Gerhard Kristensson

Modeling of Electromagnetic Interaction with Matter

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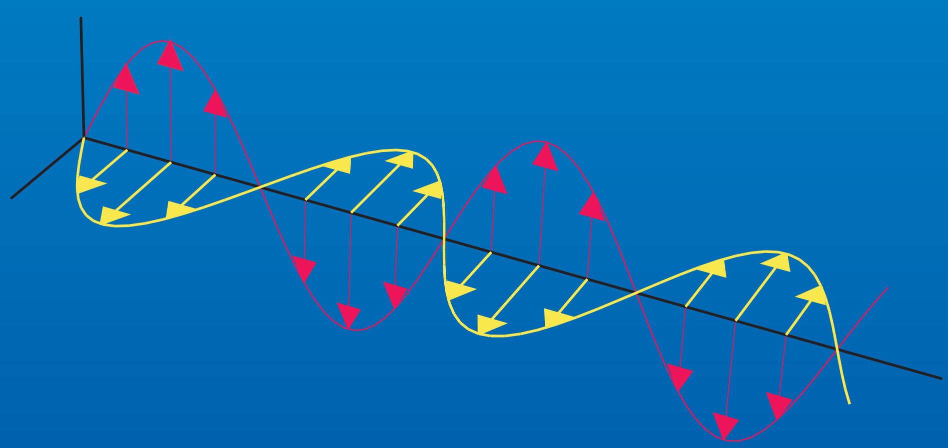

Vector operations with ∇

(1) $\nabla(\varphi + \psi) = \nabla\varphi + \nabla\psi$

(2) $\nabla(\varphi\psi) = \psi\nabla\varphi + \varphi\nabla\psi$

(3) $\nabla(\boldsymbol{a}\cdot\boldsymbol{b}) = (\boldsymbol{a}\cdot\nabla)\boldsymbol{b} + (\boldsymbol{b}\cdot\nabla)\boldsymbol{a} + \boldsymbol{a}\times(\nabla\times\boldsymbol{b}) + \boldsymbol{b}\times(\nabla\times\boldsymbol{a})$

(4) $\nabla(\boldsymbol{a}\cdot\boldsymbol{b}) = -\nabla\times(\boldsymbol{a}\times\boldsymbol{b}) + 2(\boldsymbol{b}\cdot\nabla)\boldsymbol{a} + \boldsymbol{a}\times(\nabla\times\boldsymbol{b}) + \boldsymbol{b}\times(\nabla\times\boldsymbol{a}) + \boldsymbol{a}(\nabla\cdot\boldsymbol{b}) - \boldsymbol{b}(\nabla\cdot\boldsymbol{a})$

(5) $\nabla\cdot(\boldsymbol{a}+\boldsymbol{b}) = \nabla\cdot\boldsymbol{a} + \nabla\cdot\boldsymbol{b}$

(6) $\nabla\cdot(\varphi\boldsymbol{a}) = \varphi(\nabla\cdot\boldsymbol{a}) + (\nabla\varphi)\cdot\boldsymbol{a}$

(7) $\nabla\cdot(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{b}\cdot(\nabla\times\boldsymbol{a}) - \boldsymbol{a}\cdot(\nabla\times\boldsymbol{b})$

(8) $\nabla\times(\boldsymbol{a}+\boldsymbol{b}) = \nabla\times\boldsymbol{a} + \nabla\times\boldsymbol{b}$

(9) $\nabla\times(\varphi\boldsymbol{a}) = \varphi(\nabla\times\boldsymbol{a}) + (\nabla\varphi)\times\boldsymbol{a}$

(10) $\nabla\times(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{a}(\nabla\cdot\boldsymbol{b}) - \boldsymbol{b}(\nabla\cdot\boldsymbol{a}) + (\boldsymbol{b}\cdot\nabla)\boldsymbol{a} - (\boldsymbol{a}\cdot\nabla)\boldsymbol{b}$

(11) $\nabla\times(\boldsymbol{a}\times\boldsymbol{b}) = -\nabla(\boldsymbol{a}\cdot\boldsymbol{b}) + 2(\boldsymbol{b}\cdot\nabla)\boldsymbol{a} + \boldsymbol{a}\times(\nabla\times\boldsymbol{b}) + \boldsymbol{b}\times(\nabla\times\boldsymbol{a}) + \boldsymbol{a}(\nabla\cdot\boldsymbol{b}) - \boldsymbol{b}(\nabla\cdot\boldsymbol{a})$

(12) $\nabla\cdot\nabla\varphi = \nabla^2\varphi = \Delta\varphi$

(13) $\nabla\times(\nabla\times\boldsymbol{a}) = \nabla(\nabla\cdot\boldsymbol{a}) - \nabla^2\boldsymbol{a}$

(14) $\nabla\times(\nabla\varphi) = \boldsymbol{0}$

(15) $\nabla\cdot(\nabla\times\boldsymbol{a}) = 0$

(16) $\nabla^2(\varphi\psi) = \varphi\nabla^2\psi + \psi\nabla^2\varphi + 2\nabla\varphi\cdot\nabla\psi$

(17) $\nabla r = \hat{\boldsymbol{r}}$

(18) $\nabla\times\boldsymbol{r} = \boldsymbol{0}$

(19) $\nabla\times\hat{\boldsymbol{r}} = \boldsymbol{0}$

(20) $\nabla\cdot\boldsymbol{r} = 3$

(21) $\nabla\cdot\hat{\boldsymbol{r}} = \dfrac{2}{r}$

(22) $\nabla(\boldsymbol{a}\cdot\boldsymbol{r}) = \boldsymbol{a}$, $\quad\boldsymbol{a}$ constant vector

(23) $(\boldsymbol{a}\cdot\nabla)\boldsymbol{r} = \boldsymbol{a}$

(24) $(\boldsymbol{a}\cdot\nabla)\hat{\boldsymbol{r}} = \dfrac{1}{r}\left(\boldsymbol{a} - \hat{\boldsymbol{r}}(\boldsymbol{a}\cdot\hat{\boldsymbol{r}})\right) = \dfrac{\boldsymbol{a}_\perp}{r}$

(25) $\nabla^2(\boldsymbol{r}\cdot\boldsymbol{a}) = 2\nabla\cdot\boldsymbol{a} + \boldsymbol{r}\cdot(\nabla^2\boldsymbol{a})$

(26) $\nabla u(f) = (\nabla f)\dfrac{du}{df}$

(27) $\nabla\cdot\boldsymbol{F}(f) = (\nabla f)\cdot\dfrac{d\boldsymbol{F}}{df}$

(28) $\nabla\times\boldsymbol{F}(f) = (\nabla f)\times\dfrac{d\boldsymbol{F}}{df}$

(29) $\nabla = \hat{\boldsymbol{r}}(\hat{\boldsymbol{r}}\cdot\nabla) - \hat{\boldsymbol{r}}\times(\hat{\boldsymbol{r}}\times\nabla)$

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Preface

This text is an introduction to the most important properties of electromagnetic fields and their interaction with passive materials. The main purpose of the text is to describe the theoretical principles of the modeling of these interaction phenomena. Chapter 2 gives a description of this interaction by employing the constitutive relations. The treatment is made in the time domain to avoid several pitfalls, *e.g.*, lack of causality.

The course requires a certain knowledge of basic electromagnetic field theory, for instance the basic course in electromagnetic field theory at an undergraduate level. We expect the Maxwell field equations to be known, as well as basic vector analysis, and calculations with the nabla operator ∇.

Exercises or problem are gathered at the end of each chapter. Advanced exercises are marked with a star (*). Answers to the exercises are found at the end of the book.

Chapter 1

The Maxwell equations

The foundation of the electromagnetics stands on the shoulders of the scientific giants of the 19$^{\text{th}}$ century. Stars like André Marie Ampère[1], Michael Faraday[2], and James Clerk Maxwell[3] shine brightly, see Figure 1.1. Many other scientist have contributed to the theory. A few of these giants are shown in Figure 1.2.

The physics of electromagnetic phenomena takes place in space and time. Therefore, a time dependent description is a natural starting point of modeling the electromagnetic interaction with matter. In fact, this approach is the guiding principle throughout the first part of this chapter, which is devoted to modeling of electromagnetic interaction with matter. By taking this viewpoint, we avoid some of the pitfalls that might occur if you start with a frequency domain formulation. In particular, causality is naturally included in the modeling.

The Maxwell equations are the fundamental mathematical model for all theoretical analysis of macroscopic electromagnetic phenomena. James Clerk Maxwell realized that light is an electromagnetic disturbance, and he published this result in 1864 in a paper entitled: *A dynamical theory of the electromagnetic field* [18]. His famous equations were published almost a decade later in 1873 in his textbook: *A Treatise on Electricity and Magnetism* [19, 20].

All experimental tests performed since then have confirmed this model, and, through the years, an impressive amount of evidences for the validity of these equations have been gathered in different fields of applications. However, microscopic phenomena require a more refined model including also quantum effects, but these effects are out of the scope of this book.

The Maxwell equations are the cornerstone in the analysis of macroscopic electromagnetic wave propagation phenomena.[4] The Maxwell equations in SI-units

[1] André Marie Ampère (1775–1836), French physicist.

[2] Michael Faraday (1791–1867), English chemist and physicist.

[3] James Clerk Maxwell (1831–1879), Scottish physicist and mathematician.

[4] It is out of the scope of this textbook to present a derivation of these equations. Several excellent derivations of these macroscopic equations from a microscopic formulation are found in the literature, see *e.g.*, [5, 10, 22].

Figure 1.1: The pioneers of electromagnetic theory. From left to right: André Marie Ampère (1775–1836), French physicist. Michael Faraday (1791–1867), English chemist and physicist. James Clerk Maxwell (1831–1879), Scottish physicist and mathematician.

(MKSA) are:

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}, t) = -\frac{\partial \boldsymbol{B}(\boldsymbol{r}, t)}{\partial t} \tag{1.1}$$

$$\nabla \times \boldsymbol{H}(\boldsymbol{r}, t) = \boldsymbol{J}(\boldsymbol{r}, t) + \frac{\partial \boldsymbol{D}(\boldsymbol{r}, t)}{\partial t} \tag{1.2}$$

The equation (1.1) (or the corresponding integral formulation) is used to called Faraday's law of induction , and the equation (1.2) is often called Ampère-Maxwell law. The different vector fields in the Maxwell equations are[5]:

$\boldsymbol{E}(\boldsymbol{r}, t)$	Electric field [V/m]
$\boldsymbol{H}(\boldsymbol{r}, t)$	Magnetic field [A/m]
$\boldsymbol{D}(\boldsymbol{r}, t)$	Electric flux density [As/m^2]
$\boldsymbol{B}(\boldsymbol{r}, t)$	Magnetic flux density or magnetic induction [Vs/m^2]
$\boldsymbol{J}(\boldsymbol{r}, t)$	Current density [A/m^2]

All these fields are functions of space and time, *i.e.*, space coordinates $\boldsymbol{r}$ and time t. Often these arguments are suppressed. Only when the equations and the expression can be misinterpreted, we make sure the arguments are explicitly written out.

The electric field $\boldsymbol{E}(\boldsymbol{r}, t)$ and the magnetic flux density $\boldsymbol{B}(\boldsymbol{r}, t)$ are defined by the force, $\boldsymbol{F}(t)$, on a charged particle by Lorentz' force.[6]

$$\boldsymbol{F}(t) = q \left\{ \boldsymbol{E}(\boldsymbol{r}, t) + \boldsymbol{v}(t) \times \boldsymbol{B}(\boldsymbol{r}, t) \right\} \tag{1.3}$$

[5]Sometimes we will for simplicity use the names $\boldsymbol{E}$-field, $\boldsymbol{D}$-field, $\boldsymbol{B}$-field, and $\boldsymbol{H}$-field.
[6]Hendrik Antoon Lorentz (1853–1928), Dutch physicist.

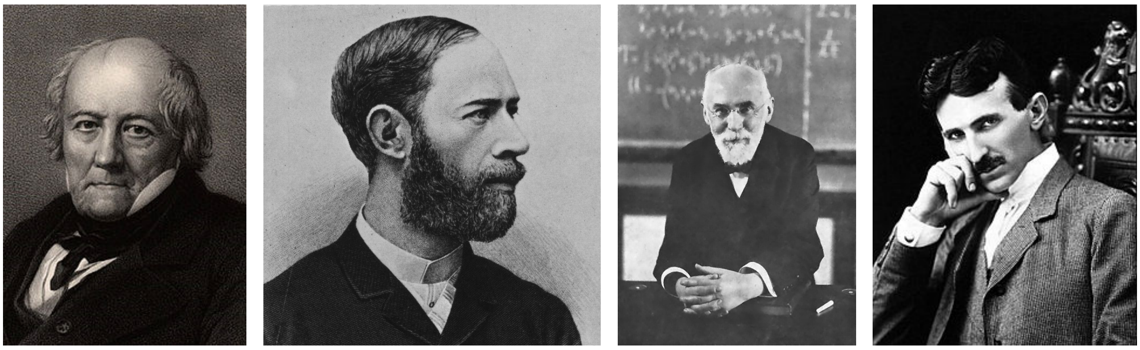


Figure 1.2: Immortal scientists of electromagnetic theory. From left to right: Jean-Baptiste Biot (1774–1862), French physicist, astronomer, and mathematician. Heinrich Rudolf Hertz (1857–1894), German physicist. Hendrik Antoon Lorentz (1853–1928), Dutch physicist. Nikola Tesla (1856–1943), Serbian inventor, mechanical engineer, and electrical engineer.

where q is the electric charge of the particle located at $\boldsymbol{r}(t)$, and $\boldsymbol{v}(t)$ is its velocity.

The free charges in the material, *e.g.*, the conduction electrons, are described by the current density $\boldsymbol{J}(\boldsymbol{r},t)$. The field contributions from bounded charges, *e.g.*, the electrons bound to the kernel of the atom, are included in the electric flux density $\boldsymbol{D}(\boldsymbol{r},t)$. In Section 2 we address the differences between the electric flux density $\boldsymbol{D}(\boldsymbol{r},t)$ and the electric field $\boldsymbol{E}(\boldsymbol{r},t)$, as well as the differences between the magnetic field $\boldsymbol{H}(\boldsymbol{r},t)$ and the magnetic flux density $\boldsymbol{B}(\boldsymbol{r},t)$, and models for the interrelations between these fields are presented.

One of the fundamental assumptions in physics is that electric charges are indestructible, *i.e.*, the sum of the charges is always constant. This invariance principle is very carefulness tested. One way of expressing the conservation of charges in mathematical terms is through the continuity law of charges

$$\nabla \cdot \boldsymbol{J}(\boldsymbol{r},t) + \frac{\partial \rho(\boldsymbol{r},t)}{\partial t} = 0 \tag{1.4}$$

Here $\rho(\boldsymbol{r},t)$ is the charge density (charge/unit volume) that is associated with the current density $\boldsymbol{J}(\boldsymbol{r},t)$. The charge density $\rho(\boldsymbol{r},t)$ therefore models the free charges of the problem. As alluded to above, the contributions from bounded charges are included in the electric flux density $\boldsymbol{D}(\boldsymbol{r},t)$ and the magnetic field $\boldsymbol{H}(\boldsymbol{r},t)$.

Two additional equations are usually associated with the Maxwell equations.

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r},t) = 0 \tag{1.5}$$

$$\nabla \cdot \boldsymbol{D}(\boldsymbol{r},t) = \rho(\boldsymbol{r},t) \tag{1.6}$$

Equation (1.5) tells us that no magnetic charges exist, and it implies that the magnetic flux is conserved. The equation (1.6) is usually called Gauss' law[7]. Under

[7]Johann Carl Friedrich Gauss (1777–1855). German mathematician.

suitable assumptions, both these equations can be derived from the equations (1.1), (1.2) and (1.4). To see this, take the divergence of (1.1) and (1.2). This implies

$$\begin{cases} \nabla \cdot \dfrac{\partial \boldsymbol{B}(\boldsymbol{r},t)}{\partial t} = 0 \\ \nabla \cdot \boldsymbol{J}(\boldsymbol{r},t) + \nabla \cdot \dfrac{\partial \boldsymbol{D}(\boldsymbol{r},t)}{\partial t} = 0 \end{cases}$$

since $\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$ for an arbitrary vector field $\boldsymbol{A}$. Interchanging the order of differentiation and using (1.4) give

$$\begin{cases} \dfrac{\partial(\nabla \cdot \boldsymbol{B}(\boldsymbol{r},t))}{\partial t} = 0 \\ \dfrac{\partial(\nabla \cdot \boldsymbol{D}(\boldsymbol{r},t) - \rho(\boldsymbol{r},t))}{\partial t} = 0 \end{cases}$$

These equations imply

$$\begin{cases} \nabla \cdot \boldsymbol{B}(\boldsymbol{r},t) = f_1(\boldsymbol{r}) \\ \nabla \cdot \boldsymbol{D}(\boldsymbol{r},t) - \rho(\boldsymbol{r},t) = f_2(\boldsymbol{r}) \end{cases}$$

where $f_1(\boldsymbol{r})$ and $f_2(\boldsymbol{r})$ are two functions that do not depend on time t, but can depend on the spatial coordinates $\boldsymbol{r}$. If the fields $\boldsymbol{B}(\boldsymbol{r},t)$, $\boldsymbol{D}(\boldsymbol{r},t)$ and $\rho(\boldsymbol{r},t)$ are identically zero before a fixed time, τ, *i.e.*,

$$\begin{cases} \boldsymbol{B}(\boldsymbol{r},t) = \boldsymbol{0} \\ \boldsymbol{D}(\boldsymbol{r},t) = \boldsymbol{0} \qquad t < \tau \\ \rho(\boldsymbol{r},t) = 0 \end{cases}$$

then the equations (1.5) and (1.6) follow. Of course, static or time-harmonic fields do not satisfy this assumption, since there is no time, τ, before which all fields are zero.[8] However, under the assumption that fields and charges do not have existed for ever, it is sufficient to use the equations (1.1), (1.2) and (1.4).

