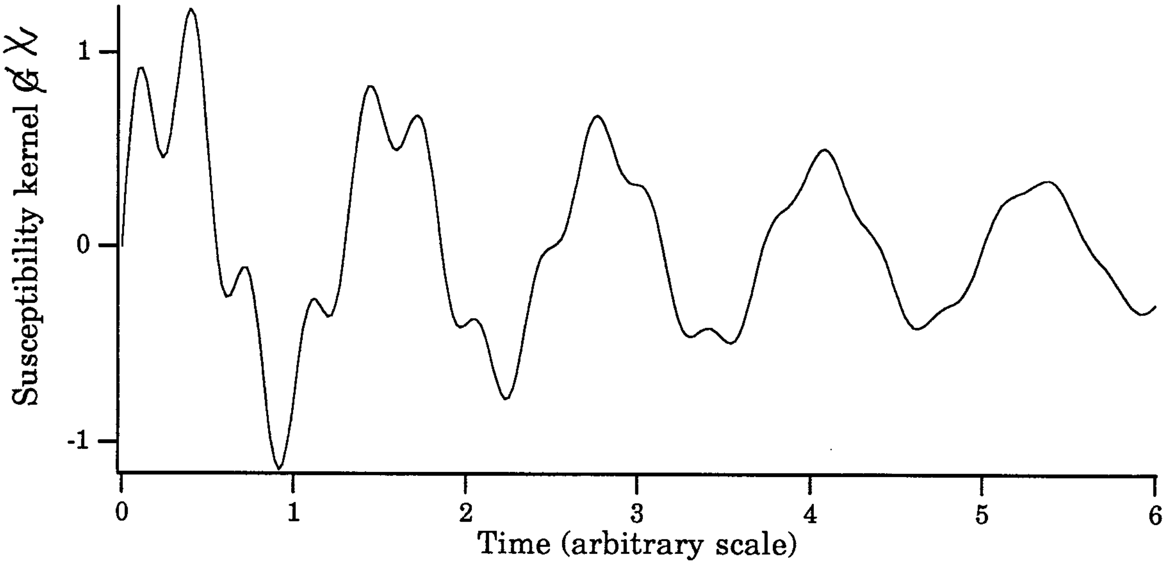
2) $X(t) = (2\alpha + \alpha t^2)\, e^{\beta t}$ $\qquad R(t) = -\frac{\alpha}{2}\, e^{(\beta - \frac{\alpha}{2})t}$

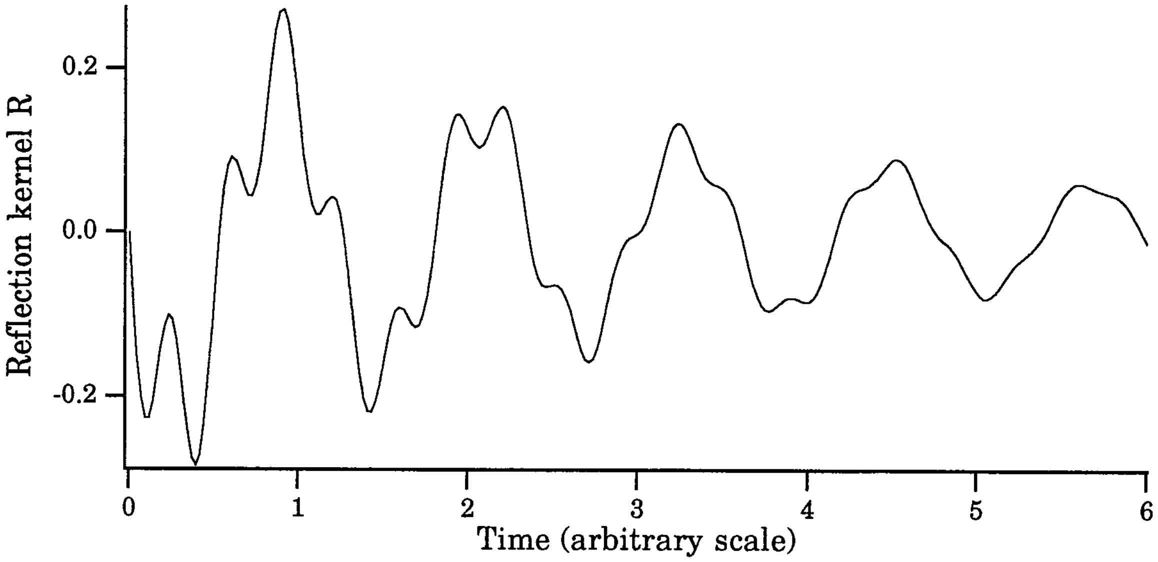
It is possible to show that the direct problem is well posed (so is the inverse due to Volterra) i.e. if $X(t)$ is bounded then

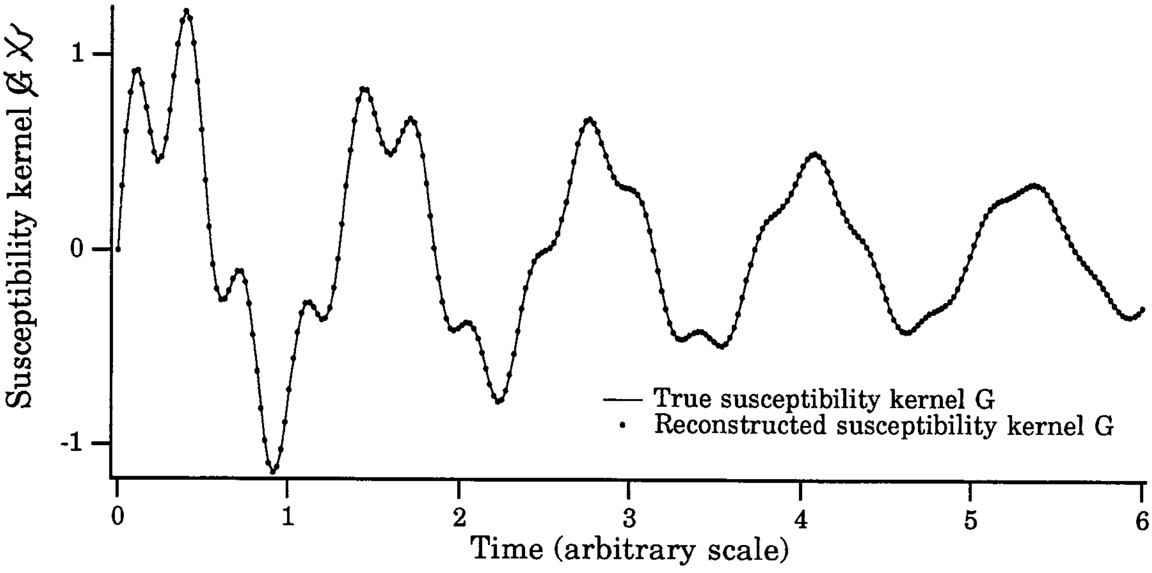
i) there exists a unique solution $R(t)$

ii) the solution depends continuously on the "data" $X(t)$.