Equations (1.1) and (1.2) contain 6 different equations — one for each vector component. Provided the current density $\boldsymbol{J}(\boldsymbol{r},t)$ is given, the Maxwell equations contain 12 unknowns — the four vector fields $\boldsymbol{E}(\boldsymbol{r},t)$, $\boldsymbol{B}(\boldsymbol{r},t)$, $\boldsymbol{D}(\boldsymbol{r},t)$ and $\boldsymbol{H}(\boldsymbol{r},t)$. The result is that we lack 6 equations in order to have as many equations as unknowns. The lacking 6 equations are called the constitutive relations and they are addressed in Section 2.

In vacuum the electric field $\boldsymbol{E}(\boldsymbol{r},t)$ and the electric flux density $\boldsymbol{D}(\boldsymbol{r},t)$ are parallel — the difference is in unit they are measured. The same holds for the magnetic flux density $\boldsymbol{B}(\boldsymbol{r},t)$ and the magnetic field $\boldsymbol{H}(\boldsymbol{r},t)$. We have

$$\begin{cases} \boldsymbol{D}(\boldsymbol{r},t) = \epsilon_0 \boldsymbol{E}(\boldsymbol{r},t) \\ \boldsymbol{B}(\boldsymbol{r},t) = \mu_0 \boldsymbol{H}(\boldsymbol{r},t) \end{cases} \tag{1.7}$$

[8] We will return to the derivation of equations (1.5) and (1.6) for time-harmonic fields in Section 3 on page 46.

where ϵ_0 and μ_0 are the permittivity and the permeability of vacuum. Numerical values of these constants are: $\epsilon_0 \approx 8.854 \cdot 10^{-12}$ As/Vm and $\mu_0 = 4\pi \cdot 10^{-7}$ Vs/Am $\approx 1.257 \cdot 10^{-6}$ Vs/Am.

Inside the material, the difference between the electric field $\boldsymbol{E}(\boldsymbol{r}, t)$ and the electric flux density $\boldsymbol{D}(\boldsymbol{r}, t)$ and between the magnetic flux density $\boldsymbol{B}(\boldsymbol{r}, t)$ and the magnetic field $\boldsymbol{H}(\boldsymbol{r}, t)$ are usually more complex. These differences are a measure of the interaction between the charges in the material and the fields. Often, two new vector fields, the polarization $\boldsymbol{P}(\boldsymbol{r}, t)$, and the magnetization $\boldsymbol{M}(\boldsymbol{r}, t)$, of the material are introduced to describe the differences between these fields. The definitions of these fields are

$$\begin{cases} \boldsymbol{P}(\boldsymbol{r}, t) = \boldsymbol{D}(\boldsymbol{r}, t) - \epsilon_0 \boldsymbol{E}(\boldsymbol{r}, t) \\ \boldsymbol{M}(\boldsymbol{r}, t) = \dfrac{1}{\mu_0} \boldsymbol{B}(\boldsymbol{r}, t) - \boldsymbol{H}(\boldsymbol{r}, t) \end{cases} \tag{1.8}$$

The vector field $\boldsymbol{P}(\boldsymbol{r}, t)$ is a measure of how much the bounded charges in the material are displaced relative their unaffected positions. These phenomena include both permanent and induced polarization. The largest contributions come from displacements of the center of charge for the positive and the negative charges in the material, but also other higher order effects contribute. In an analogous manner, the magnetization $\boldsymbol{M}(\boldsymbol{r}, t)$ is a measure of the bounded currents in the material. The origin of this field can also be both permanent or induced.

The modeling of the polarization and the magnetization effects of the material is equivalent to specify the constitutive relations of the material, and it implies that 6 additional equations characterizing the material are given. In Section 2, we analyze several different models for the polarization and the magnetization of the material in detail. In the next two subsections, we investigate additional consequences of the Maxwell equations, namely the boundary conditions and energy conservation.

1.1 Boundary conditions at interfaces

At an interface between two different materials some components of the electromagnetic field are discontinuous. The way they vary is a consequence of the Maxwell equations, and in this section we give a simple derivation of the boundary conditions the fields must satisfy. Only surfaces that are fixed in time (no moving surfaces) are treated here.

The Maxwell equations, as they were presented in (1.1) and (1.2), assume that all field quantities are differentiable functions of space and time. At an interface between two media, the fields, as already mentioned above, are discontinuous functions of the spatial variables. Therefore, we need to reformulate the Maxwell equations in such a way that the equations can be interpreted in a weaker, more general, sense. The aim is to obtain equations that are also valid for fields that are not differentiable at all points in space.

Let V be an arbitrary (simply connected) volume with bounding surface S and unit outward normal vector $\hat{\boldsymbol{\nu}}$, see Figure 1.3. We start by assuming the fields $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{D}$, and $\boldsymbol{H}$ are continuously differentiable in V, and then we see how the

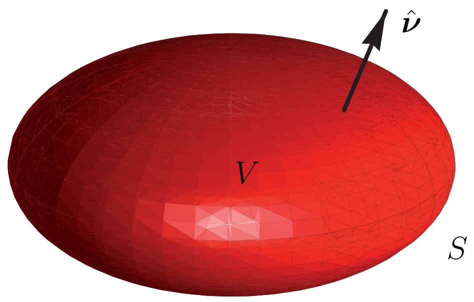


Figure 1.3: Geometry of integration.

differentiability property can be relaxed. Integrate the Maxwell equations, (1.1), (1.2), (1.5), and (1.6), over the volume V. We get

$$\begin{aligned}
&\iiint_V \nabla\times\boldsymbol{E}\,\mathrm{d}v = -\iiint_V \frac{\partial\boldsymbol{B}}{\partial t}\,\mathrm{d}v\\
&\iiint_V \nabla\times\boldsymbol{H}\,\mathrm{d}v = \iiint_V \boldsymbol{J}\,\mathrm{d}v + \iiint_V \frac{\partial\boldsymbol{D}}{\partial t}\,\mathrm{d}v\\
&\iiint_V \nabla\cdot\boldsymbol{B}\,\mathrm{d}v = 0\\
&\iiint_V \nabla\cdot\boldsymbol{D}\,\mathrm{d}v = \iiint_V \rho\,\mathrm{d}v
\end{aligned}$$

where $\mathrm{d}v$ is the volume measure ($\mathrm{d}v = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$).

The following two integration theorems for vector fields are now useful:

$$\begin{aligned}
&\iiint_V \nabla\cdot\boldsymbol{A}\,\mathrm{d}v = \iint_S \boldsymbol{A}\cdot\hat{\boldsymbol{\nu}}\,\mathrm{d}S\\
&\iiint_V \nabla\times\boldsymbol{A}\,\mathrm{d}v = \iint_S \hat{\boldsymbol{\nu}}\times\boldsymbol{A}\,\mathrm{d}S
\end{aligned}$$

where $\boldsymbol{A}$ is a continuously differentiable vector field in V, and $\mathrm{d}S$ is the surface element of S. The first theorem is usually called the divergence theorem or the Gauss theorem[9] and the other Gauss' analogous theorem, see Problem 1.1.

The result after interchanging the derivation w.r.t. time t and integration (volume

[9]Distinguish between the Gauss law, (1.6), and the Gauss theorem.

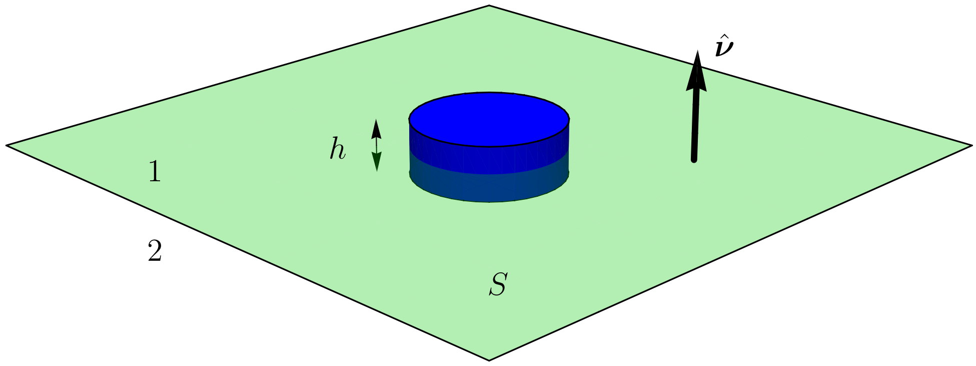


Figure 1.4: Interface between two different media 1 and 2.

V is fixed in time and we assume all field to be sufficiently regular) is

$$\begin{cases} \iint\limits_S \hat{\boldsymbol{\nu}} \times \boldsymbol{E} \, \mathrm{d}S = -\frac{\mathrm{d}}{\mathrm{d}t} \iiint\limits_V \boldsymbol{B} \, \mathrm{d}v \\ \iint\limits_S \hat{\boldsymbol{\nu}} \times \boldsymbol{H} \, \mathrm{d}S = \iiint\limits_V \boldsymbol{J} \, \mathrm{d}v + \frac{\mathrm{d}}{\mathrm{d}t} \iiint\limits_V \boldsymbol{D} \, \mathrm{d}v \\ \iint\limits_S \boldsymbol{B} \cdot \hat{\boldsymbol{\nu}} \, \mathrm{d}S = 0 \\ \iint\limits_S \boldsymbol{D} \cdot \hat{\boldsymbol{\nu}} \, \mathrm{d}S = \iiint\limits_V \rho \, \mathrm{d}v \end{cases} \tag{1.9}$$

In a domain V, where the fields $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{D}$, and $\boldsymbol{H}$ are continuously differentiable, these integral expressions are equivalent to the differential equations in (1.1), (1.2), (1.5), and (1.6). We have proved this equivalence one way and in the other direction we do the analysis in a reversed direction and use the fact that the volume V is simply connected and arbitrary.

The integral formulation, (1.9), has, however, the advantage that the fields do not have to be differentiable in the spatial variables to make sense. In this respect, the integral formulation is more general than the differential formulation in (1.1), (1.2), (1.5), and (1.6). The fields $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{D}$ and $\boldsymbol{H}$, that satisfy the equations in (1.9) are called weak solutions to the Maxwell equations, in the case the fields are not continuously differentiable, and (1.1), (1.2), (1.5), and (1.6) lack meaning.

The integral expressions (1.9) are now employed to a special volume, V_h, that intersects the interface, S, between two different media, see Figure 1.4. The unit normal, $\hat{\boldsymbol{\nu}}$, of the interface, S, is directed from medium 2 into medium 1. We assume all electromagnetic fields $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{D}$ and $\boldsymbol{H}$, and their time derivatives, to have finite limit values in the limit from both sides of the interface. For the electric field, these limit values in medium 1 and 2 are denoted $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$, respectively, and similar notation, with index 1 and 2, is adopted for the other fields. The current density $\boldsymbol{J}$ and the charge density ρ, however, can assume infinite values at the interface for perfectly conducting surfaces.[10] It is convenient to introduce a surface current density $\boldsymbol{J}_S$ and surface charge density ρ_S as a limit process

$$\begin{cases} \boldsymbol{J}_S = h\boldsymbol{J} \\ \rho_S = h\rho \end{cases}$$

where h is the thickness of the layer that contains the charges close to the surface. We assume that this thickness approaches zero, and that $\boldsymbol{J}$ and ρ approach infinity in such a way that $\boldsymbol{J}_S$ and ρ_S have well defined limits in this process. The surface current density $\boldsymbol{J}_S$ is assumed to be a tangential field to the surface S. We let the height of the volume V_h be h, and the area on the upper and lower part of the bounding surface of V_h be a, which is small compared to the variations of the fields and the curvature of the surface S.

The two terms containing time derivatives in (1.9), *i.e.*,

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V_h} \boldsymbol{B}\,\mathrm{d}v, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V_h} \boldsymbol{D}\,\mathrm{d}v$$

both approach zero as $h \to 0$, since the fields $\boldsymbol{B}$ and $\boldsymbol{D}$ and their time derivatives are assumed to be finite at the interface. Moreover, the contributions from all lateral sides (areas $\sim h$) of the surface integrals on the left-hand side in (1.9) approach zero as $h \to 0$. The contribution from the upper part (unit normal $\hat{\boldsymbol{\nu}}$) and lower part (unit normal $-\hat{\boldsymbol{\nu}}$) are proportional to the area a, provided the area is chosen sufficiently small, and the mean value theorem for integrals is used. The contributions from the upper and the lower parts of the surface integrals in the limit $h \to 0$ are

$$\begin{cases} a\left[\hat{\boldsymbol{\nu}} \times (\boldsymbol{E}_1 - \boldsymbol{E}_2)\right] = \boldsymbol{0} \\ a\left[\hat{\boldsymbol{\nu}} \times (\boldsymbol{H}_1 - \boldsymbol{H}_2)\right] = ah\boldsymbol{J} = a\boldsymbol{J}_S \\ a\left[\hat{\boldsymbol{\nu}} \cdot (\boldsymbol{B}_1 - \boldsymbol{B}_2)\right] = 0 \\ a\left[\hat{\boldsymbol{\nu}} \cdot (\boldsymbol{D}_1 - \boldsymbol{D}_2)\right] = ah\rho = a\rho_S \end{cases}$$

Simplify these expressions by dividing with the area a. The result is

$$\begin{cases} \hat{\boldsymbol{\nu}} \times (\boldsymbol{E}_1 - \boldsymbol{E}_2) = \boldsymbol{0} \\ \hat{\boldsymbol{\nu}} \times (\boldsymbol{H}_1 - \boldsymbol{H}_2) = \boldsymbol{J}_S \\ \hat{\boldsymbol{\nu}} \cdot (\boldsymbol{B}_1 - \boldsymbol{B}_2) = 0 \\ \hat{\boldsymbol{\nu}} \cdot (\boldsymbol{D}_1 - \boldsymbol{D}_2) = \rho_S \end{cases} \tag{1.10}$$

These boundary conditions prescribe how the electromagnetic fields are related to each other on each side of the interface (the unit normal $\hat{\boldsymbol{\nu}}$ is directed from medium 2 into medium 1). We formulate these boundary conditions in words.

[10] This is of course an idealization of a situation where the density assumes very high values in a macroscopically thin layer.

- The tangential components of the electric field $\boldsymbol{E}$ are continuous across the interface S.
- The tangential components of the magnetic field $\boldsymbol{H}$ are discontinuous over the interface S. The size of the discontinuity is $\boldsymbol{J}_S$. If the surface current density is zero, *e.g.*, if the material has finite conductivity,[11,12] the tangential components of the magnetic field are continuous across the interface S.
- The normal component of the magnetic flux density $\boldsymbol{B}$ is continuous across the interface S.
- The normal component of the electric flux density $\boldsymbol{D}$ is discontinuous across the interface S. The size of the discontinuity is ρ_S. If the surface charge density is zero, the normal component of the electric flux density is continuous across the interface.

In Figure 1.5 we illustrate the typical variations in the normal components of the electric and the magnetic flux densities as a function of the distance across the interface between two media.

A special case of (1.10) is the case where medium 2 is a perfectly conducting material, which is a model of a material with freely moving charges, *e.g.*, many metals. In material 2, the fields are zero, and we get from (1.10)

$$\begin{cases} \hat{\boldsymbol{\nu}} \times \boldsymbol{E}_1 = \boldsymbol{0} \\ \hat{\boldsymbol{\nu}} \times \boldsymbol{H}_1 = \boldsymbol{J}_S \\ \hat{\boldsymbol{\nu}} \cdot \boldsymbol{B}_1 = 0 \\ \hat{\boldsymbol{\nu}} \cdot \boldsymbol{D}_1 = \rho_S \end{cases} \tag{1.11}$$

where $\boldsymbol{J}_S$ and ρ_S are the surface current density and surface charge density on the surface S, respectively.

1.2 Energy conservation and Poynting's theorem

Energy conservation is shown from the Maxwell equations (1.1) and (1.2).

$$\begin{cases} \nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t} \\ \nabla \times \boldsymbol{H} = \boldsymbol{J} + \dfrac{\partial \boldsymbol{D}}{\partial t} \end{cases}$$

Make a scalar multiplication of the first equation with $\boldsymbol{H}$ and the second equation with $\boldsymbol{E}$ and subtract. The result is

$$\boldsymbol{H} \cdot (\nabla \times \boldsymbol{E}) - \boldsymbol{E} \cdot (\nabla \times \boldsymbol{H}) + \boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{E} \cdot \boldsymbol{J} = 0$$

[11] The conductivity of a material is introduced below.

[12] This is an implication of the assumption that the electric field $\boldsymbol{E}$ is finite close to the interface. We have $\boldsymbol{J}_S = h\boldsymbol{J} = h\sigma\boldsymbol{E} \to \boldsymbol{0}$, as $h \to 0$.

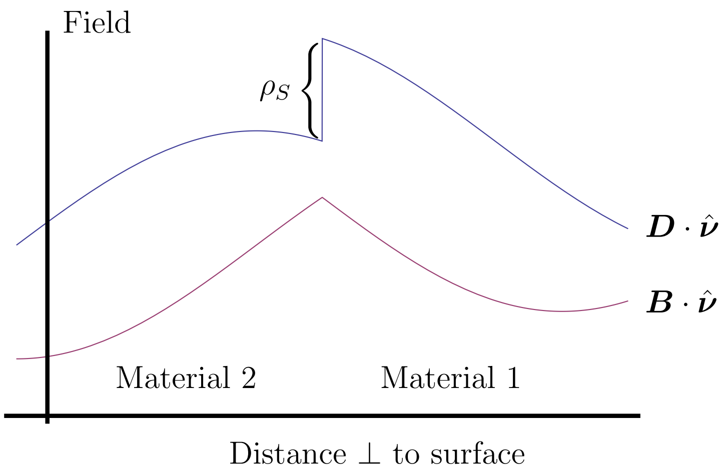


Figure 1.5: The variation of $\boldsymbol{B}\cdot\hat{\boldsymbol{\nu}}$ and $\boldsymbol{D}\cdot\hat{\boldsymbol{\nu}}$ at the interface between two different material 1 and 2.

We rewrite this expression with the use of the differential rule of the nabla-operator $\nabla\cdot(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{b}\cdot(\nabla\times\boldsymbol{a}) - \boldsymbol{a}\cdot(\nabla\times\boldsymbol{b})$. We have

$$\nabla\cdot(\boldsymbol{E}\times\boldsymbol{H}) + \boldsymbol{H}\cdot\frac{\partial\boldsymbol{B}}{\partial t} + \boldsymbol{E}\cdot\frac{\partial\boldsymbol{D}}{\partial t} + \boldsymbol{E}\cdot\boldsymbol{J} = 0$$

The vector product of the electric and the magnetic field plays a special role, and we introduce Poynting's vector,[13] $\boldsymbol{S} = \boldsymbol{E}\times\boldsymbol{H}$. We get Poynting's theorem.

$$\nabla\cdot\boldsymbol{S} + \boldsymbol{H}\cdot\frac{\partial\boldsymbol{B}}{\partial t} + \boldsymbol{E}\cdot\frac{\partial\boldsymbol{D}}{\partial t} + \boldsymbol{E}\cdot\boldsymbol{J} = 0 \tag{1.12}$$

Poynting's vector $\boldsymbol{S}$ gives the power per unit area of the electromagnetic field or the power flow in the direction of the vector $\boldsymbol{S}$. This becomes clearer if we integrate (1.12) over a simply connected volume V, bounded by the surface S and with unit outward normal vector $\hat{\boldsymbol{\nu}}$, see Figure 1.3, and use the divergence theorem. We get

$$\begin{aligned}\iint_S \boldsymbol{S}\cdot\hat{\boldsymbol{\nu}}\,\mathrm{d}S &= \iiint_V \nabla\cdot\boldsymbol{S}\,\mathrm{d}v\\ &= -\iiint_V \left[\boldsymbol{H}\cdot\frac{\partial\boldsymbol{B}}{\partial t} + \boldsymbol{E}\cdot\frac{\partial\boldsymbol{D}}{\partial t}\right]\mathrm{d}v - \iiint_V \boldsymbol{E}\cdot\boldsymbol{J}\,\mathrm{d}v\end{aligned} \tag{1.13}$$

The terms are interpreted in the following way:

The left-hand side:

$$\iint_S \boldsymbol{S}\cdot\hat{\boldsymbol{\nu}}\,\mathrm{d}S$$

This is the total power radiated out of the bounding surface S, *i.e.*, the energy per time unit, carried by the electromagnetic field.

[13] John Henry Poynting (1852–1914), English physicist.

The right-hand side: The power flow through the surface S is compensated by two different contributions. The first volume integral on the right-hand side

$$\iiint_V \left[\boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} \right] \mathrm{d}v$$

gives the power stored in the electromagnetic field in the volume V. Included in this contribution is the power needed to polarize and magnetize the material in V. The second volume integral in (1.13)

$$\iiint_V \boldsymbol{E} \cdot \boldsymbol{J} \, \mathrm{d}v$$

gives the work per unit time, *i.e.*, the power, that the electric field does on the charges in V.

This interpretation implies that (1.13) expresses power balance in the volume V, *i.e.*,

$$\begin{aligned} &\text{Through } S \text{ radiated power} + \text{power consumption in } V \\ &\qquad = - \text{ power bounded to the electromagnetic field in } V \end{aligned}$$

In the derivation above, we assumed that the volume V did not cut any surface where the fields vary discontinuously, *e.g.*, an interface between two media. We now prove that this assumption is no severe restriction, and the assumption can easily be relaxed. If the surface S is an interface between two media, see Figure 1.4, Poynting's vector in medium 1 close to the interface is

$$\boldsymbol{S}_1 = \boldsymbol{E}_1 \times \boldsymbol{H}_1$$

and Poynting's vector close to the interface in medium 2 is

$$\boldsymbol{S}_2 = \boldsymbol{E}_2 \times \boldsymbol{H}_2$$

The boundary conditions at the interface given by (1.10) read

$$\begin{cases} \hat{\boldsymbol{\nu}} \times \boldsymbol{E}_1 = \hat{\boldsymbol{\nu}} \times \boldsymbol{E}_2 \\ \hat{\boldsymbol{\nu}} \times \boldsymbol{H}_1 = \hat{\boldsymbol{\nu}} \times \boldsymbol{H}_2 + \boldsymbol{J}_S \end{cases}$$

A cyclic permutation of the vectors and the use of the boundary conditions imply

$$\begin{aligned} \hat{\boldsymbol{\nu}} \cdot \boldsymbol{S}_1 &= \hat{\boldsymbol{\nu}} \cdot (\boldsymbol{E}_1 \times \boldsymbol{H}_1) = \boldsymbol{H}_1 \cdot (\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_1) = \boldsymbol{H}_1 \cdot (\hat{\boldsymbol{\nu}} \times \boldsymbol{E}_2) \\ &= -\boldsymbol{E}_2 \cdot (\hat{\boldsymbol{\nu}} \times \boldsymbol{H}_1) = -\boldsymbol{E}_2 \cdot (\hat{\boldsymbol{\nu}} \times \boldsymbol{H}_2 + \boldsymbol{J}_S) \\ &= \hat{\boldsymbol{\nu}} \cdot (\boldsymbol{E}_2 \times \boldsymbol{H}_2) - \boldsymbol{E}_2 \cdot \boldsymbol{J}_S = \hat{\boldsymbol{\nu}} \cdot \boldsymbol{S}_2 - \boldsymbol{E}_2 \cdot \boldsymbol{J}_S \end{aligned}$$

By integration of this expression over the interface S we obtain power conservation over the surface S as expressed as

$$\iint_S \boldsymbol{S}_1 \cdot \hat{\boldsymbol{\nu}} \, \mathrm{d}S = \iint_S \boldsymbol{S}_2 \cdot \hat{\boldsymbol{\nu}} \, \mathrm{d}S - \iint_S \boldsymbol{E}_2 \cdot \boldsymbol{J}_S \, \mathrm{d}S \tag{1.14}$$

where the surface S is an arbitrary part of the interface. Note that the unit normal $\hat{\boldsymbol{\nu}}$ is directed from medium 2 into medium 1.

The last surface integral in (1.14) gives the work per unit time the electric field does on the charge at the interface. If there are no surface currents at the interface, the normal component of Poynting's vector is continuous across the interface. It is irrelevant which electric field we use in the last surface integral in (1.14) since the surface current density $\boldsymbol{J}_S$ is parallel to the interface S and the tangential components of the electric field are continuous across the interface, *i.e.*,

$$\iint_S \boldsymbol{E}_1 \cdot \boldsymbol{J}_S \, \mathrm{d}S = \iint_S \boldsymbol{E}_2 \cdot \boldsymbol{J}_S \, \mathrm{d}S$$

Problems for Chapter 1

1.1 Show the following analogous theorem of Gauss' theorem:

$$\iiint_V \nabla \times \boldsymbol{A} \, \mathrm{d}v = \iint_S \hat{\boldsymbol{\nu}} \times \boldsymbol{A} \, \mathrm{d}S$$

by the use of Gauss' theorem

$$\iiint_V \nabla \cdot \boldsymbol{A} \, \mathrm{d}v = \iint_S \boldsymbol{A} \cdot \hat{\boldsymbol{\nu}} \, \mathrm{d}S$$

1.2 A finite volume contains a magnetic material with magnetization $\boldsymbol{M}$. In the absence of current density (free charges), $\boldsymbol{J} = \boldsymbol{0}$, show that the static magnetic field, $\boldsymbol{H}$, and the magnetic flux density, $\boldsymbol{B}$, satisfy

$$\iiint_{\mathbb{R}^3} \boldsymbol{B} \cdot \boldsymbol{H} \, \mathrm{d}v = 0$$

where the integration is over all space.

1.3 An infinitely long, straight conductor of circular cross section (radius a) consists of a material with finite conductivity σ. In the conductor a static current I is flowing. The current density $\boldsymbol{J}$ is assumed to be homogeneous over the cross section of the conductor. Compute the terms in Poynting's theorem and show that power balance holds for a volume V, which consists of a finite portion l of the conductor, see Figure 1.6.

***1.4** A capacitor is modeled by two circular plates (radius a, distance d between the plates), see Figure 1.7. A time harmonic current is applied to the capacitor. The

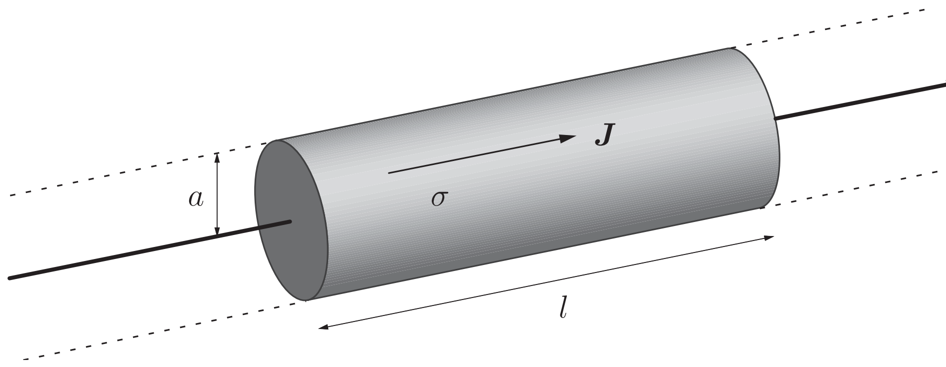


Figure 1.6: The geometry of the Problem 1.3. The figure shows a finite portion (length l) of the conductor with circular cross section (radius a). The conductor consists of a material with finite conductivity σ.

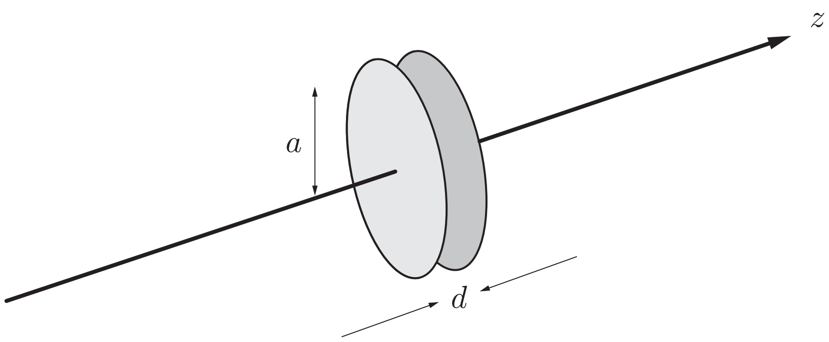


Figure 1.7: The geometry of the Problem 1.4.

medium between the plates is vacuous and we neglect all effects from the edges. Determine by the Ansatz

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{r},t) &= \hat{\boldsymbol{z}}E(\rho,\omega)\cos(\omega t+\alpha)\\ \boldsymbol{H}(\boldsymbol{r},t) &= \hat{\boldsymbol{\phi}}H(\rho,\omega)\cos(\omega t+\beta)\end{aligned}$$

the electric and the magnetic field, respectively, between the plates of the capacitor, *i.e.*, determine $E(\rho,\omega)$ and $H(\rho,\omega)$ and the phase β (expressed in α). Show that Poynting's theorem holds for the volume V between the plates.

Hint: Show that the electric field $E(\rho,\omega)$ satisfies

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial E(\rho,\omega)}{\partial\rho}\right)+\frac{\omega^2}{c_0^2}E(\rho,\omega)=0$$

where c_0 is the speed of light in vacuum.

*__1.5__ Using the results of Problem 1.4 show that the capacitance C and the inductance L

of the capacitor are

$$\begin{cases} L = \dfrac{d\mu_0}{8\pi}\left(1 + \dfrac{\xi^2}{12} + O(\xi^4)\right) \\ C = \dfrac{\pi a^2 \epsilon_0}{d}\left(1 + O(\xi^4)\right) \end{cases}$$

where $\xi = \omega a/c_0$ is the dimensionless (frequency) parameter. Moreover, find an expression of the resonance frequency ω_r of the "circuit". What is the explicit resonance frequency for a capacitor with $a = 1$ cm?

Chapter 2

Constitutive relations

As already mentioned in Section 1, the Maxwell equations (1.1) and (1.2) are not complete. The equations contain twelve field quantities ($\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{D}$, and $\boldsymbol{H}$), but there are only six equations. The remaining six equations, the so called constitutive relations will be treated in this section. This section deals with general time dependent fields in the same way as in Section 1. The special conditions that hold for time harmonic fields are addressed in Section 3.

The Maxwell equations contains the fields $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{D}$, and $\boldsymbol{H}$ and their sources. These equations model the dynamics of the fields, but how the fields are related to each other is independent information. This information is, crudely speaking, the dynamics of the charges in the material. In general, the constitutive relations is a relation between two pairs of fields, *e.g.*, $\{\boldsymbol{D}, \boldsymbol{B}\}$ and $\{\boldsymbol{E}, \boldsymbol{H}\}$.

$$\{\boldsymbol{D}, \boldsymbol{B}\} = \boldsymbol{F}(\{\boldsymbol{E}, \boldsymbol{H}\}) \tag{2.1}$$

This particular form of the constitutive relations emphasizes the fields $\boldsymbol{E}$ and $\boldsymbol{H}$, and the transformation (2.1) can be used to eliminate the flux densities, $\boldsymbol{D}$ and $\boldsymbol{B}$, from the Maxwell equations. The electric and the magnetic fields, $\boldsymbol{E}$ and $\boldsymbol{H}$, which define Poynting's vector, are central in this relation. In wave propagation problems, this particular form of the constitutive relations is appropriate, since power flow is an important quantity in these problems.

Other combinations of the fields in the constitutive relations often occur in the literature. Frequently, a relation between the pairs $\{\boldsymbol{D}, \boldsymbol{H}\}$ and $\{\boldsymbol{E}, \boldsymbol{B}\}$ is used. This relation emphasizes the fields $\boldsymbol{E}$ and $\boldsymbol{B}$, defined in the Lorentz force, (1.3). There are also other reasons, based upon the theory of special relativity, that make this transformation between $\{\boldsymbol{D}, \boldsymbol{H}\}$ and $\{\boldsymbol{E}, \boldsymbol{B}\}$ preferable. In this book we use the constitutive relations given in (2.1), due to the fact that the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ define Poynting's vector.