Numerical example

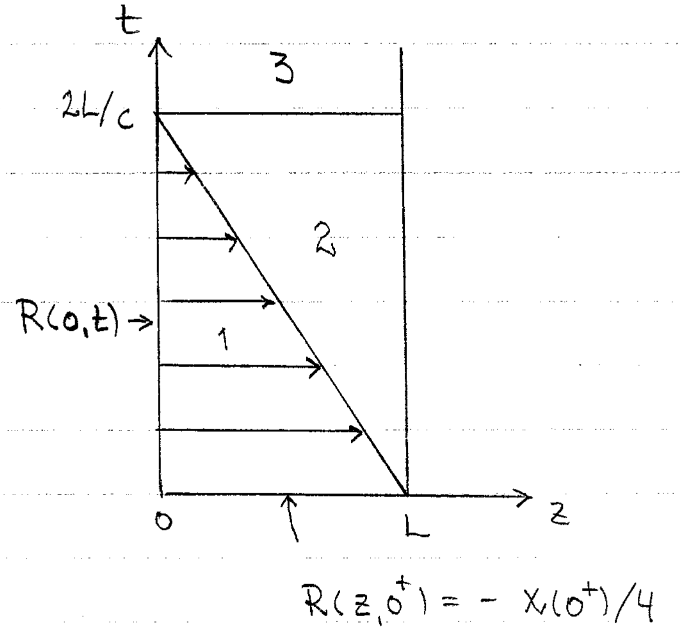






2.7 Finite slab

As for the infinite slab, $R(z,t)$ is independent of z in region 1



To solve the problem in region 2 & 3 write the R-equation in integral form.

$$2c\frac{d}{dz}R(z,t-2z/c) = \chi'(t-2z/c)$$

$$+(\chi(0)+\chi'*)(2R+R*R)(z,t-2z/c)$$

Integrate from z to $z+h$ and let $t \to t+2z/c$

$$2c[R(z+h,t-2h/c)-R(z,t)] = -\frac{c}{2}[\chi(t-2h/c)-\chi(t)]$$

$$+\int_z^{z+h}\{(\chi_0+\chi'*)(2R+R*R)(z',t+2(z-z')/c)\}\,dz'$$

2.8 Some details on the direct problem

Discretize: in z: $z_i = ih$; $i = 0, 1, \dots, N$ $(Nh = L)$
in t: $t_j = 2jh/c$; $j = 0, 1, \dots, J$

$$\begin{cases} R_{ij} \equiv R(z_i, t_j) \\ X_j \equiv X(t_j) \\ X'_j \equiv X'(t_j) \end{cases}$$

Trapezoidal rule on imbedding equation

$$\begin{aligned} 2c\,[R_{i+1,j-1} - R_{i,j}] &= \\ &= \tfrac{1}{2} h\,[X'_{j-1} + X'_j] + h X_0 [R_{i+1,j-1} + R_{i,j}] \\ &\quad + h^2 X_0 [A_{i+1,j-1} + A_{i,j}]/c \\ &\quad + 2h^2 [B_{i+1,j-1} + B_{i,j}]/c \\ &\quad + 2h^3 [C_{i+1,j-1} + C_{i,j}]/c^2 \qquad (*) \end{aligned}$$

$$\begin{cases} A_{i,j} \equiv \sum\limits_{k=1}^{j} R_{i,j-k} R_{i,k} \qquad \left((R * R)_{i,j} = \frac{2h}{c} \sum\limits_{k=1}^{j} R_{i,k} R_{i,j-k} \right) \\ B_{i,j} \equiv \frac{1}{2} (X'_j R_{i,0} + X'_0 R_{i,j}) + \sum\limits_{k=1}^{j} X'_{j-k} R_{i,k} \\ C_{i,j} \equiv \frac{1}{2} X'_0 A_{i,j} + \sum\limits_{k=1}^{j-1} X'_{j-k} A_{i,k} \end{cases}$$

In region 1 (No variation in z !)

$$\left.\begin{array}{l} 4R_{0,j} + X_j + 4hE_j/c + 4h^2F_j/c^2 = 0 \qquad (**) \\ \left\{\begin{array}{l} E_j = \frac{1}{2}(X_jR_{0,0} + X_0R_{0,j}) + \sum_{k=1}^{j-1} X_{j-k}R_{0,k} \quad . \\ F_j = \frac{1}{2}X_0A_{0,j} + \sum_{k=1}^{j-1} X_{j-k}A_{0,k} \end{array}\right. \end{array}\right\} \begin{array}{l} j= \\ 0,1,\dots,N \end{array}$$

Algorithm: (X_j given, find $R_{0,j}$, $j=0,1,2,\dots$)

1) Use $(**)$ and solve for $R_{0,j}$, $j=0,1,2,\dots,N$.

2) $R_{i,j} = R_{0,j}$; $j=0,1,\dots,N$, $i=0,1,\dots,N-j$

3) $R_{i,N-i}$ is used as an approximation of $R(z,2(L-z)/c^{-})$

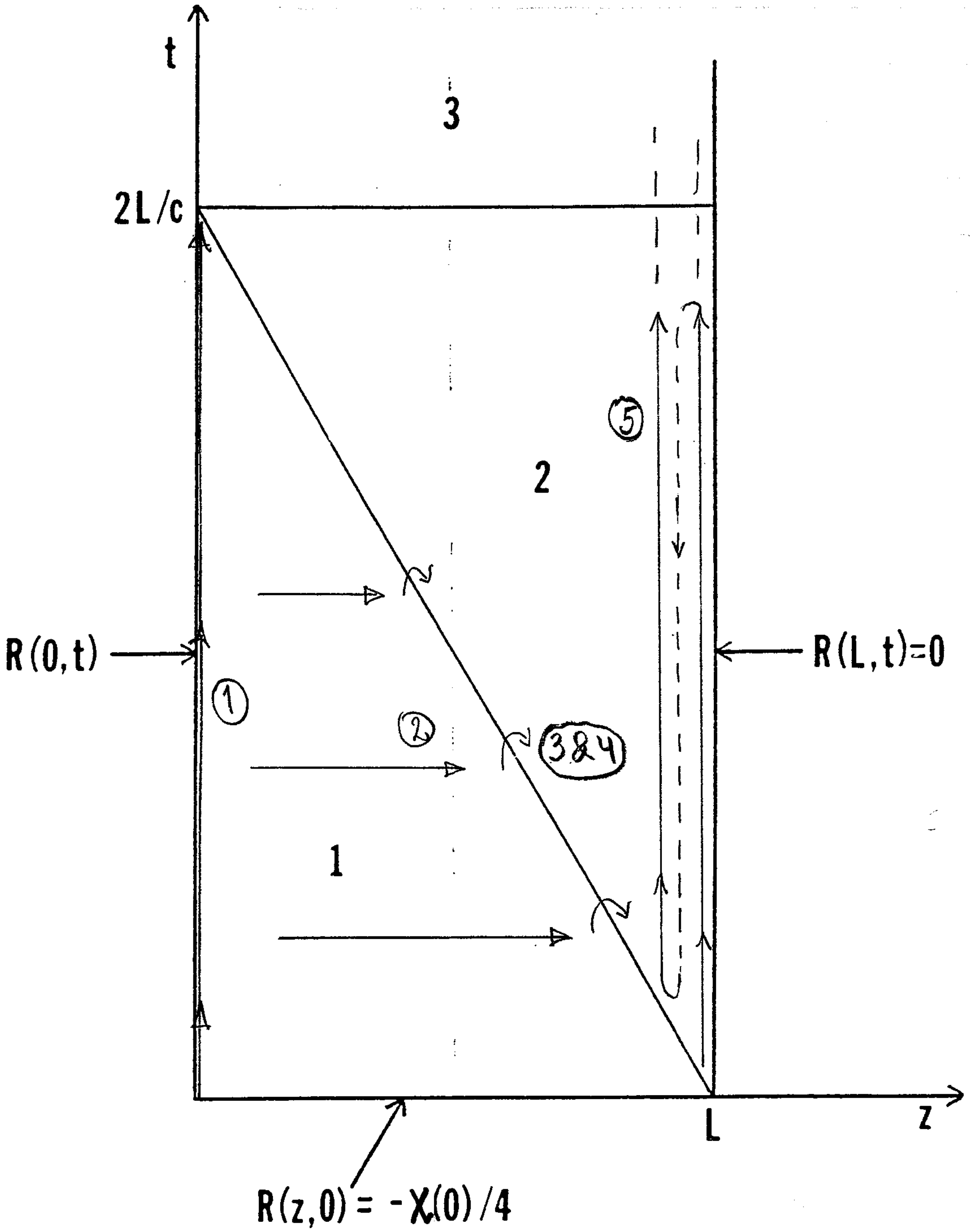
4) $[R] = \frac{X_0}{4}\exp(-X_0(L-z)/c)$ to obtain

$R_{i,N-i}$ in region 2.

5) Use $(*)$ from right to left with boundary condition

$R_{N,j} = 0$, $j=0,1,2,3,\dots,J$

Route in the direct problem



2.9 The inverse algorithm

($R_{0,j}$ given, find X_j ; $j=0,1,2,\dots$)

1) Use (**) to solve for X_j , $j=0,1,2,\dots,N$

2) $R_{i,j}=R_{0,j}$; $j=0,1,\dots,N$; $i=0,1,\dots,N-j$

3) Compute X_j' via a difference scheme ; $j=0,1,\dots,N$

4) $[R]=\frac{X_0}{4}\exp(-X_0(L-z)/c)$ to obtain

$R_{i,N-i}$ in region 2.

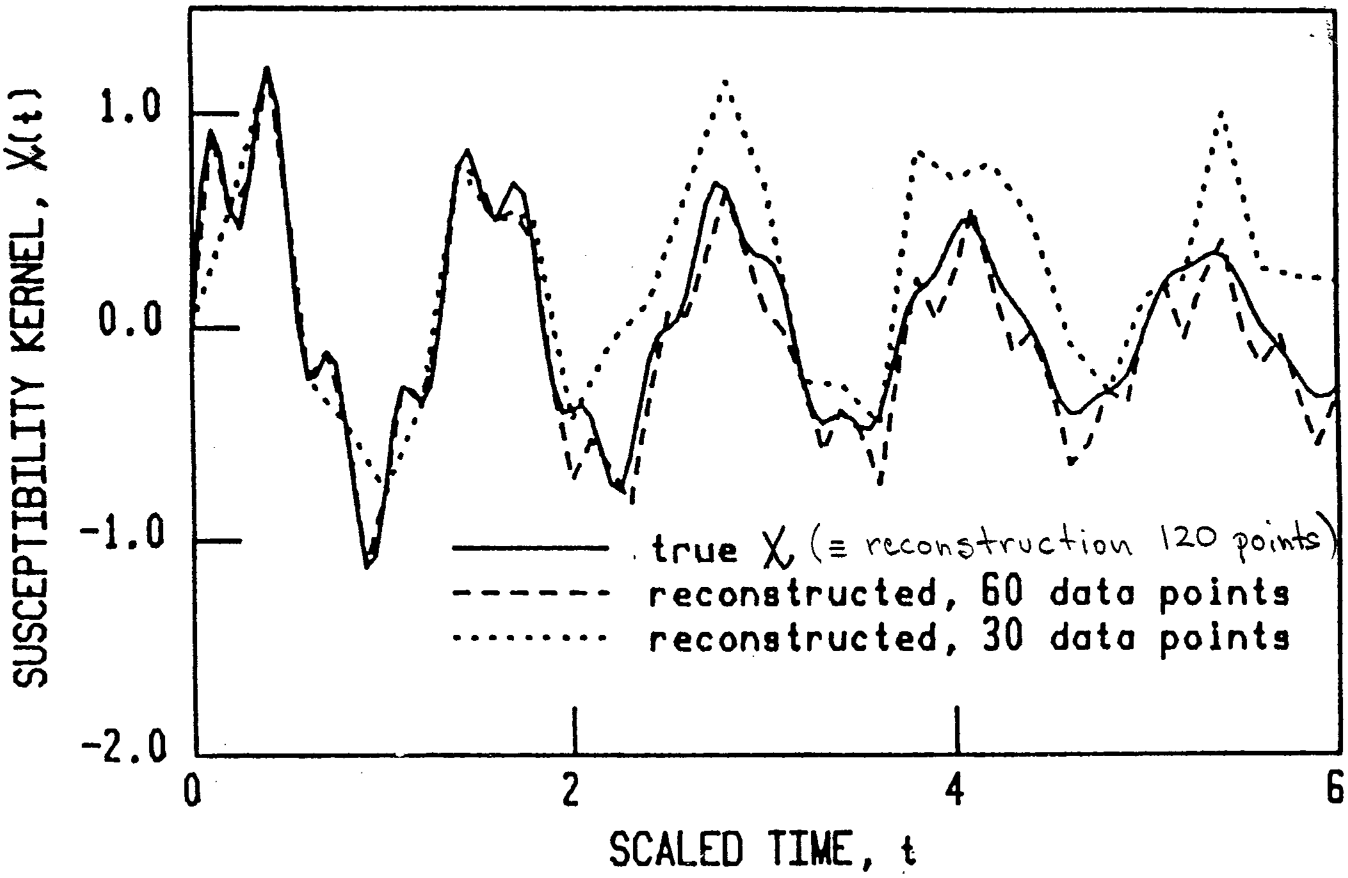
5) Use (*) from left to right to determine

X_j' , $j=N+1,\dots$

once $R_{i,j}$ is determined in region 2 (use (*))

2.10 Numerical examples

1/ $\chi(t) = e^{-0.2t} \sin 1.6\pi t + 0.5\, e^{-0.5t} \sin 6\pi t \quad , \quad 0 < t < 6$