The mapping $\boldsymbol{F}$ in (2.1) has to satisfy certain general requirements or physical assumptions to be a candidate for a realistic model. These physical assumptions or requirements are discussed in this section. The special simplifications that occur for time harmonic fields are analyzed in Section 3. The first subsection deals with isotropic media with dispersion. In a subsequent subsection, these results are generalized and models for general linear media are stated.

2.1 Isotropic media with dispersion

As an introduction to the more general constitutive relations that are treated in Section 2.3, we first analyze the simpler case with an isotropic media.

An isotropic, dispersive material is the most simple example of a constitutive relation between the fields. The isotropy implies that the material has identical properties in all directions and as a consequence of this there is no coupling between different components of the fields.[1] Moreover, we assume that there is no coupling between the electric fields, $\boldsymbol{E}$ and $\boldsymbol{D}$, and the magnetic fields, $\boldsymbol{H}$ and $\boldsymbol{B}$. The mapping in (2.1) then becomes two separate mappings that do not couple. For simplicity, we also assume that the material is non-magnetic, *i.e.*, the magnetic fields satisfy the vacuum relations, (1.7). Equation (2.1) then simplifies to

$$\begin{cases} \boldsymbol{D} = \boldsymbol{F}(\boldsymbol{E}) \\ \boldsymbol{B} = \mu_0 \boldsymbol{H} \end{cases}$$

We now state a series of requirements that the mapping $\boldsymbol{F}$ has to satisfy as a function of time t at every point, $\boldsymbol{r}$, in the medium. All other dependence of the macroscopic variables, such as temperature or pressure, are not explicitly indicated to avoid cumbersome notation. Specifically, no functional dependence, except time t, is written out, unless it is required for the understanding. To this end, fields and other quantities do depend on the space variables $\boldsymbol{r}$, even if this is not written out explicitly.

The following assumptions on the mapping $\boldsymbol{F}$ are required in this textbook:

1. The mapping $\boldsymbol{F}$ is linear in the field $\boldsymbol{E}$, *i.e.*, for all real constants α and β and for all fields $\boldsymbol{E}$ and $\boldsymbol{E}'$ we have

$$\boldsymbol{F}(\alpha\boldsymbol{E} + \beta\boldsymbol{E}') = \alpha\boldsymbol{F}(\boldsymbol{E}) + \beta\boldsymbol{F}(\boldsymbol{E}')$$

2. The mapping $\boldsymbol{F}$ is causal, *i.e.*, for all τ, and for every field $\boldsymbol{E}$ such that $\boldsymbol{E}(t) = 0$, for $t < \tau$ we have

$$\boldsymbol{F}(\boldsymbol{E})(t) = \boldsymbol{0} \quad \text{for } t < \tau$$

3. The mapping $\boldsymbol{F}$ is invariant under time translations, *i.e.*, for every pair of fields $\boldsymbol{E}$ and $\boldsymbol{D}$, where the fields are related by $\boldsymbol{D}(t) = \boldsymbol{F}(\boldsymbol{E})(t)$, and every τ and pair of fields $\boldsymbol{E}'$ and $\boldsymbol{D}'$, defined by $\boldsymbol{E}'(t) = \boldsymbol{E}(t-\tau)$ and $\boldsymbol{D}'(t) = \boldsymbol{F}(\boldsymbol{E}')(t)$, we have

$$\boldsymbol{D}'(t) = \boldsymbol{D}(t-\tau)$$

[1]Note that an isotropic material does not imply that the material is homogeneous, *i.e.*, the material properties are the same at all points in the material. The isotropic properties of the material is a microscopic property, but the inhomogeneous variation of the material is a variation on a macroscopic length scale.

The property 1), of course, exclude all nonlinear phenomena that occur in many electromagnetic applications, but since most materials show linear behavior at sufficiently weak field strength, this is not a serious limitation. Causality, as it is stated in item 2), implies that there is no reaction (the electric flux density $\boldsymbol{D}$ or polarization $\boldsymbol{P}$) before its cause (the electric field $\boldsymbol{E}$). The property 3) implies that the material does not change (age), and the same result is obtained in an experiment that is repeated at a later time. These assumptions are the foundations to the constitutive relations used this book and all constitutive relations must satisfy these requirements to be physically sound.[2]

The starting point to the realization of the mapping $\boldsymbol{F}$ is the following Ansatz between the polarization of the material $\boldsymbol{P}$ and the electric field $\boldsymbol{E}$:

$$\boldsymbol{P}(t) = \epsilon_0 \int_{-\infty}^{\infty} \chi(t,t')\boldsymbol{E}(t')\ \mathrm{d}t'$$

The function $\chi(t,t')$ has the unit frequency. The electric flux density $\boldsymbol{D}$ then is, see (1.8),

$$\boldsymbol{D}(t) = \epsilon_0 \left\{ \boldsymbol{E}(t) + \int_{-\infty}^{\infty} \chi(t,t')\boldsymbol{E}(t')\ \mathrm{d}t' \right\}$$

This integral is, of course, linear, and, furthermore, it is in agreement with the assumption that the material is isotropic (no coupling between different vector components). The electric field is assumed to be zero before a given fixed time τ, *i.e.*, $\boldsymbol{E}(t) = \boldsymbol{0}$, $t < \tau$. We now address the requirements stated in items 2) and 3).

Causality, item 2), immediately implies that the function $\chi(t,t')$ have to be zero when $t < t'$, *i.e.*,

$$\chi(t,t') = 0, \quad t < t'$$

since $\boldsymbol{D}(t) = \boldsymbol{0}$, $t < \tau$. The range of integration therefore is limited to the interval $(-\infty, t]$. We have

$$\boldsymbol{D}(t) = \epsilon_0 \left\{ \boldsymbol{E}(t) + \int_{-\infty}^{t} \chi(t,t')\boldsymbol{E}(t')\ \mathrm{d}t' \right\}$$

This relation between the electric flux density $\boldsymbol{D}$ and the electric field $\boldsymbol{E}$ implies that $\boldsymbol{D}$ depends on the entire time history of the electric field — in other words, the material has a memory or show dispersion. This memory is characterized or modeled by the function $\chi(t,t')$.

In order to investigate what conditions the time invariance imply, we create two field pairs $\{\boldsymbol{D},\boldsymbol{E}\}$ and $\{\boldsymbol{D}',\boldsymbol{E}'\}$, defined by

$$\begin{cases} \boldsymbol{D}(t) = \epsilon_0 \left\{ \boldsymbol{E}(t) + \int_{-\infty}^{t} \chi(t,t')\boldsymbol{E}(t')\ \mathrm{d}t' \right\} \\ \boldsymbol{D}'(t) = \epsilon_0 \left\{ \boldsymbol{E}'(t) + \int_{-\infty}^{t} \chi(t,t')\boldsymbol{E}'(t')\ \mathrm{d}t' \right\} \end{cases}$$

[2] In fact, an additional assumption is also made, but we prefer not to explicitly list this assumption in order to avoid some technical mathematical machinery that is not used in this textbook. The cause, $\boldsymbol{D}$, must depend continuously on the electric field $\boldsymbol{E}$, *i.e.*, a small variation in the electric field cannot cause arbitrary large variation in the electric flux density. For more details, see [12].

where $\boldsymbol{E}'(t) = \boldsymbol{E}(t-\tau)$. Time invariance, property 3), implies $\boldsymbol{D}'(t) = \boldsymbol{D}(t-\tau)$. If the first integral is evaluated at time $t-\tau$ the two integrals are

$$\begin{cases} \boldsymbol{D}(t-\tau) = \epsilon_0 \left\{ \boldsymbol{E}(t-\tau) + \int_{-\infty}^{t-\tau} \chi(t-\tau, t')\boldsymbol{E}(t')\, \mathrm{d}t' \right\} \\ \boldsymbol{D}(t-\tau) = \epsilon_0 \left\{ \boldsymbol{E}(t-\tau) + \int_{-\infty}^{t} \chi(t, t')\boldsymbol{E}(t'-\tau)\, \mathrm{d}t' \right\} \end{cases}$$

Equating the two expressions implies

$$\int_{-\infty}^{t-\tau} \chi(t-\tau, t')\boldsymbol{E}(t')\, \mathrm{d}t' = \int_{-\infty}^{t} \chi(t, t')\boldsymbol{E}(t'-\tau)\, \mathrm{d}t' = \int_{-\infty}^{t-\tau} \chi(t, t'+\tau)\boldsymbol{E}(t')\, \mathrm{d}t'$$

where we have made a coordinate transformation in the last integral. This implies

$$\int_{-\infty}^{t-\tau} \left(\chi(t-\tau, t') - \chi(t, t'+\tau)\right) \boldsymbol{E}(t')\, \mathrm{d}t' = \boldsymbol{0}$$

The field $\boldsymbol{E}$ is here arbitrary, which implies

$$\chi(t-\tau, t') = \chi(t, t'+\tau)$$

for all t, τ and t', or equivalently[3]

$$\chi(t, t') = \chi(t-t', 0)$$

We observe that the function $\chi(t, t')$ is only a function of the difference in time, $t-t'$. Therefore, the constitutive relations of the material can be written[4]

$$\begin{cases} \boldsymbol{D}(t) = \epsilon_0 \left\{ \boldsymbol{E}(t) + \int_{-\infty}^{t} \chi(t-t')\boldsymbol{E}(t')\, \mathrm{d}t' \right\} \\ \boldsymbol{B}(t) = \mu_0 \boldsymbol{H}(t) \end{cases} \tag{2.2}$$

The function $\chi(t)$ is not defined for negative times t, but it is natural to extend the domain of definition to the entire real axis by defining $\chi(t) = 0$ for $t < 0$. This is in agreement with the causality property.

The function $\chi(t)$ is called the susceptibility function of the material (unit frequency), and it gives the polarization of the medium at a delta function excitation. To see this, let $\boldsymbol{E}(t) = \boldsymbol{E}_0\delta(t)$. We then have

$$\boldsymbol{D}(t) = \underbrace{\epsilon_0\delta(t)\boldsymbol{E}_0}_{\text{Momentaneous response}} + \underbrace{\epsilon_0\chi(t)\boldsymbol{E}_0}_{\text{Transient field}}$$

The susceptibility function $\chi(t)$ is the mathematical model of the memory properties of the medium and its dispersive effects.

[3]Take *e.g.*, $t' = 0$ and exchange $\tau \to t'$.

[4]We prefer to keep the notion χ even if it formally is not the same function as above.

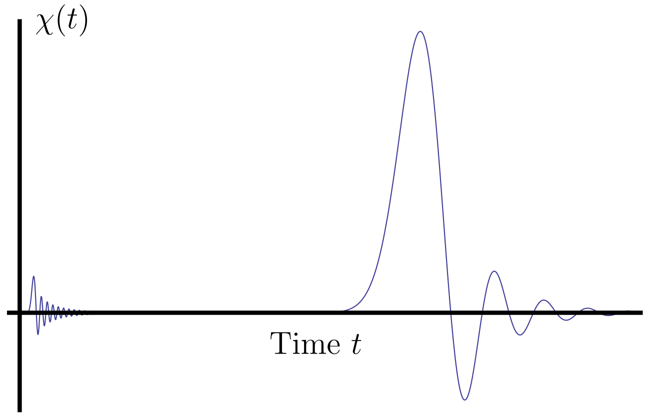


Figure 2.1: The susceptibility function $\chi(t)$ divided in two terms which exemplifies the optical response of the material. The time scale is arbitrary.

2.1.1 Optical response

If there are several physical processes that contribute to the electrical properties of the material, there are often several different time scales involved. For example, the interaction of the electric field with electrons, which are light, is a fast process compared to the more slow processes that occur when the more heavy atoms or molecules interact with the field. Usually, the polarization of the material contains a contribution which originates from very fast processes in the material. This contribution is usually called the optical response and can be modeled if we divide the susceptibility function $\chi(t)$ in a sum of two terms $\chi_1(t)$ and $\chi_2(t)$, *i.e.*,

$$\chi(t) = \chi_1(t) + \chi_2(t)$$

where $\chi_1(t)$ varies with the faster time scale compared to $\chi_2(t)$, which models the slower variations in the polarization effects of the material. We visualize this division in Figure 2.1. If the electric field only varies slowly compared to the variations in $\chi_1(t)$ it is convenient to include the effects of $\chi_1(t)$ as an instant contribution, similar to the first term in (2.2). The electric field $\boldsymbol{E}$ is then — compared to the variations in $\chi_1(t)$ — approximately constant. Therefore, we can take the electric field $\boldsymbol{E}$ outside the integrations and we obtain:

$$\begin{aligned}
\boldsymbol{D}(t) &= \epsilon_0 \left\{ \boldsymbol{E}(t) + \int_{-\infty}^{t} \left[\chi_1(t-t') + \chi_2(t-t')\right] \boldsymbol{E}(t')\ \mathrm{d}t' \right\} \\
&= \epsilon_0 \left\{ \boldsymbol{E}(t) + \int_{-\infty}^{t} \chi_1(t-t')\ \mathrm{d}t' \boldsymbol{E}(t) + \int_{-\infty}^{t} \chi_2(t-t')\boldsymbol{E}(t')\ \mathrm{d}t' \right\} \\
&= \epsilon_0 \left\{ \boldsymbol{E}(t) + \int_{0}^{\infty} \chi_1(t')\ \mathrm{d}t' \boldsymbol{E}(t) + \int_{-\infty}^{t} \chi_2(t-t')\boldsymbol{E}(t')\ \mathrm{d}t' \right\}
\end{aligned}$$

It is convenient to introduce the following notation:

$$\epsilon = 1 + \int_0^\infty \chi_1(t)\ \mathrm{d}t$$

which implies

$$\begin{cases} \boldsymbol{D}(t) = \epsilon_0 \left\{ \epsilon \boldsymbol{E}(t) + \int_{-\infty}^{t} \chi(t-t')\boldsymbol{E}(t')\, \mathrm{d}t' \right\} \\ \boldsymbol{B}(t) = \mu_0 \boldsymbol{H}(t) \end{cases} \tag{2.3}$$

where the index 2 on the susceptibility function is dropped for convenience.

Another equivalent way of introducing the optical response is to model the fast processes as a delta function contribution. To this end, we model the susceptibility function as

$$[\epsilon - 1]\, \delta(t) + \chi(t)$$

which in (2.2) also gives (2.3), but more directly. The first term in this expression is a model of the momentaneous response of the medium to an electric field excitation. In most situations, the origin of this term is due to charges with small inertia.

Comment 2.1
There is an inherent conflict with the concept of optical response in the sense that all macroscopic material models break down at the length scale associated with very fast transients. When the length scale of the fields becomes of the order of the size of the constitutive part of the material, *e.g.*, nanoscale, the material does not behave as a bulk material any longer, but the fields see the constitutive parts of the material as individual scatterers and not as a homogeneous material. Nevertheless, the concept of optical response is appropriate when used and employed with care. At higher frequencies still, the entire classical modeling of materials becomes inaccurate — this is the realm of quantum phenomena.

2.1.2 Conductivity

The difference between bounded and free charges is apparent for static (time independent) fields. Bounded charges polarize the medium, and free charges contributes to the currents in the medium. This difference is wiped out for general time dependent fields as the analysis in this section shows.

In general, there are two kind of current densities, $\boldsymbol{J}_{\mathrm{inf}}$ and $\boldsymbol{J}_{\mathrm{imp}}$ — the impressed current density and the induced current density, respectively. The impressed current density are supplied by external sources, which are controlled externally and they are not caused by the existing electromagnetic fields. The induced currents, on the other hand, are caused by the present electromagnetic fields. It is the latter, the induced current density, that we now address and model.

Several media, especially many metals, have easily movable charges. Ohm's law with conductivity σ

$$\boldsymbol{J} = \sigma \boldsymbol{E} \tag{2.4}$$

is often used as a model of charge transport in these media. A somewhat more general set of constitutive relations than (2.3) and (2.4) is

$$\begin{cases} \boldsymbol{D}(t) = \epsilon_0 \left\{ \epsilon \boldsymbol{E}(t) + \int_{-\infty}^{t} \chi(t-t')\boldsymbol{E}(t') \, \mathrm{d}t' \right\} \\ \boldsymbol{J}(t) = \sigma \boldsymbol{E}(t) + \epsilon_0 \int_{-\infty}^{t} \Sigma(t-t')\boldsymbol{E}(t') \, \mathrm{d}t' \\ \boldsymbol{B}(t) = \mu_0 \mu \boldsymbol{H}(t) \end{cases} \tag{2.5}$$

In addition to the instantaneous term σ between the current density $\boldsymbol{J}$ and the electric field $\boldsymbol{E}$, The extended Ohm's law contains a term that models memory effects or dispersion effects. This model has, in addition to the dispersive effects described by the susceptibility function $\chi(t)$ (bounded charges), also dispersive effects of the free charges in the function $\Sigma(t)$. The coupling between the $\boldsymbol{B}$- and the $\boldsymbol{H}$-fields is also more general than in (2.3). The real constant μ is a measure of the momentaneous magnetic properties of the medium.

In this section, we prove that the effect of easily mobile charges, which we model by the conductivity σ and the function $\Sigma(t)$, in fact can be included in the susceptibility function $\chi(t)$. Physically, this means that we classify the easily mobile charges as bound charges. This is of course an arbitrary choice, provided that all quantities that could be observed physically, such as electric and magnetic fields, are unaffected by this rearrangement. Conversely, we can prove that all dispersive effects, that is modeled by the susceptibility function $\chi(t)$, can be transferred to an effective conductivity and a dispersive term. In this case, we classify the bound charges modeled by the susceptibility $\chi(t)$, as easily mobile charges with Ohm's law. This is also possible, provided that the physical quantities are unaffected by this rearrangement. To see this, we prove that there is non-uniqueness in the constitutive relations as they are formulated in (2.5).

The constitutive relations in (2.5) are not uniquely determined, but every choice of the constitutive relations on the form

$$\begin{cases} \boldsymbol{D}(t) = \epsilon_0 \left\{ \epsilon \boldsymbol{E}(t) + \int_{-\infty}^{t} \left(\chi(t-t') + f(t-t')\right) \boldsymbol{E}(t') \, \mathrm{d}t' \right\} \\ \boldsymbol{J}(t) = \left(\sigma - \epsilon_0 f(0^+)\right) \boldsymbol{E}(t) + \epsilon_0 \int_{-\infty}^{t} \left(\Sigma(t-t') - \frac{\partial f(t-t')}{\partial t} \right) \boldsymbol{E}(t') \, \mathrm{d}t' \\ \boldsymbol{B}(t) = \mu_0 \mu \boldsymbol{H}(t) \end{cases}$$

where $f(t)$ is an arbitrary (differentiable) function ($f(t) = 0$, $t < 0$ and $f(0^+) = \lim_{t\to 0, t>0} f(t)$), gives the same right-hand side in Ampère's law. We see this immediately by insertion in Ampère's law.

$$\begin{aligned} \nabla \times \boldsymbol{H}(t) &= \boldsymbol{J} + \frac{\partial \boldsymbol{D}(t)}{\partial t} \\ &= \left(\sigma - \epsilon_0 f(0^+)\right) \boldsymbol{E}(t) + \epsilon_0 \int_{-\infty}^{t} \left(\Sigma(t-t') - \frac{\partial f(t-t')}{\partial t} \right) \boldsymbol{E}(t') \, \mathrm{d}t' \\ &\quad + \epsilon_0 \left\{ \epsilon \frac{\partial \boldsymbol{E}(t)}{\partial t} + \frac{\partial}{\partial t} \int_{-\infty}^{t} \left(\chi(t-t') + f(t-t')\right) \boldsymbol{E}(t') \, \mathrm{d}t' \right\} \end{aligned}$$

Differentiation of the integral leads to

$$\begin{aligned}
\nabla \times \boldsymbol{H}(t) &= \left(\sigma - \epsilon_0 f(0^+)\right) \boldsymbol{E}(t) + \epsilon_0 \int_{-\infty}^{t} \left(\Sigma(t-t') - \frac{\partial f(t-t')}{\partial t}\right) \boldsymbol{E}(t') \, \mathrm{d}t' \\
&\quad + \epsilon_0 \left\{ \epsilon \frac{\partial \boldsymbol{E}(t)}{\partial t} + \frac{\partial}{\partial t} \int_{-\infty}^{t} \chi(t-t') \boldsymbol{E}(t') \, \mathrm{d}t' \right\} \\
&\quad + \epsilon_0 f(0^+) \boldsymbol{E}(t) + \epsilon_0 \int_{-\infty}^{t} \frac{\partial f(t-t')}{\partial t} \boldsymbol{E}(t') \, \mathrm{d}t' \\
&= \sigma \boldsymbol{E}(t) + \epsilon_0 \int_{-\infty}^{t} \Sigma(t-t') \boldsymbol{E}(t') \, \mathrm{d}t' \\
&\quad + \epsilon_0 \left\{ \epsilon \frac{\partial \boldsymbol{E}(t)}{\partial t} + \frac{\partial}{\partial t} \int_{-\infty}^{t} \chi(t-t') \boldsymbol{E}(t') \, \mathrm{d}t' \right\}
\end{aligned}$$

The function $f(t)$ does not affect Ampère's law. Every choice of $f(t)$ gives the same right-hand side in Ampère's law.

Each choice of $f(t)$ is a reclassification of bound charges (modeled by the electric flux density $\boldsymbol{D}$) and the free charges (modeled by the charge density $\boldsymbol{J}$). To illustrate this reclassification and the non-uniqueness in the constitutive relations, we show two special cases in the next two examples.

Example 2.1
The dispersion model: In this example we show that the constitutive relations given by (2.5) can be transformed such that $\boldsymbol{J} = \boldsymbol{0}$, *i.e.*, the whole contribution from the charge density $\boldsymbol{J}$ is included in the electric flux density $\boldsymbol{D}$. The obtain this, choose the function $f(t)$ as

$$f(t) = H(t) \left[\frac{\sigma}{\epsilon_0} + \int_0^t \Sigma(t') \, \mathrm{d}t' \right]$$

where $H(t)$ is the Heaviside step function. This implies

$$\begin{cases} f(0^+) = \dfrac{\sigma}{\epsilon_0} \\ f'(t) = \Sigma(t), \quad t > 0 \end{cases}$$

The constitutive relations then become

$$\begin{cases} \boldsymbol{D}(t) = \epsilon_0 \left\{ \epsilon \boldsymbol{E}(t) + \displaystyle\int_{-\infty}^{t} \left(\chi(t-t') + f(t-t')\right) \boldsymbol{E}(t') \, \mathrm{d}t' \right\} \\ \boldsymbol{J}(t) = \boldsymbol{0} \\ \boldsymbol{B}(t) = \mu_0 \mu \boldsymbol{H}(t) \end{cases}$$

This set of constitutive relations are called the dispersion model, since the contribution from the current density $\boldsymbol{J}$ is zero. These constitutive relations are equivalent to the ones in (2.5), in that they give the same right-hand side expression in Ampère's law.

Therefore, there is no loss of generality to let the contribution of the free charges, *i.e.*, Ohm's law, (2.4), be included in the susceptibility function $\chi_{\text{new}}(t)$

$$\chi_{\text{new}}(t) = H(t) \frac{\sigma}{\epsilon_0} + \chi(t)$$

and the effects of the easily mobile charges have been absorbed in the susceptibility function. This new susceptibility function is a rearrangement of the free charges such that they are now included in the contribution of the bound charges. ∎

Example 2.2
The conductivity model: We can also choose the function $f(t)$ such that another extreme is obtained, the conductivity model. Choose the function $f(t)$ as

$$f(t) = -\chi(t)$$

We get the following constitutive relations

$$\begin{cases} \boldsymbol{D}(t) = \epsilon_0\epsilon\boldsymbol{E}(t) \\ \boldsymbol{J}(t) = \left(\sigma + \epsilon_0\chi(0^+)\right)\boldsymbol{E}(t) + \epsilon_0 \int_{-\infty}^{t} \left(\Sigma(t-t') + \frac{\partial\chi(t-t')}{\partial t}\right)\boldsymbol{E}(t')\,\mathrm{d}t' \\ \boldsymbol{B}(t) = \mu_0\mu\boldsymbol{H}(t) \end{cases} \tag{2.6}$$

In this particular set of constitutive relations, we have a susceptibility function $\chi(t)$ that is zero. All effects of dispersion is here collected into the current density $\boldsymbol{J}$. These constitutive relations are equivalent with the ones given in (2.5). ∎

Of course, other choices of the function $f(t)$ are possible, which are a mixture of these two extreme cases. In this book, the dispersion model is usually used.

We end the analysis of isotropic materials by inverting the relation between the electric flux density $\boldsymbol{D}(\boldsymbol{r}, t)$ and the electric field $\boldsymbol{E}(\boldsymbol{r}, t)$. The relation (2.3), *i.e.*,

$$\boldsymbol{D}(t) = \epsilon_0\left\{\epsilon\boldsymbol{E}(t) + \int_{-\infty}^{t}\chi(t-t')\boldsymbol{E}(t')\,\mathrm{d}t'\right\}$$

can be inverted by applying the resolvent of the susceptibility function $\chi(t)$. The resolvent kernel $\Upsilon(t)$ satisfies the resolvent equation

$$\frac{1}{\epsilon}\chi(t) + \epsilon\Upsilon(t) + \int_{0}^{t}\chi(t-t')\Upsilon(t')\,\mathrm{d}t' = 0 \tag{2.7}$$

This solution is uniquely soluble for $t > 0$, and for $t < 0$, $\Upsilon(t) = 0$. In terms of the resolvent kernel, the inverse is

$$\boldsymbol{E}(t) = \frac{1}{\epsilon_0}\left\{\frac{1}{\epsilon}\boldsymbol{D}(t) + \int_{-\infty}^{t}\Upsilon(t-t')\boldsymbol{D}(t')\,\mathrm{d}t'\right\} \tag{2.8}$$

This relation is easily verified by inserting the constitutive relation into (2.8) and using the resolvent equation, (2.7).

2.2 Examples

We exemplify the constitutive relations in this subsection. For convenience, the dispersion model is used. In a first example, we describe Debye's model, or the relaxation model, which is used to model interaction of the electromagnetic field in polar liquids, *i.e.*, liquids with molecules with permanent electric dipole moment. In a second example Lorentz' model, or the resonance model, is described. This model is often used for electromagnetic interaction in solids.

2.2.1 Debye's model

The first example is Debye's model[5] for a dispersive material. This model is adequate for polar liquids, which have molecules with permanent electric dipole moment.

We assume the molecules (or the atoms if that is appropriate) have a permanent electric dipole moment $\boldsymbol{p}$, which normally is arbitrarily oriented due to thermal fluctuations, see Figure 2.2 (left figure). The polarization of the material $\boldsymbol{P}$ is defined as the total electric dipole moment per unit volume, *i.e.*,

$$\boldsymbol{P} = \lim_{\Delta V \to 0} \frac{\sum_i \boldsymbol{p}_i}{\Delta V}$$

where the sum over the index i is over all molecules located inside the volume ΔV. In an undisturbed state without electric field we have $\boldsymbol{P} = \boldsymbol{0}$. The polarization $\boldsymbol{P}$ changes due to two competing processes.[6]

1. One process striving to align the polarization $\boldsymbol{P}$ parallel to the applied electric field $\boldsymbol{E}$.[7] We assume that the rate of changes in $\boldsymbol{P}$ due to this process is proportional to $\epsilon_0 \alpha \boldsymbol{E}$. The frequency $\alpha > 0$ is a measure of this change.

2. The second competing process trying to disorient the polarization. If $\tau > 0$ is the relaxation time for this process, the rate of changes in $\boldsymbol{P}$ are assumed to be proportional to $-\boldsymbol{P}/\tau$.

Debye's model for molecules with permanent electric dipole moment is illustrated in Figure 2.2.

In total the rate of changes in the polarization $\boldsymbol{P}$ is then

$$\boxed{\frac{\mathrm{d}\boldsymbol{P}(t)}{\mathrm{d}t} = \epsilon_0 \alpha \boldsymbol{E}(t) - \frac{\boldsymbol{P}(t)}{\tau}} \tag{2.9}$$

This equation models the interaction of the electromagnetic field with the charges of the material, in this case the permanent electric dipoles of the material. We also call this equation the dynamics of the charges in the material.

From the previous section we know that the dispersive effects of the material can be written as, see (2.3) and (1.8) (notice that we are using the dispersion model)

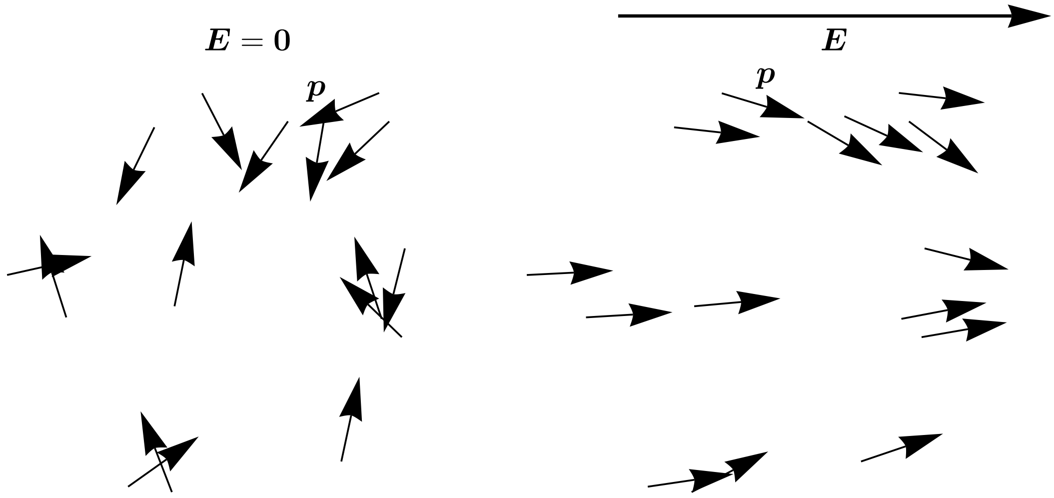
$$\boldsymbol{P}(t) = \boldsymbol{D}(t) - \epsilon_0 \boldsymbol{E}(t) = \epsilon_0 \left\{ (\epsilon - 1)\, \boldsymbol{E}(t) + \int_{-\infty}^{t} \chi(t - t') \boldsymbol{E}(t')\ \mathrm{d}t' \right\}$$

---

[5] Peter Joseph William Debye (1884–1966), Dutch physicist and physical chemist.

[6] The intensity of these processes depend strongly on many exterior parameters, *e.g.*, the temperature T. In this treatment we assume that these exterior parameters are fixed. The temperature dependence is investigated in Example 2.2 on page 54.

[7] Strictly speaking, the electric field $\boldsymbol{E}$, which affects the dipoles, is usually not identical to the exterior field, but it is modified due to the presence of the media. Suitable corrections for the difference between this local field and the exterior field can be made, *i.e.*, the Clausius-Mossotti (Lorenz-Lorentz) law.



Without external electric field With external electric field

Figure 2.2: Molecules with permanent electric dipole moment with and without an aligning exterior electric field $\boldsymbol{E}$.

The time derivative of this expression is

$$\frac{1}{\epsilon_0}\frac{\mathrm{d}\boldsymbol{P}(t)}{\mathrm{d}t}=(\epsilon-1)\,\frac{\mathrm{d}\boldsymbol{E}(t)}{\mathrm{d}t}+\chi(0^+)\boldsymbol{E}(t)+\int_{-\infty}^{t}\chi'(t-t')\boldsymbol{E}(t')\;\mathrm{d}t'$$

where $\chi'(t)$ is the time derivative of $\chi(t)$ and $\chi(0^+)=\lim_{t\to 0, t>0}\chi(t)$, which might be non-zero, *i.e.*, $\chi(t)$ is discontinuous at zero. Insert in (2.9) and collect terms. We get

$$\begin{aligned}&(\epsilon-1)\,\frac{\mathrm{d}\boldsymbol{E}(t)}{\mathrm{d}t}+\left(\chi(0^+)-\alpha+\frac{1}{\tau}\,(\epsilon-1)\right)\boldsymbol{E}(t)\\&+\int_{-\infty}^{t}\left(\chi'(t-t')+\frac{1}{\tau}\chi(t-t')\right)\boldsymbol{E}(t')\;\mathrm{d}t'=\mathbf{0}\end{aligned}$$

The field $\boldsymbol{E}$ is arbitrary, which implies that the coefficient in front of each term must be zero, *i.e.*,

$$\begin{cases}\epsilon-1=0\\ \chi(0^+)-\alpha+\dfrac{1}{\tau}\,(\epsilon-1)=0\\ \chi'(t)+\dfrac{1}{\tau}\chi(t)=0\end{cases}$$

If we simplify, we get

$$\begin{cases}\epsilon=1\\ \chi(0^+)=\alpha\\ \chi'(t)+\dfrac{1}{\tau}\chi(t)=0\end{cases}$$

The first condition, $\epsilon = 1$, implies that there is no optical response,[8] and from the last two conditions on $\chi(t)$ we easily get

$$\chi(t) = \alpha \mathrm{e}^{-t/\tau} \qquad t \geq 0$$

The final expressions for Debye's model is

$$\begin{cases} \boldsymbol{D}(t) = \epsilon_0 \left\{ \boldsymbol{E}(t) + \int_{-\infty}^{t} \chi(t-t')\boldsymbol{E}(t') \,\mathrm{d}t' \right\} \\ \chi(t) = H(t)\alpha \mathrm{e}^{-t/\tau} \end{cases} \tag{2.10}$$

The susceptibility function $\chi(t)$ for this model is an exponentially decreasing function.