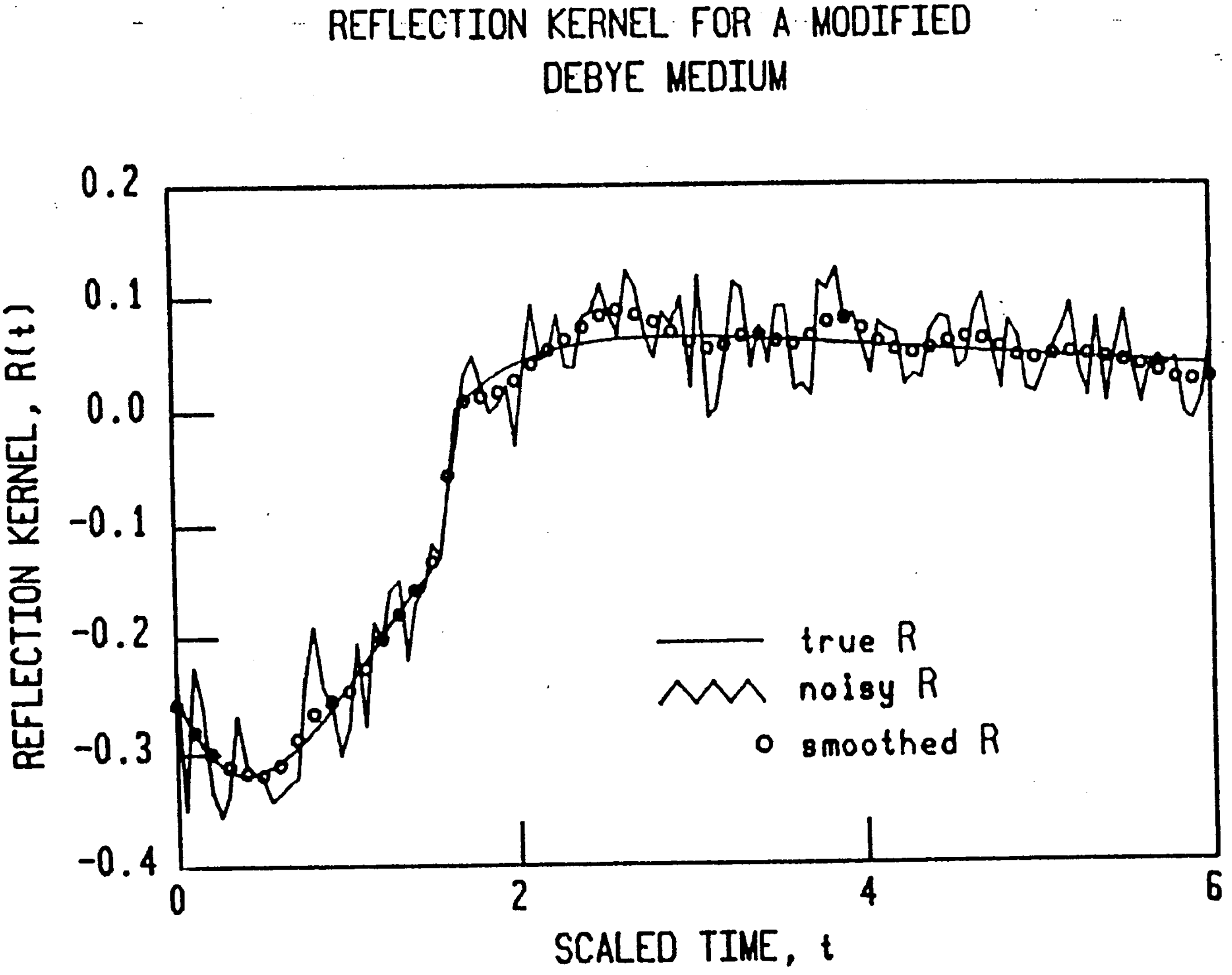


RECONSTRUCTION OF THE SUSCEPTIBILITY KERNEL
FOR A TWO RESONANCE MODEL

$L = 0.8$

$t = 6$ corresponds to $\frac{6}{1.6} = 3.75$ round trips

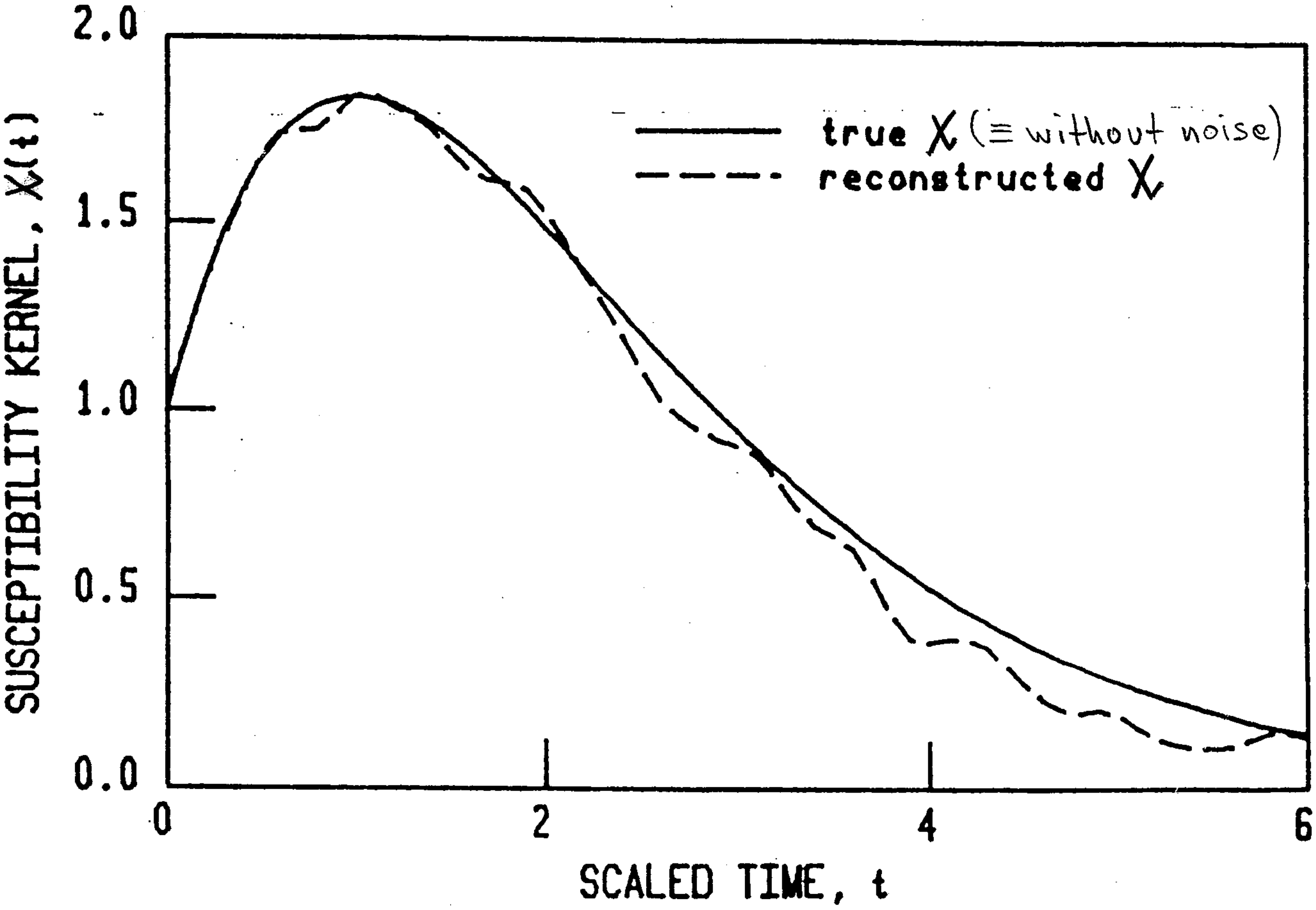
2) $\chi(t) = (1+3t+t^2)e^{-t}$, $0 < t < 6$



L=0.8

s/n = 7.8

RECONSTRUCTION OF THE SUSCEPTIBILITY KERNEL USING NOISY DATA



3. Green function approach

3.1 Repetition of basic equations

The medium is modeled by a susceptibility kernel $\chi(t)$ (cf. p. 160)

$$\boxed{\bar{D}(\bar{r},t) = \epsilon_0 \left\{ \bar{E}(\bar{r},t) + \int_{-\infty}^{t} \chi(t-t')\,\bar{E}(t')\,dt' \right\}}$$

The "wave equation" for a transversely polarized electric field is (cf. p. 174)

$$\boxed{E_{zz} - c^{-2}\left(E_{tt} + \chi * E_{tt} \right) = 0}$$

Wave splitting as before (cf. p. 176)

$$\boxed{E^{\pm}(z,t) = \frac{1}{2}\left[E(z,t) \mp c\int_{-\infty}^{t} E_z(z,t')\,dt' \right]}$$

$c = 1/\sqrt{\mu_0\epsilon_0}$ = speed of light in vacuum

The "dynamics" of $E^{\pm}(z,t)$ are (cf. p. 178)

$$\boxed{\partial_z \begin{pmatrix} E^+ \\ E^- \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} E^+ \\ E^- \end{pmatrix}}$$

$$\begin{cases} \alpha = -\delta = -\left(\partial_t + \frac{1}{2} \chi * \partial_t\right)/c \\ \beta = -\gamma = -\frac{1}{2} \chi * \partial_t / c \end{cases}$$

Up to this point everything is identical to chapter D.2..