Example 2.3
The constitutive relations in equation (2.10) are given in the dispersion model, see Example 2.1, *i.e.*,

$$\begin{cases} \boldsymbol{D}(t) = \epsilon_0 \left\{ \boldsymbol{E}(t) + \int_{-\infty}^{t} \mathrm{e}^{-(t-t')/\tau} \boldsymbol{E}(t') \,\mathrm{d}t' \right\} \\ \boldsymbol{J}(t) = \boldsymbol{0} \end{cases}$$

In Example 2.2 we showed that with a special choice of the function $f(t)$ all effects on the charges in the material could be included in Ohm's law. The rewrite the constitutive relations (2.10) in the conductivity model, we choose $f(t) = -\chi(t)$. In this case we get

$$f(t) = -H(t)\alpha \mathrm{e}^{-t/\tau}$$

and

$$\begin{cases} f(0^+) = -\alpha \\ \dfrac{\partial f(t)}{\partial t} = \dfrac{\alpha}{\tau}\mathrm{e}^{-t/\tau}, \quad t > 0 \end{cases}$$

Example 2.2 now gives the constitutive relations for a Debye material in the conductivity model, see (2.6)

$$\begin{cases} \boldsymbol{D}(t) = \epsilon_0 \boldsymbol{E}(t) \\ \boldsymbol{J}(t) = \epsilon_0 \alpha \boldsymbol{E}(t) - \epsilon_0 \dfrac{\alpha}{\tau} \int_{-\infty}^{t} \mathrm{e}^{-(t-t')/\tau} \boldsymbol{E}(t') \,\mathrm{d}t' \end{cases}$$

∎

2.2.2 Lorentz' model

We assume the material has bound charges (usually electrons), which interact with the nucleus of the atom. The atoms are usually arranged in a lattice structure, but they do not necessarily have to be that. Amorphous materials are also possible.

The charges have charge q and mass m and are assumed to be affected by three different forces:

[8] This is expected, since we have not assumed any other processes, and therefore all interaction is resolved.

1. An electric force $\boldsymbol{F}_1 = q\boldsymbol{E}$ from the applied electric field $\boldsymbol{E}$.[9]

2. A restoring force proportional to the displacement of the charge from its equilibrium, *i.e.*, a harmonic force, $\boldsymbol{F}_2 = -m\omega_0^2\boldsymbol{r}$, where $\omega_0 \geq 0$ is the harmonic frequency, and $\boldsymbol{r}$ is the displacement of the charge from its equilibrium.

3. A frictional force proportional to the velocity of the charge $\boldsymbol{r}'(t)$, $\boldsymbol{F}_3 = -m\nu\boldsymbol{r}'(t)$, where $\nu \geq 0$ is the collision frequency.

We assume that the dynamics of the charges is described by the laws of classic mechanic. Newton's acceleration law gives

$$m\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 = q\boldsymbol{E} - m\omega_0^2\boldsymbol{r} - m\nu\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}$$

or

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} + \nu\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} + \omega_0^2\boldsymbol{r} = \frac{q}{m}\boldsymbol{E}$$

Introduce the polarization $\boldsymbol{P}$ of the material defined by

$$\boldsymbol{P} = Nq\boldsymbol{r}$$

where N is the number charges per unit volume.[10] Rewrite the dynamics in the polarization vector as

$$\boxed{\frac{\mathrm{d}^2\boldsymbol{P}(t)}{\mathrm{d}t^2} + \nu\frac{\mathrm{d}\boldsymbol{P}(t)}{\mathrm{d}t} + \omega_0^2\boldsymbol{P}(t) = \frac{Nq^2}{m}\boldsymbol{E}(t)} \tag{2.11}$$

In the same way as for Debye's model, we introduce the dispersion of the material as (2.3) and (1.8).

$$\boldsymbol{P}(t) = \boldsymbol{D}(t) - \epsilon_0\boldsymbol{E}(t) = \epsilon_0\left\{(\epsilon - 1)\,\boldsymbol{E}(t) + \int_{-\infty}^{t}\chi(t-t')\boldsymbol{E}(t')\,\mathrm{d}t'\right\}$$

The first and second derivative of this expression is

$$\begin{aligned}
\frac{1}{\epsilon_0}\frac{\mathrm{d}\boldsymbol{P}(t)}{\mathrm{d}t} &= (\epsilon - 1)\,\frac{\mathrm{d}\boldsymbol{E}(t)}{\mathrm{d}t} + \chi(0^+)\boldsymbol{E}(t) + \int_{-\infty}^{t}\chi'(t-t')\boldsymbol{E}(t')\,\mathrm{d}t' \\
\frac{1}{\epsilon_0}\frac{\mathrm{d}^2\boldsymbol{P}(t)}{\mathrm{d}t^2} &= (\epsilon - 1)\,\frac{\mathrm{d}^2\boldsymbol{E}(t)}{\mathrm{d}t^2} + \chi(0^+)\frac{\mathrm{d}\boldsymbol{E}(t)}{\mathrm{d}t} + \chi'(0^+)\boldsymbol{E}(t) \\
&\quad + \int_{-\infty}^{t}\chi''(t-t')\boldsymbol{E}(t')\,\mathrm{d}t'
\end{aligned}$$

where $\chi'(t)$ and $\chi''(t)$ are the first and second derivative of $\chi(t)$ w.r.t. time, respectively, and $\chi(0^+) = \lim_{t\to 0, t>0}\chi(t)$ and $\chi'(0^+) = \lim_{t\to 0, t>0}\chi'(t)$, which might be

[9]See comment under Footnote 7 on page 24. If the medium is not dense, *e.g.*, a gas, the difference is small.

[10]We assume this quantity is constant in time, which is an approximation.

non-zero. Enter these expressions in the dynamics, (2.11), and collect terms. We get

$$\begin{aligned}&(\epsilon-1)\frac{\mathrm{d}^2\boldsymbol{E}(t)}{\mathrm{d}t^2}+\big(\chi(0^+)+\nu(\epsilon-1)\big)\frac{\mathrm{d}\boldsymbol{E}(t)}{\mathrm{d}t}\\&+\left(\chi'(0^+)+\nu\chi(0^+)+\omega_0^2(\epsilon-1)-\frac{Nq^2}{m\epsilon_0}\right)\boldsymbol{E}(t)\\&+\int_{-\infty}^{t}\big(\chi''(t-t')+\nu\chi'(t-t')+\omega_0^2\chi(t-t')\big)\boldsymbol{E}(t')\ \mathrm{d}t'=\mathbf{0}\end{aligned}$$

The field $\boldsymbol{E}$ is arbitrary, and therefore the coefficients in front of every term must be identically zero, *i.e.*,

$$\begin{cases}\epsilon-1=0\\ \chi(0^+)+\nu(\epsilon-1)=0\\ \chi'(0^+)+\nu\chi(0^+)+\omega_0^2(\epsilon-1)-\omega_{\mathrm{p}}^2=0\\ \chi''(t)+\nu\chi'(t)+\omega_0^2\chi(t)=0\end{cases}$$

where

$$\omega_{\mathrm{p}}=\sqrt{\frac{Nq^2}{m\epsilon_0}}$$

is the plasma frequency of the material. We get

$$\begin{cases}\epsilon=1\\ \chi(0^+)=0\\ \chi'(0^+)=\omega_{\mathrm{p}}^2\\ \chi''(t)+\nu\chi'(t)+\omega_0^2\chi(t)=0\end{cases}$$

The first expression, $\epsilon=1$, indicates that we have no optical response, which is expected since there are no unresolved processes in the material. The other conditions on $\chi(t)$ is a initial value problem for a second order ordinary differential equation in time with the unique solution

$$\chi(t)=\frac{\omega_{\mathrm{p}}^2}{\nu_0}\mathrm{e}^{-\frac{\nu t}{2}}\sin\nu_0 t\qquad t\geq 0$$

where $\nu_0^2=\omega_0^2-\nu^2/4$.

The final expression of the constitutive relations for Lorentz' model is

$$\begin{cases}\boldsymbol{D}(t)=\epsilon_0\left\{\boldsymbol{E}(t)+\displaystyle\int_{-\infty}^{t}\chi(t-t')\boldsymbol{E}(t')\ \mathrm{d}t'\right\}\\ \chi(t)=H(t)\dfrac{\omega_{\mathrm{p}}^2}{\nu_0}\mathrm{e}^{-\frac{\nu t}{2}}\sin\nu_0 t\end{cases}\tag{2.12}$$

where we explicitly have set $\chi(t)=0$, $t<0$ by introducing the Heaviside function $H(t)$. An example of the the susceptibility function for Lorentz' model is given in Figure 2.3.

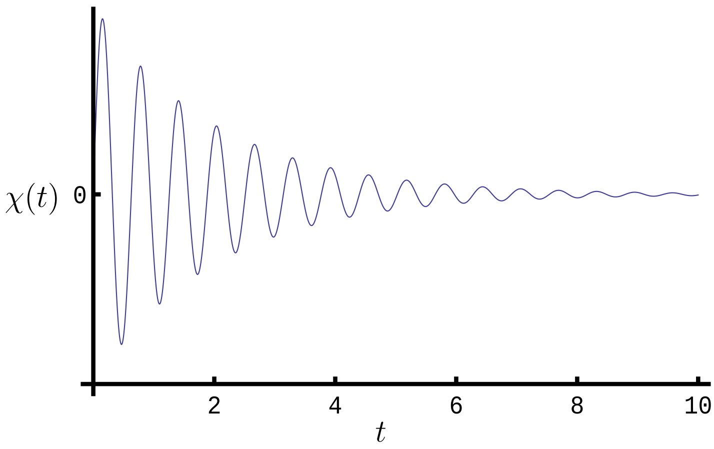


Figure 2.3: An example of a susceptibility function $\chi(t)$ for Lorentz' model. The time scale is in arbitrary units.

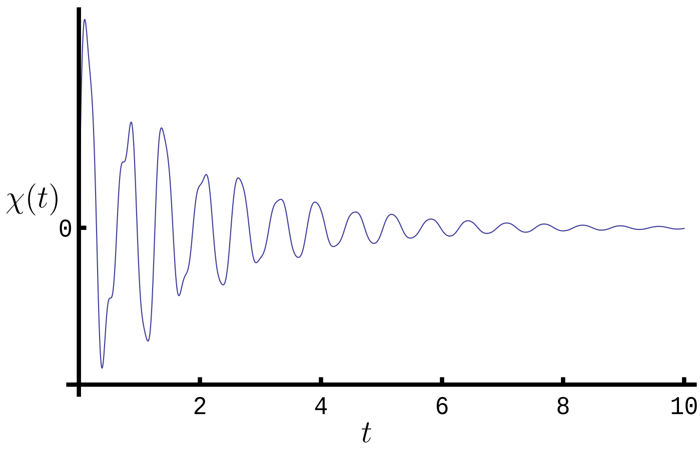


Figure 2.4: An example of a susceptibility function $\chi(t)$ for Lorentz' model with three resonance frequencies. The time scale is in arbitrary units.

In a more general situation, several different processes contribute to the susceptibility function, where each contribution is a resonance model derived above. The processes have different frequencies $\omega_{\mathrm{p}i}$, ω_{0i} and ν_i. The susceptibility function for the general resonance model is a sum of all these contributions.

$$\chi(t) = H(t) \sum_{i=1}^{M} \frac{\omega_{\mathrm{p}i}^2}{\sqrt{\omega_{0i}^2 - \nu_i^2/4}} \mathrm{e}^{-\frac{\nu_i t}{2}} \sin \sqrt{\omega_{0i}^2 - \nu_i^2/4}t$$

An example on a susceptibility function for Lorentz' model with several frequencies is depicted in Figure 2.4.

If the friction on the charges can be neglected, *i.e.*, $\nu \to 0$, then $\nu_0 = \omega_0$ and (2.12) is simplified into

$$\chi(t) = H(t) \frac{\omega_{\mathrm{p}}^2}{\omega_0} \sin \omega_0 t$$

In this case $\chi(t)$ is an undamped sinusoidal function.

On the other hand, if the restoring force can be neglected[11], *i.e.*, $\omega_0 \to 0$, then $\nu_0 = \mathrm{i}\nu/2$ and (2.12) becomes

$$\chi(t) = H(t)\frac{\omega_\mathrm{p}^2}{\nu}\left(1 - \mathrm{e}^{-\nu t}\right)$$

Without the restoring force the electron[12] is a free conducting electron, which is not bounded to any nucleus. In this case it is possible to identify a conductivity σ of the material as in Ohm's law, see (2.4) or more generally (2.6). We start with $\sigma = \Sigma(t) = 0$ in our conductivity model, *i.e.*,

$$\begin{cases} \boldsymbol{D}(t) = \epsilon_0 \epsilon \boldsymbol{E}(t) \\ \boldsymbol{J}(t) = \epsilon_0 \chi(0^+)\boldsymbol{E}(t) + \epsilon_0 \displaystyle\int_{-\infty}^{t} \frac{\partial \chi(t-t')}{\partial t} \boldsymbol{E}(t')\ \mathrm{d}t' \end{cases}$$

and we wish to identify the susceptibility function $\chi(t)$ with an conductivity in this equation. In our case $\epsilon = 1$, $\chi(0^+) = 0$ and $\chi'(t) = H(t)\omega_\mathrm{p}^2\mathrm{e}^{-\nu t}$ and we get

$$\boldsymbol{J}(t) = \epsilon_0 \omega_\mathrm{p}^2 \int_{-\infty}^{t} \mathrm{e}^{-\nu(t-t')} \boldsymbol{E}(t')\ \mathrm{d}t'$$

If the field $\boldsymbol{E}(t')$ varies slowly on a time scale $1/\nu$, we can approximate the integral by evaluating the field $\boldsymbol{E}(t')$ at the time t and move the field outside the integral.[13] We get

$$\boldsymbol{J}(t) = \epsilon_0 \omega_\mathrm{p}^2 \boldsymbol{E}(t) \int_{-\infty}^{t} \mathrm{e}^{-\nu(t-t')}\ \mathrm{d}t' = \frac{\epsilon_0 \omega_\mathrm{p}^2}{\nu} \boldsymbol{E}(t)$$

A direct comparison in this special case gives the conductivity of the material

$$\sigma = \frac{\epsilon_0 \omega_\mathrm{p}^2}{\nu}$$

We immediately see that if the friction of the electrons increases (ν increases) this implies that the conductivity decreases (resistivity $= 1/\sigma$ increases), and vice versa. This observation is intuitively well motivated.

Example 2.4

Suppose a medium is described by the Lorentz model with negligible losses (collision frequency $\nu \approx 0$). The susceptibility function, which is a model of the dispersive effects in the medium, is then

$$\chi(t) = H(t)\frac{\omega_p^2}{\omega_0}\sin\omega_0 t$$

[11]This special case is often denoted Drude's model, and it was an early model for conduction of charges in metals.

[12]We assume that the charges in the material are electrons.

[13]This very analogous to the argument used in Section 2.1.1 where we identified the optical response of the material.

In many practical situations it is more convenient to work with constitutive relations that do not contain any convolution integrals, but only contain the field evaluated at the time t.