The difference lies in the way $E^{\pm}$ are connected. The one relevant in this section is (cf. p. 100)

$$\boxed{\begin{aligned} E^+(z,t+z/c) &= \tau(z) \left\{ E^+(0,t) + \int_{-\infty}^{t} G^+(z,t-t') E^+(0,t')\, dt' \right\} \\ E^-(z,t+z/c) &= \tau(z) \int_{-\infty}^{t} G^-(z,t-t') E^+(0,t')\, dt' \end{aligned}}$$

where $\tau(z) = \exp\{-\chi(0) z/2c\}$

(This is seen from the result on p 100 with the transformation $x = z/L$, $s = tc/L$, $\ell = L/c$, $B = -\chi(0) \frac{L}{c}$)

Again

$$\boxed{\begin{aligned} G^-(0,t) &= R(t) = \text{Physical reflection kernel} \\ G^+(L,t) &= T(t) = \text{Physical transmission kernel} \end{aligned}}$$

$$\boxed{\begin{aligned} G^+(0,t) &= 0 \\ G^-(L,t) &= 0 \end{aligned}}$$

3.2 Differential equations for $G^\pm$

Assume $G^\pm(z,t)$ are continuous everywhere except along $t = d_\pm(z)$.

Proceed as before

$$\frac{d}{dz} E^+(z, t + z/c) = -\frac{\chi(0)}{2c} \cancelto{②}{E^+(z, t+z/c)} +$$

$$+ \tau(z) \Big\{ \int_{-\infty}^{t} G_z^+(z, t-t') E^+(0,t') dt' - d_+'(z) \left[G^+(z, d_+(z)\right] E^+(0, t - d_+(z)) \Big]$$

$$\frac{d}{dz} E^+(z, t + z/c) = E_z^+(z, t+z/c) + \frac{1}{c} E_t^+(z, t+z/c)$$

$$= -\frac{1}{c} \cancelto{①}{E_t^+(z, t+z/c)} - \frac{1}{2c} \int_{-\infty}^{t} \chi(t-t') E_t^+(z, t'+z/c) dt'$$

$$- \frac{1}{2c} \int_{-\infty}^{t} \chi(t-t') E_t^-(z, t'+z/c) dt' + \frac{1}{c} \cancelto{①}{E_t^+(z, t+z/c)}$$

$$= -\frac{1}{2c} \chi(0) E^+(z, t+z/c) - \frac{1}{2c} \int_{-\infty}^{t} \chi'(t-t') E^+(z, t'+z/c) dt'$$

$$- \frac{1}{2c} \chi(0) E^-(z, t+z/c) - \frac{1}{2c} \int_{-\infty}^{t} \chi'(t-t') E^-(z, t'+z/c) dt'$$

$$= -\frac{1}{2c} \chi(0) \cancelto{②}{E^+(z, t+z/c)} - \frac{1}{2c} \tau(z) \Big\{ \int_{-\infty}^{t} \chi'(t-t') \Big[E^+(0,t')$$

$$+ \int_{-\infty}^{t'} G^+(z, t'-t'') E^+(0,t'') dt'' \Big] dt' + \chi(0) \int_{-\infty}^{t} G^-(z, t-t') E^+(0,t') dt'$$

$$+ \int_{-\infty}^{t} \chi'(t-t') \int_{-\infty}^{t'} G^-(z, t'-t'') E^+(0,t'') dt'' dt' \Big\}$$

Balance terms!

$$\begin{cases} d_+'(z) = 0 \quad \Rightarrow \quad t = \text{constant} \\ G_z^+ = -\frac{1}{2c}\chi' - \frac{1}{2c}\chi' * G^+ - \frac{1}{2c}\chi(0)\, G^- - \frac{1}{2c}\chi' * G^- \\ \quad\; = -\frac{1}{2c}\left\{ \chi' + \chi' * G^+ + \partial_t(\chi * G^-) \right\} \end{cases}$$

Similarly,

$$\frac{d}{dz} E^-(z, t+z/c) = -\frac{\chi(0)}{2c}\,\tau(z) \int_{-\infty}^{t} G^-(z, t-t')\, E^+(0, t')\, dt'$$

$$+ \tau(z) \left\{ \int_{-\infty}^{t} G_z^-(z, t-t')\, E^+(0, t')\, dt' - d_-'(z) \left[G^-(z, d_-(z)) \right] E^+(0, t-d_-(z)) \Big] \right\}$$

$$\frac{d}{dz} E^-(z, t+z/c) = E_z^-(z, t+z/c) + \frac{1}{c} E_t^-(z, t+z/c)$$

$$= \quad \frac{1}{2c} \int_{-\infty}^{t} \chi(t-t')\, E_t^+(z, t'+z/c)\, dt' + \frac{2}{c} E_t^-(z, t+z/c)$$

$$+ \frac{1}{2c} \int_{-\infty}^{t} \chi(t-t')\, E_t^-(z, t'+z/c)\, dt' = \quad \frac{1}{2c} \left\{ \chi(0)\, E^+(z, t+z/c) \right.$$

$$+ \int_{-\infty}^{t} \chi'(t-t')\, E^+(z, t'+z/c)\, dt' + \chi(0)\, E^-(z, t+z/c)$$

$$\left. + \int_{-\infty}^{t} \chi'(t-t')\, E^-(z, t'+z/c)\, dt' \right\} + \frac{2}{c} E_t^-(z, t+z/c) =$$

$$
\begin{aligned}
= \quad & \tfrac{1}{2c}\,\tau(z)\Big\{ \chi(0)\Big[E^{+}(0,t) + \int_{-\infty}^{t} G^{+}(z,t-t')\,E^{+}(0,t')\,dt'\Big] \\
& + \int_{-\infty}^{t} \chi'(t-t')\Big[E^{+}(0,t') + \int_{-\infty}^{t'} G^{+}(z,t'-t'')\,E^{+}(0,t'')\,dt''\Big]\,dt' \\
& + \chi(0)\int_{-\infty}^{t} G^{-}(z,t-t')\,E^{+}(0,t')\,dt' \\
& + \int_{-\infty}^{t} \chi'(t-t')\int_{-\infty}^{t'} G^{-}(z,t'-t'')\,E^{+}(0,t'')\,dt''\,dt' + 4\cdot G^{-}(z,0)\,E^{+}(0,t) \\
& + 4\cdot\int_{-\infty}^{t} G^{-}_{t}(z,t-t')\,E^{+}(0,t')\,dt' + 4\,\big[G^{-}(z,d_{-}(z))\big]\,E^{+}(0,t-d_{-}(z))\Big\}
\end{aligned}
$$

Balance terms !

$$
\left\{
\begin{aligned}
& d'_{-}(z) = -{}^{2}/_{c} \quad \Longrightarrow \quad t = -{}^{2z}/_{c} + \text{constant} \\
& G^{-}(z,0) = -\tfrac{1}{4}\,\chi(0) \\
& 2cG^{-}_{z} - 4G^{-}_{t} = \chi(0)\,G^{-} + \chi(0)\,G^{+} + \chi' + \chi' * G^{+} \\
& \qquad + \chi(0)\,G^{-} + \chi' * G^{-} = \chi(0)\,G^{-} + \\
& \qquad + \partial_{t}\{\chi + \chi * G^{-} + \chi + G^{+}\}
\end{aligned}
\right.
$$