Integration by parts of the electric flux density $\boldsymbol{D}(t)$ can be expressed as (provided the series converges):

$$\begin{aligned}\boldsymbol{D}(t) =&\epsilon_0 \left\{ \boldsymbol{E}(t) + \int_{-\infty}^{t} \chi(t-t')\boldsymbol{E}(t')\ \mathrm{d}t' \right\}\\ =&\epsilon_0 \left\{ \boldsymbol{E}(t) + \frac{\omega_p^2}{\omega_0} \int_{-\infty}^{t} \sin\omega_0(t-t')\boldsymbol{E}(t')\ \mathrm{d}t' \right\} = \epsilon_0 \sum_{n=0}^{\infty} A_n \boldsymbol{E}^{(n)}(t)\end{aligned}$$

where

$$A_n = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \delta_{n,0} + (-1)^{n/2} \dfrac{\omega_{\mathrm{p}}^2}{\omega_0^{n+2}}, & \text{if } n \text{ is even} \end{cases}$$

The first term $A_0 = 1 + \omega_{\mathrm{p}}^2/\omega_0^2 = \epsilon$ acts as an optical response. To estimate the error that we use if we replace the entire series by just the optical response, assume the electric field $\boldsymbol{E}(t)$ oscillates harmonically, say $\boldsymbol{E}(t) = \boldsymbol{E}_0\cos(\omega t + \alpha)$, with angular frequency $\omega = 1\cdot 10^{10}$ rad/s and that the resonance frequency and the plasma frequency of the Lorentz material are $\omega_0 = 1\cdot 10^{14}$ rad/s, and $\omega_p = 3\cdot 10^{14}$ rad/s, respectively. The relative error in the $\boldsymbol{D}(t)$-field if $\boldsymbol{D}(t) = \epsilon_0\epsilon\boldsymbol{E}(t)$ is used instead of the complete expression is

$$\frac{\left|A_0\boldsymbol{E}(t) - \sum_{n=0}^{\infty} A_n \boldsymbol{E}^{(n)}(t)\right|}{\left|\sum_{n=0}^{\infty} A_n \boldsymbol{E}^{(n)}(t)\right|} = \frac{\frac{\omega_{\mathrm{p}}^2}{\omega_0^2}\sum_{n=1}^{\infty}\frac{\omega^{2n}}{\omega_0^{2n}}}{1 + \frac{\omega_{\mathrm{p}}^2}{\omega_0^2} + \frac{\omega_{\mathrm{p}}^2}{\omega_0^2}\sum_{n=1}^{\infty}\frac{\omega^{2n}}{\omega_0^{2n}}} = \frac{9\frac{10^{-8}}{1-10^{-8}}}{10 + 9\frac{10^{-8}}{1-10^{-8}}} \approx 9\cdot 10^{-9}$$

The error that is made by approximating the convolution integral by an optical response $\epsilon = 10$ is negligible for this frequency ω. For a higher frequency the error becomes much larger. Compare the result of this example with the definition and analysis of the optical response on page 19. ■

2.3 General linear media with dispersion

So far, we have neglected the magnetic effects and any coupling between electric and magnetic phenomena. If these effects are essential, we follow the same line of analysis as in Section 2.1. We don't give the details of this analysis here, but refer to the literature [12].

A general set of linear constitutive relations, which allows for coupling between the electric and the magnetic fields, is the following Ansatz:

$$\begin{cases} \boldsymbol{D}(t) = \epsilon_0\Big\{ \boldsymbol{\epsilon}\cdot\boldsymbol{E}(t) + \displaystyle\int_{-\infty}^{t} \boldsymbol{\chi}_{\mathrm{ee}}(t-t')\cdot\boldsymbol{E}(t')\ \mathrm{d}t' \\ \qquad\qquad + \eta_0 \displaystyle\int_{-\infty}^{t} \boldsymbol{\chi}_{\mathrm{em}}(t-t')\cdot\boldsymbol{H}(t')\ \mathrm{d}t' \Big\} \\ \boldsymbol{B}(t) = \dfrac{1}{c_0}\Big\{ \displaystyle\int_{-\infty}^{t} \boldsymbol{\chi}_{\mathrm{me}}(t-t')\cdot\boldsymbol{E}(t')\ \mathrm{d}t' \\ \qquad\qquad + \eta_0\boldsymbol{\mu}\cdot\boldsymbol{H}(t) + \eta_0\displaystyle\int_{-\infty}^{t} \boldsymbol{\chi}_{\mathrm{mm}}(t-t')\cdot\boldsymbol{H}(t')\ \mathrm{d}t' \Big\} \end{cases} \tag{2.13}$$

or in Cartesian components ($i = 1, 2, 3$):

$$\begin{cases} D_i(t) = \epsilon_0 \displaystyle\sum_{j=1}^{3} \Big\{ \epsilon_{ij} E_j(t) + \int_{-\infty}^{t} \chi_{\mathrm{ee}ij}(t-t') E_j(t')\ \mathrm{d}t' \\ \qquad\qquad + \eta_0 \displaystyle\int_{-\infty}^{t} \chi_{\mathrm{em}ij}(t-t') H_j(t')\ \mathrm{d}t' \Big\} \\ B_i(t) = \dfrac{1}{c_0} \displaystyle\sum_{j=1}^{3} \Big\{ \int_{-\infty}^{t} \chi_{\mathrm{me}ij}(t-t') E_j(t')\ \mathrm{d}t' \\ \qquad\qquad + \eta_0 \mu_{ij} H_j(t) + \eta_0 \displaystyle\int_{-\infty}^{t} \chi_{\mathrm{mm}ij}(t-t') H_j(t')\ \mathrm{d}t' \Big\} \end{cases}$$

We have here introduced the wave impedance and the wave front velocity in vacuum $\eta_0 = \sqrt{\mu_0/\epsilon_0}$, and $c_0 = \sqrt{1/\epsilon_0\mu_0}$, respectively. In this way all fields have the same unit (V/m).

At this stage, it is convenient to employ a six-dimensional notation, as an alternative to the more commonly used three-dimensional formulation. This notation makes the expressions very compact. To this end, define[14]

$$\mathsf{e}(\boldsymbol{r}, t) = \begin{pmatrix} \boldsymbol{E}(\boldsymbol{r}, t) \\ \eta_0 \boldsymbol{H}(\boldsymbol{r}, t) \end{pmatrix}, \qquad \mathsf{d}(\boldsymbol{r}, t) = c_0 \begin{pmatrix} \eta_0 \boldsymbol{D}(\boldsymbol{r}, t) \\ \boldsymbol{B}(\boldsymbol{r}, t) \end{pmatrix} \tag{2.14}$$

In this notation, Maxwell equations, (1.1) and (1.2) have the form

$$\mathsf{D} \cdot \mathsf{e}(\boldsymbol{r}, t) = \frac{1}{c_0} \frac{\partial}{\partial t} \mathsf{d}(\boldsymbol{r}, t)$$

where the Maxwell operator D is defined as[15]

$$\mathsf{D} \cdot \mathsf{e}(\boldsymbol{r}, t) = \begin{pmatrix} \mathbf{0} & \nabla \times \mathbf{I}_3 \\ -\nabla \times \mathbf{I}_3 & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{E}(\boldsymbol{r}, t) \\ \eta_0 \boldsymbol{H}(\boldsymbol{r}, t) \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \eta_0 \nabla \times \boldsymbol{H}(\boldsymbol{r}, t) \\ -\nabla \times \boldsymbol{E}(\boldsymbol{r}, t) & \mathbf{0} \end{pmatrix}$$

We also define two six-dimensional dyadics or matrices

$$\mathsf{A} = \begin{pmatrix} \boldsymbol{\epsilon} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\mu} \end{pmatrix}, \qquad \mathsf{M}(\boldsymbol{r}, t) = \begin{pmatrix} \boldsymbol{\chi}_{\mathrm{ee}}(\boldsymbol{r}, t) & \boldsymbol{\chi}_{\mathrm{em}}(\boldsymbol{r}, t) \\ \boldsymbol{\chi}_{\mathrm{me}}(\boldsymbol{r}, t) & \boldsymbol{\chi}_{\mathrm{mm}}(\boldsymbol{r}, t) \end{pmatrix} \tag{2.15}$$

where A is the six-dimensional optical response.

The constitutive relations in (2.13) now read at each fixed point $\boldsymbol{r}$ in space

$$\mathsf{d}(t) = \mathsf{A} \cdot \mathsf{e}(t) + \int_{-\infty}^{t} \mathsf{M}(t-t') \cdot \mathsf{e}(t')\ \mathrm{d}t' = \mathsf{A} \cdot \mathsf{e}(t) + \int_{0}^{\infty} \mathsf{M}(t') \cdot \mathsf{e}(t-t')\ \mathrm{d}t' \tag{2.16}$$

[14] Six-dimensional vectors and dyadics are written in sans serif bold face to distinguish them from the ordinary three-dimensional vectors.

[15] The identity dyadic in three dimensions is denoted $\mathbf{I}_3$, and in two dimensions (the x-y-plane) it is denoted $\mathbf{I}_2$. The concept of a dyadic is summarized in Appendix A.

Type	$\boldsymbol{\epsilon}, \boldsymbol{\chi}_{\text{ee}}, \boldsymbol{\mu}, \boldsymbol{\chi}_{\text{mm}}$	$\boldsymbol{\chi}_{\text{em}}, \boldsymbol{\chi}_{\text{me}}$
Isotropic	All $\sim \mathbf{I}_3$	Both $\mathbf{0}$
Anisotropic	Some not $\sim \mathbf{I}_3$	Both $\mathbf{0}$
Biisotropic	All $\sim \mathbf{I}_3$	Both $\sim \mathbf{I}_3$
Bianisotropic	All other cases	

Table 2.1: Classification of media w.r.t. their response to electromagnetic fields and their constitutive relations.

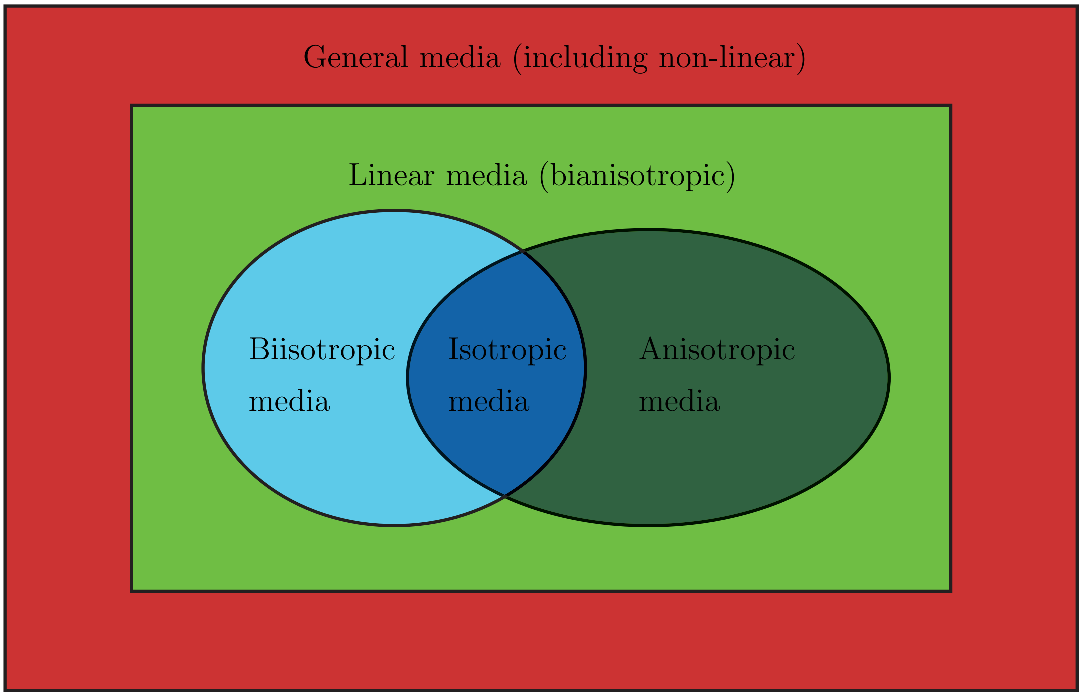


Figure 2.5: Schematic classification of media w.r.t. their constitutive relations. The area of each class is not proportional to the size of the class.

The Ansatz in (2.13) is very well motivated by the analysis for isotropic media in Section 2.1. We are referring to this general Ansatz as the constitutive relations for dispersive bianisotropic media. The dispersion of the material is modeled by four (generalized) dyadic-valued susceptibility functions, $\boldsymbol{\chi}_{\text{kk}'}(t)$, k,k'=e,m, each, in general, having nine independent components. The susceptibility functions all have the unit frequency. The optical response of the material is modeled by $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$, which both are dimensionless dyadics. Notice that there are no optical response between the $\boldsymbol{D}$ and $\boldsymbol{H}$ and between $\boldsymbol{B}$ and $\boldsymbol{E}$.

A material is classified w.r.t. its constitutive relations as in Table 2.1, see also Figure 2.5. Media for which $\boldsymbol{\chi}_{\text{em}}(t) = \boldsymbol{\chi}_{\text{me}}(t) = 0$ are called dispersive anisotropic material, and media where all quantities are proportional to the unit dyadic $\mathbf{I}_3$ are called dispersive isotropic or dispersive biisotropic material.

In an bianisotropic material, (2.13), the electric and magnetic effects are coupled in a way that is more general than in an anisotropic material. This coupling between the electric and magnetic effects is modeled by the dyadic-valued functions $\boldsymbol{\chi}_{\text{em}}(t)$ and $\boldsymbol{\chi}_{\text{me}}(t)$. In way a magnetic field $\boldsymbol{H}$ gives rise to an electric flux density $\boldsymbol{D}$ or equivalently an electric polarization $\boldsymbol{P}$ in the material. Conversely, an electric field $\boldsymbol{E}$ gives rise to a magnetization $\boldsymbol{M}$ in the material. This is not possible in an anisotropic material.

Already during the 19th century, optical active media were known for optical frequencies. The properties of these media can be explained with bianisotropic or bi-isotropic constitutive relations.[16] There are several examples of optical active media, *e.g.*, quarts and different sugar solutions. About 1960 Russian scientists discovered that several magnetic crystals, *e.g.*, Cr_2O_3, have similar properties. These materials are referred to as magneto-electric material. Similar properties are desirable in the microwave regime, but then they have to be manufactured.

Example 2.5
An important, non-trivial, example of a more general constitutive relation is the cold plasma that is developed in Problem 2.6, under the influence of a static magnetic flux density $\boldsymbol{B} = B_0\hat{\boldsymbol{z}}$. The result is

$$\begin{aligned}\boldsymbol{\chi}(t) = \frac{\omega_{\text{p}}^2}{\nu^2+\omega_g^2} H(t) \Big\{ &\mathbf{I}_2 \left(\omega_g \mathrm{e}^{-\nu t}\sin\omega_g t + \nu\left(1-\mathrm{e}^{-\nu t}\cos\omega_g t\right)\right) \\ &- \mathbf{J}\left(\omega_g\left(1-\mathrm{e}^{-\nu t}\cos\omega_g t\right) - \nu \mathrm{e}^{-\nu t}\sin\omega_g t\right) + \hat{\boldsymbol{z}}\hat{\boldsymbol{z}}\frac{\nu^2+\omega_g^2}{\nu}\left(1-\mathrm{e}^{-\nu t}\right)\Big\}\end{aligned}$$

where ν is the collision frequency, $\mathbf{I}_2$ the identity operator in the x-y-plane, and $\mathbf{J} = \hat{\boldsymbol{z}}\times\mathbf{I}_2$, which is a rotation of $\pi/2$ along the z-axis, and the gyrotropic frequency, ω_g, and, ω_{p}, the plasma frequency of the material, are defined by

$$\omega_g = \frac{qB_0}{m}, \qquad \omega_{\text{p}} = \sqrt{\frac{Nq^2}{m\epsilon_0}}$$

and N is the number of charges per unit of volume in the plasma, and m and q are the mass and the charge of the charges, respectively.

In the limit of vanishing collision frequency, $\nu \to 0$, these expressions become

$$\boldsymbol{\chi}(t) = \frac{\omega_{\text{p}}^2}{\omega_g} H(t)\left\{\mathbf{I}_2 \sin\omega_g t - \mathbf{J}\left(1-\cos\omega_g t\right) + \hat{\boldsymbol{z}}\hat{\boldsymbol{z}}\omega_g t\right\}$$

On the other hand, vanishing gyrotropic frequency, $\omega_g \to 0$, leads to a recovery of Lorentz' model with vanishing restoring force, *i.e.*,

$$\boldsymbol{\chi}(t) = \frac{\omega_{\text{p}}^2}{\nu} H(t)\left(1-\mathrm{e}^{-\nu t}\right)\mathbf{I}_3$$

[16] Media, which constitutive components are not identical to their mirror image, show bian-isotropic effects. In nature, materials that are not invariant under mirror reflection occur both on microscopic (molecular) and macroscopic level. Interestingly, there is a dominance of right-handed oriented natural material. On the microscopic level, we find such examples in *e.g.*, the chromosomes, and on the macroscopic level in *e.g.*, shells and vines. The reason for this asymmetry between right-handed and left-handed oriented media can be traced to the nucleus of the atoms and the weak interaction and its symmetry breaking properties at mirror reflection.