Propagation of singularity arguments show

$G^{\pm}(z,t)$ is continuous everywhere except for

$G^-(z,t)$ along $t = 2(L-z)/c$ where

$$[G^-(z, 2(L-z)/c] = \frac{X(0)}{4} \exp(-X(0)(L-z)/c)$$

Integration gives $G^+(z,0)$

$$G^+_z(z,0) = -\frac{1}{2c}\left(X'(0) - \frac{1}{4}X^2(0)\right)$$

In summary

$$2cG^+_z(z,t) = -\partial_t\left\{X(t) + \int_0^t X(t-t')G^-(z,t')dt'\right\} - \int_0^t X'(t-t')G^+(z,t')dt'$$

$$2cG^-_z(z,t) - 4G^-_t(z,t) = X(0)G^-(z,t)$$
$$+\partial_t\left\{X(t) + \int_0^t X(t-t')G^-(z,t')dt' + \int_0^t X(t-t')G^+(z,t')dt'\right\}$$

$$G^+(z,0) = -\frac{z}{2c}\left(X'(0) - \frac{1}{4}X^2(0)\right)$$

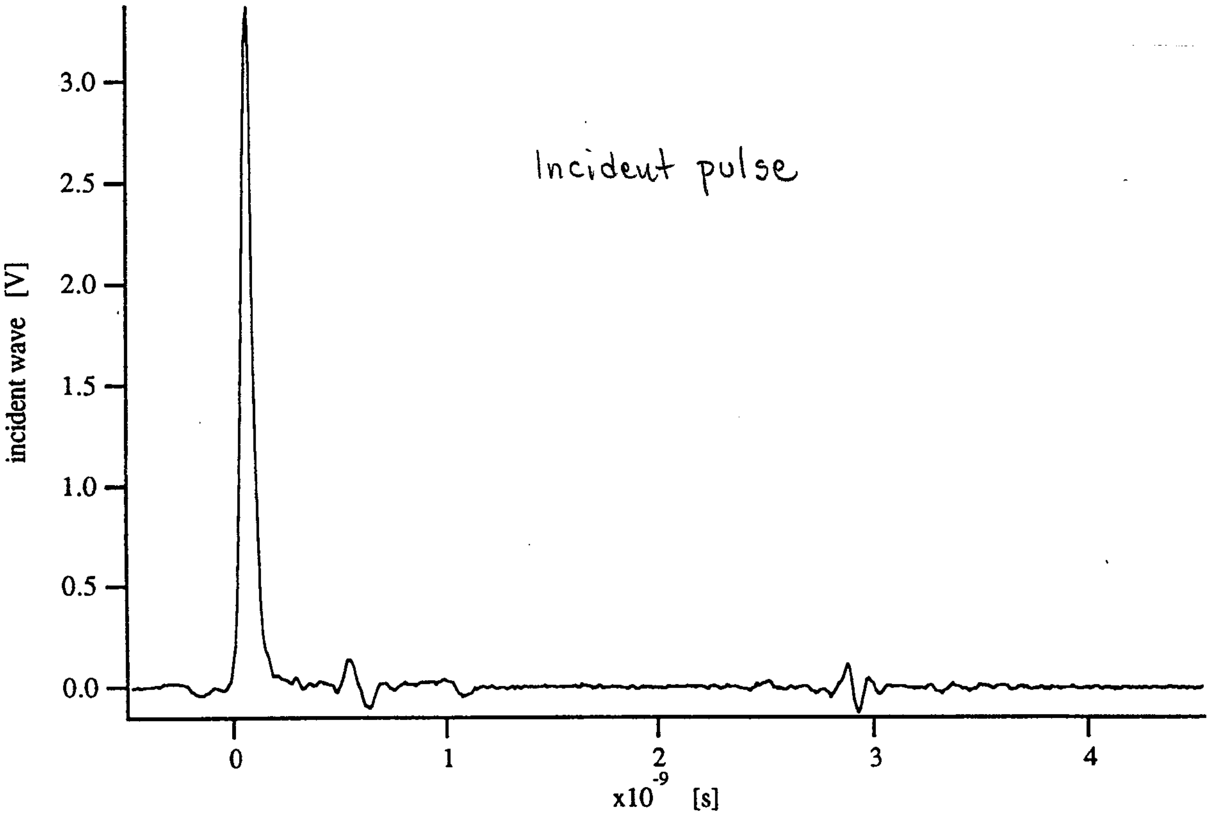
$$G^-(z,0) = -\frac{1}{4}X(0)$$

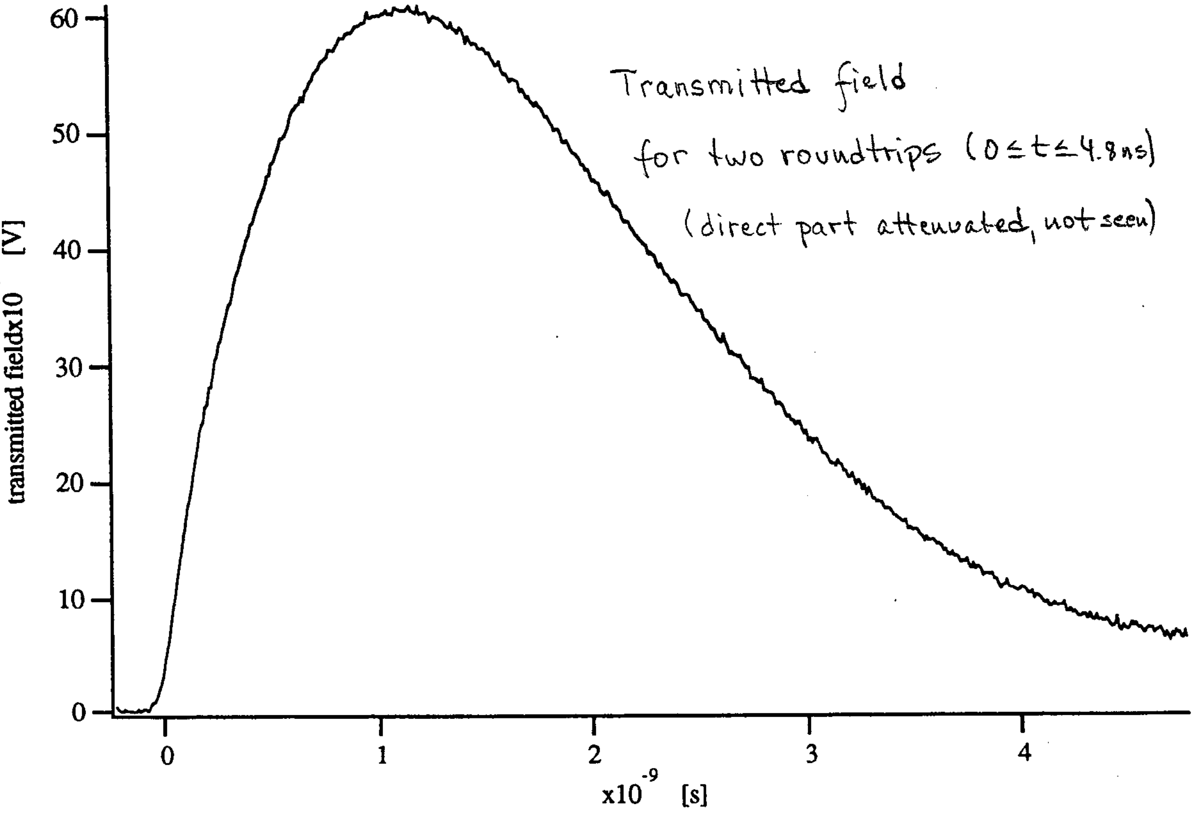
$$G^-(L,t) = G^+(0,t) = 0$$

$$[G^-(z, 2(L-z)/c)] = \frac{X(0)}{4}\exp(-X(0)(L-z)/c)$$

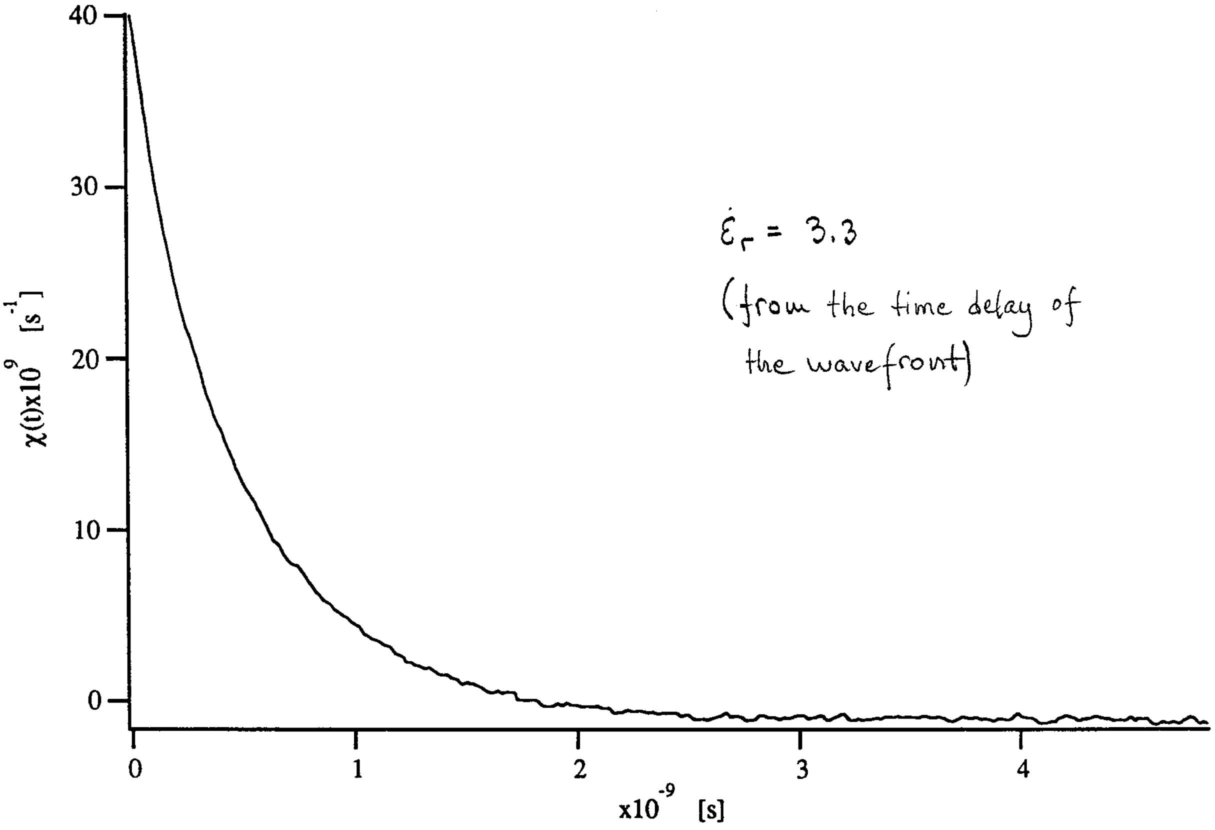
4. An experimental example

Transmission experiment with 1-buthanol.





Reconstruction



$\varepsilon_r = 3.3$

(from the time delay of the wavefront)

This reconstruction of $\chi(t)$ fits very well with a Debye model

$$\chi(t) = \alpha e^{-t/\tau}$$

$$\begin{cases} \alpha = 4 \cdot 10^{10} \, s^{-1} \\ \tau = 0.5 \, ns \end{cases}$$

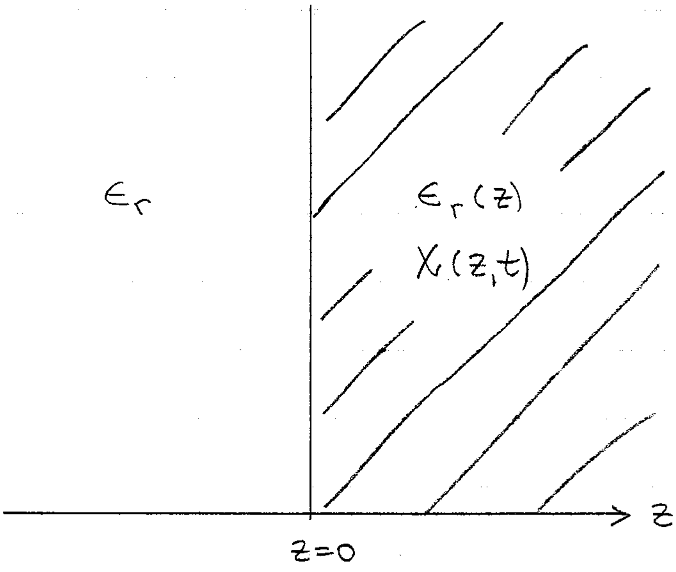
5 Non-reflecting media

Is it possible to find a medium which gives no reflected field irrespectively how it is excited?

What media have reflection kernel $R(t) \equiv 0$?

Assume the medium has the following const. rel.

$$\begin{cases} \bar{D}(z,t) = \epsilon_0 \left[\epsilon_r(z) \bar{E}(z,t) + \int_{-\infty}^{t} \chi(z,t-t') \bar{E}(z,t') dt' \right] \\ \bar{B}(z,t) = \mu_0 \bar{H}(z,t) \end{cases}$$



Only half space problems are considered here.

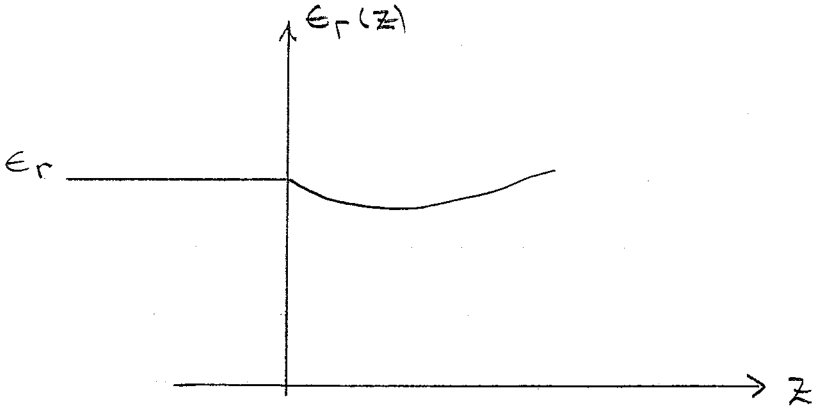
First requirement: The surface $z=0$ must be impedance matched, i.e.

$$\epsilon_r(0) = \epsilon_r$$

Then there is no hard echo from the interface.

Second requirement: $\epsilon_r(z)$ must be a decreasing function at $z=0$.

$$\epsilon_r'(0) \leq 0$$



Consequence: No non-reflecting media are possible if vacuum outside. (Magnetic properties have to be included).

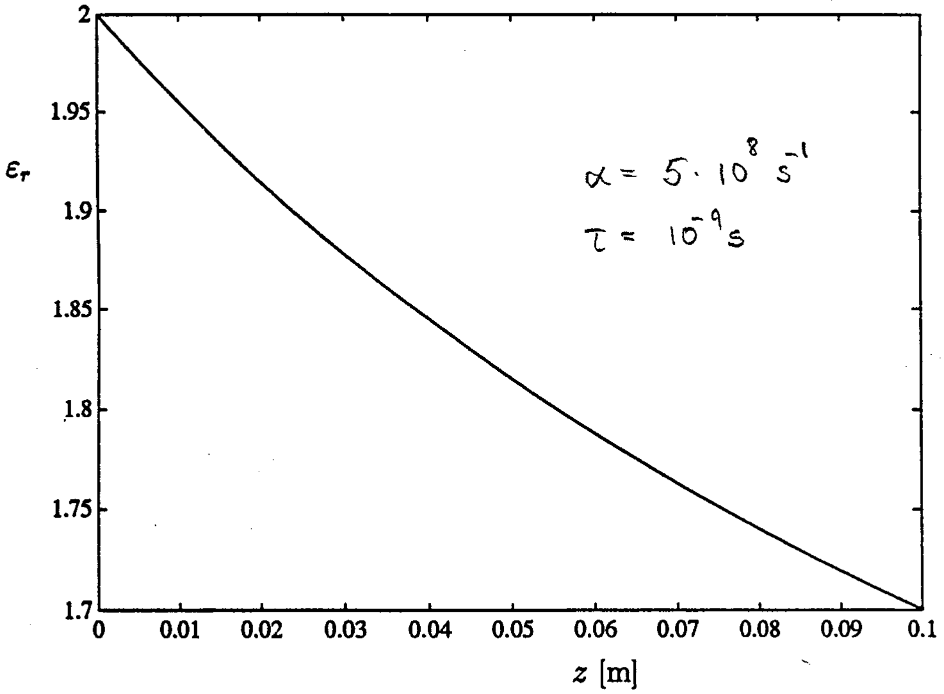
Numerical experiments

1/ Assume the medium is dispersive with

$$\chi_r(t) = \alpha e^{-t/\tau}$$

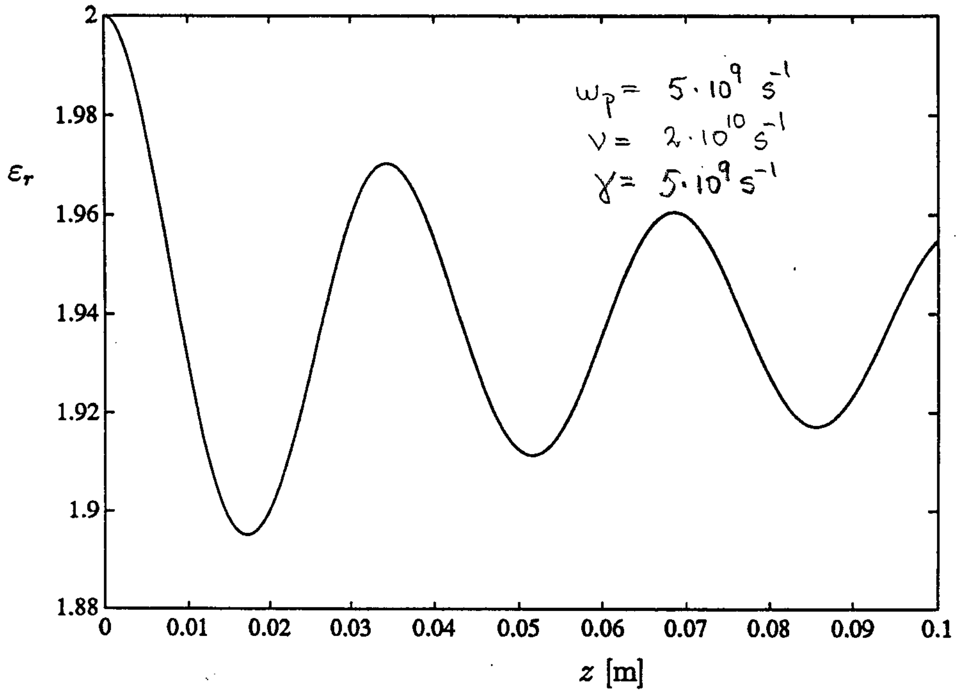
i.e. a Debye medium.

Now determine the permittivity $\epsilon_r(z)$ such that there is no reflection, $R(t) \equiv 0$.

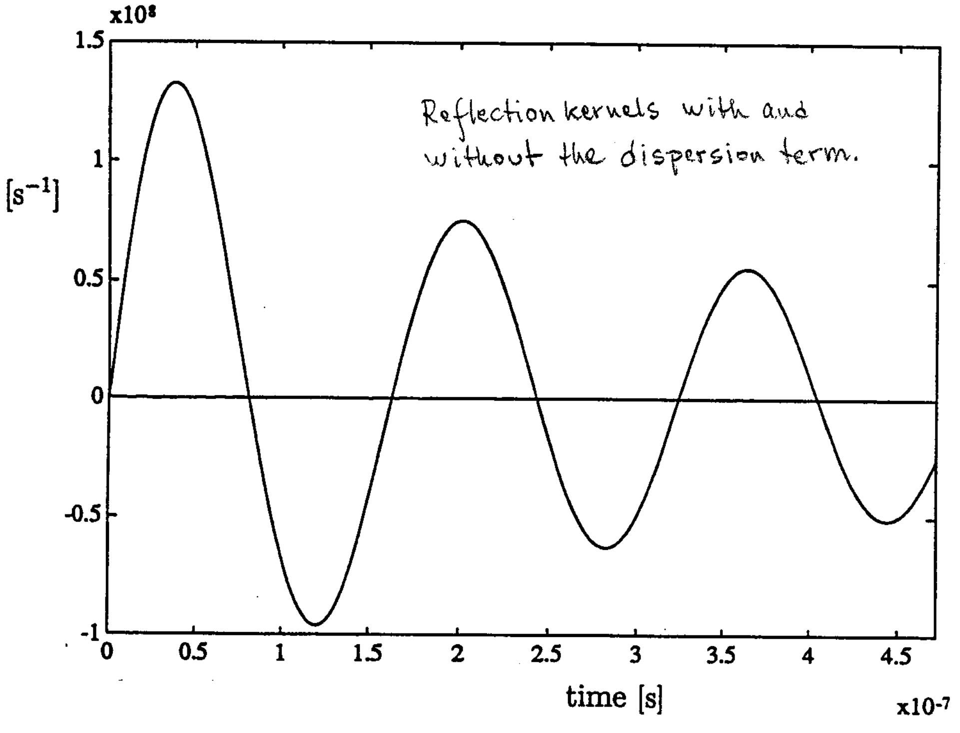


2) Assume a Lorentz medium

$$\chi(t) = \omega_p^2 \frac{\sin \nu t}{\nu} e^{-\gamma t}$$



$\omega_p = 5 \cdot 10^9\ s^{-1}$
$\nu = 2 \cdot 10^{10}\ s^{-1}$
$\gamma = 5 \cdot 10^9\ s^{-1}$



Reflection kernels with and without the dispersion term.