$\blacksquare$

Example 2.6

In 1937 Condon[17] suggested a constitutive relation that models biisotropic (chiral or optical activity) effects [4]. In this textbook, the constitutive relations for a biisotropic medium are

$$\begin{aligned}
\boldsymbol{D}(t) &= \epsilon_0 \left\{ \epsilon \boldsymbol{E}(t) + \int_{-\infty}^{t} \chi_{\mathrm{ee}}(t-t')\boldsymbol{E}(t')\ \mathrm{d}t' + \eta_0 \int_{-\infty}^{t} \chi_{\mathrm{em}}(t-t')\boldsymbol{H}(t')\ \mathrm{d}t' \right\} \\
\boldsymbol{B}(t) &= \frac{1}{c_0} \left\{ \int_{-\infty}^{t} \chi_{\mathrm{me}}(t-t')\boldsymbol{E}(t')\ \mathrm{d}t' + \eta_0 \mu \boldsymbol{H}(t) + \eta_0 \int_{-\infty}^{t} \chi_{\mathrm{mm}}(t-t')\boldsymbol{H}(t')\ \mathrm{d}t' \right\}
\end{aligned}$$

Condon generalized Lorentz' model by adding a force proportional to the time derivative of the magnetic field $\boldsymbol{H}$. Using the notation in Section 2.2.2, the equation of the polarization $\boldsymbol{P}$ is

$$\frac{d^2}{dt^2}\boldsymbol{P} + \nu \frac{d}{dt}\boldsymbol{P} + \omega_0^2 \boldsymbol{P} = \epsilon_0 \left(\omega_{\mathrm{p}}^2 \boldsymbol{E} + \omega_{\mathrm{c}} \eta_0 \frac{\partial \boldsymbol{H}}{\partial t} \right) \tag{2.17}$$

where $\omega_{\mathrm{p}} = \sqrt{\frac{Nq^2}{m\epsilon_0}}$ and ω_{c} are the plasma frequency and a frequency modeling the optical activity of the material, respectively.

Proceed as in Section 2.2.2 by differentiating the polarization $\boldsymbol{P}$. We get

$$\begin{aligned}
\frac{1}{\epsilon_0}\frac{\mathrm{d}\boldsymbol{P}(t)}{\mathrm{d}t} &= (\epsilon - 1)\frac{\mathrm{d}\boldsymbol{E}(t)}{\mathrm{d}t} + \chi_{\mathrm{ee}}(0^+)\boldsymbol{E}(t) + \int_{-\infty}^{t} \chi'_{\mathrm{ee}}(t-t')\boldsymbol{E}(t')\ \mathrm{d}t' \\
&\quad + \eta_0 \chi_{\mathrm{em}}(0^+)\boldsymbol{H}(t) + \eta_0 \int_{-\infty}^{t} \chi'_{\mathrm{em}}(t-t')\boldsymbol{H}(t')\ \mathrm{d}t' \\
\frac{1}{\epsilon_0}\frac{\mathrm{d}^2\boldsymbol{P}(t)}{\mathrm{d}t^2} &= (\epsilon - 1)\frac{\mathrm{d}^2\boldsymbol{E}(t)}{\mathrm{d}t^2} + \chi_{\mathrm{ee}}(0^+)\frac{\mathrm{d}\boldsymbol{E}(t)}{\mathrm{d}t} + \chi'_{\mathrm{ee}}(0^+)\boldsymbol{E}(t) + \int_{-\infty}^{t} \chi''_{\mathrm{ee}}(t-t')\boldsymbol{E}(t')\ \mathrm{d}t' \\
&\quad + \eta_0 \chi_{\mathrm{em}}(0^+)\frac{\mathrm{d}\boldsymbol{H}(t)}{\mathrm{d}t} + \eta_0 \chi'_{\mathrm{em}}(0^+)\boldsymbol{H}(t) + \eta_0 \int_{-\infty}^{t} \chi''_{\mathrm{em}}(t-t')\boldsymbol{H}(t')\ \mathrm{d}t'
\end{aligned}$$

Insert into the equation of dynamics, (2.17), and collect terms

$$\begin{aligned}
(\epsilon - 1)\frac{\mathrm{d}^2\boldsymbol{E}(t)}{\mathrm{d}t^2} &+ \left(\chi_{\mathrm{ee}}(0^+) + \nu(\epsilon - 1)\right)\frac{\mathrm{d}\boldsymbol{E}(t)}{\mathrm{d}t} \\
&+ \left(\chi'_{\mathrm{ee}}(0^+) + \nu\chi_{\mathrm{ee}}(0^+) + \omega_0^2(\epsilon - 1) - \omega_{\mathrm{p}}^2\right)\boldsymbol{E}(t) \\
&+ \int_{-\infty}^{t} \left(\chi''_{\mathrm{ee}}(t-t') + \nu\chi'_{\mathrm{ee}}(t-t') + \omega_0^2\chi_{\mathrm{ee}}(t-t')\right)\boldsymbol{E}(t')\ \mathrm{d}t' \\
&+ \eta_0\left(\chi_{\mathrm{em}}(0^+) - \omega_{\mathrm{c}}\right)\frac{\mathrm{d}\boldsymbol{H}(t)}{\mathrm{d}t} + \eta_0\left(\chi'_{\mathrm{em}}(0^+) + \nu\chi_{\mathrm{em}}(0^+) + \omega_0^2\right)\boldsymbol{H}(t) \\
&+ \eta_0 \int_{-\infty}^{t} \left(\chi''_{\mathrm{em}}(t-t') + \nu\chi'_{\mathrm{em}}(t-t') + \omega_0^2\chi_{\mathrm{em}}(t-t')\right)\boldsymbol{H}(t')\ \mathrm{d}t' = \boldsymbol{0}
\end{aligned}$$

[17] Edward Uhler Condon (1902–1974), American physicist.

Treating the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ as independent fields lead to

$$\begin{cases} \epsilon - 1 = 0 \\ \chi_{\mathrm{ee}}(0^+) + \nu\,(\epsilon - 1) = 0 \\ \chi'_{\mathrm{ee}}(0^+) + \nu\chi_{\mathrm{ee}}(0^+) + \omega_0^2\,(\epsilon - 1) - \omega_{\mathrm{p}}^2 = 0 \\ \chi''_{\mathrm{ee}}(t) + \nu\chi'_{\mathrm{ee}}(t) + \omega_0^2\chi_{\mathrm{ee}}(t) = 0 \end{cases} \qquad \begin{cases} \chi_{\mathrm{em}}(0^+) - \omega_{\mathrm{c}} = 0 \\ \chi'_{\mathrm{em}}(0^+) + \nu\chi_{\mathrm{em}}(0^+) = 0 \\ \chi''_{\mathrm{em}}(t) + \nu\chi'_{\mathrm{em}}(t) + \omega_0^2\chi_{\mathrm{em}}(t) = 0 \end{cases}$$

The solutions to these problems are

$$\begin{cases} \chi_{\mathrm{ee}}(t) = H(t)\dfrac{\omega_{\mathrm{p}}^2}{\nu_0}e^{-\frac{\nu t}{2}}\sin\nu_0 t \\ \chi_{\mathrm{em}}(t) = H(t)\omega_{\mathrm{c}}e^{-\frac{\nu t}{2}}\left(\cos\nu_0 t - \dfrac{\nu}{2\nu_0}\sin\nu_0 t\right) \end{cases}$$

where $\nu_0^2 = \omega_0^2 - \nu^2/4$. This analysis determines the susceptibility functions $\chi_{\mathrm{ee}}(t)$ and $\chi_{\mathrm{em}}(t)$. If the material is non-magnetic $\chi_{\mathrm{mm}}(t) = 0$, and if the material satisfy additional conditions, such as the reciprocity condition developed and analyzed in Section 3.5 on page 73, we have also $\chi_{\mathrm{me}}(t) = -\chi_{\mathrm{em}}(t)$. ■

We end the analysis on constitutive relations by inverting the relation between the electric flux density $\boldsymbol{D}(t)$ and the electric field $\boldsymbol{E}(t)$ for an anisotropic material. The constitutive relations, (2.13), for an anisotropic material is

$$\boldsymbol{D}(t) = \epsilon_0\left\{\boldsymbol{\epsilon}\cdot\boldsymbol{E}(t) + \int_{-\infty}^{t}\boldsymbol{\chi}(t-t')\cdot\boldsymbol{E}(t')\,\mathrm{d}t'\right\}$$

This relation can be inverted by applying the resolvent of the susceptibility dyadic $\boldsymbol{\chi}(t)$. The resolvent kernel $\boldsymbol{\Upsilon}(t)$ satisfies the resolvent equation

$$\boldsymbol{\epsilon}^{-1}\cdot\boldsymbol{\chi}(t) + \boldsymbol{\Upsilon}(t)\cdot\boldsymbol{\epsilon} + \int_0^t \left(\boldsymbol{\Upsilon}(t-t')\cdot\boldsymbol{\epsilon}\right)\cdot\left(\boldsymbol{\epsilon}^{-1}\cdot\boldsymbol{\chi}(t')\right)\,\mathrm{d}t' = \mathbf{0} \tag{2.18}$$

This solution is uniquely soluble for $t > 0$, and for $t < 0$, $\boldsymbol{\Upsilon}(t) = \mathbf{0}$. In terms of the resolvent kernel, the inverse is

$$\boldsymbol{E}(t) = \frac{1}{\epsilon_0}\left\{\boldsymbol{\epsilon}^{-1}\cdot\boldsymbol{D}(t) + \int_{-\infty}^{t}\boldsymbol{\Upsilon}(t-t')\cdot\boldsymbol{D}(t')\,\mathrm{d}t'\right\} \tag{2.19}$$

This relation is easily verified by inserting the constitutive relation into (2.19) and using the resolvent equation, (2.18).

2.4 Energy and passivity

The classification of a passive material in this section follow closely Karlsson and Kristensson [11, 12].

In Section 1.2 we identified the power bounded in the electromagnetic field in the volume V as, see (1.12)

$$\iiint_V \left[\boldsymbol{H}\cdot\frac{\partial\boldsymbol{B}}{\partial t} + \boldsymbol{E}\cdot\frac{\partial\boldsymbol{D}}{\partial t}\right]\,\mathrm{d}v$$

The integrand defines the power density of the material, and the total energy stored per volume therefore is

$$w(\boldsymbol{r}, t) = \int_{-\infty}^{t} \left[\boldsymbol{H}(t') \cdot \frac{\partial \boldsymbol{B}(t')}{\partial t'} + \boldsymbol{E}(t') \cdot \frac{\partial \boldsymbol{D}(t')}{\partial t'} \right] \, \mathrm{d}t'$$

To proceed, it is convenient to employ the six-dimensional notation in (2.14), (2.15), and (2.16). In this notation, the Poynting theorem (without sources, dispersive model) reads

$$\nabla \cdot \boldsymbol{S}(\boldsymbol{r}, t) + \frac{1}{\eta_0} \mathsf{e}(\boldsymbol{r}, t) \cdot \frac{\partial \mathsf{d}(\boldsymbol{r}, t)}{\partial t} = 0$$

and the total energy stored per volume at a point $\boldsymbol{r}$ is

$$w(t) = \frac{1}{\eta_0} \int_{-\infty}^{t} \mathsf{e}(t') \cdot \mathsf{d}'(t') \, \mathrm{d}t' = w_{\mathrm{em}}(t) + w_{\mathrm{d}}(t)$$

where the instantaneous part of the total energy stored is

$$\begin{aligned} w_{\mathrm{em}}(t) &= \frac{1}{\eta_0} \int_{-\infty}^{t} \mathsf{e}(t') \cdot (\mathsf{A} \cdot \mathsf{e}'(t')) \, \mathrm{d}t' \\ &= \frac{1}{4\eta_0} \mathsf{e}(t) \cdot \left(\left(\mathsf{A} + \mathsf{A}^t \right) \cdot \mathsf{e}(t) \right) + \frac{1}{2\eta_0} \int_{-\infty}^{t} \mathsf{e}(t') \cdot \left(\left(\mathsf{A} - \mathsf{A}^t \right) \cdot \mathsf{e}'(t') \right) \, \mathrm{d}t' \end{aligned} \tag{2.20}$$

and the dispersive part of the total energy stored is

$$\begin{aligned} w_{\mathrm{d}}(t) &= \frac{1}{\eta_0} \int_{-\infty}^{t} \mathsf{e}(t') \cdot \int_{0}^{\infty} \mathsf{M}(t'') \cdot \mathsf{e}'(t' - t'') \, \mathrm{d}t'' \, \mathrm{d}t' \\ &= \frac{1}{\eta_0} \int_{-\infty}^{t} \mathsf{e}(t') \cdot \frac{\partial}{\partial t'} \left(\int_{-\infty}^{t'} \mathsf{M}(t' - t'') \cdot \mathsf{e}(t'') \, \mathrm{d}t'' \right) \mathrm{d}t' \\ &= \frac{1}{\eta_0} \int_{-\infty}^{t} \mathsf{e}(t') \cdot \left(\mathsf{M}(0) \cdot \mathsf{e}(t') + \int_{-\infty}^{t'} \mathsf{M}'(t' - t'') \cdot \mathsf{e}(t'') \, \mathrm{d}t'' \right) \mathrm{d}t' \end{aligned} \tag{2.21}$$

The medium is classified as a passive material at the fixed point $\boldsymbol{r}$, if

$$w(t) = w_{\mathrm{em}}(t) + w_{\mathrm{d}}(t) \geq 0$$

for every continuously differentiable, compactly supported vector $\mathsf{e}(t)$.

The motivation behind this definition is clear from the Poynting theorem, which implies that a passive material satisfies

$$- \int_{-\infty}^{t} \iint_{S_{\boldsymbol{r}}} \boldsymbol{S}(\boldsymbol{r}, t') \cdot \hat{\boldsymbol{\nu}} \, \mathrm{d}S \, \mathrm{d}t' \geq 0$$

for all electromagnetic fields $\mathsf{e}(t)$. Here $S_{\boldsymbol{r}}$ is the bounding surface of an arbitrary, open neighborhood $V_{\boldsymbol{r}}$ of the point $\boldsymbol{r}$. This integral states that the power is always consumed inside $S_{\boldsymbol{r}}$ for all excitations.

We apply this definition to a continuously differentiable field e, which satisfies $\mathsf{e}(t) = \mathsf{0}$, and, moreover, each component assumes non-zero values only in a neighborhood of $\tau < t$, more precisely, $\mathsf{e}(t) = \mathsf{0}$, $t \notin [\tau - \epsilon, \tau + \epsilon]$, where ϵ is sufficiently small. From (2.20) and (2.21), the total energy stored per volume is

$$w(t) = \frac{1}{2\eta_0} \int_{\tau-\epsilon}^{\tau+\epsilon} \mathsf{e}(t') \cdot \left((\mathsf{A} - \mathsf{A}^t) \cdot \mathsf{e}'(t') \right) \, \mathrm{d}t' \\ + \frac{1}{\eta_0} \int_{\tau-\epsilon}^{\tau+\epsilon} \mathsf{e}(t') \cdot \int_{t'-\tau-\epsilon}^{t'-\tau+\epsilon} \mathsf{M}(t'') \cdot \mathsf{e}'(t' - t'') \, \mathrm{d}t'' \, \mathrm{d}t' \geq 0$$

The first integral is of the order $O(\epsilon)$ and the second integral is of order $O(\epsilon^2)$ as $\tau \to 0$. Therefore, the first integral dominates, and in the limit $\epsilon \to 0$ we get

$$\frac{1}{2} \int_{\tau-\epsilon}^{\tau+\epsilon} \mathsf{e}(t') \cdot \left((\mathsf{A} - \mathsf{A}^t) \cdot \mathsf{e}'(t') \right) \, \mathrm{d}t' \geq 0$$

for all fields e. Now e and e' can be chosen independently, which implies that

$$\mathsf{A} = \mathsf{A}^t$$

and A has to be symmetric for a passive material, *i.e.*,

$$w_{\mathrm{em}}(t) = \frac{1}{2\eta_0} \mathsf{e}(t) \cdot (\mathsf{A} \cdot \mathsf{e}(t)) = \frac{1}{2\eta_0} \mathsf{e}(t) \cdot \mathsf{A} \cdot \mathsf{e}(t)$$

Since this quantity always is positive, we conclude that the optical response A, in addition of being symmetric, also is a positive definite dyadic.

The total power stored becomes

$$w(t) = \frac{1}{2\eta_0} \mathsf{e}(t) \cdot \mathsf{A} \cdot \mathsf{e}(t) + \frac{1}{\eta_0} \int_{-\infty}^{t} \mathsf{e}(t') \cdot \int_0^\infty \mathsf{M}(t'') \cdot \mathsf{e}'(t' - t'') \, \mathrm{d}t'' \, \mathrm{d}t'$$

Again, for a continuously differentiable field e, which satisfies $\mathsf{e}(t) = \mathsf{0}$, and which assumes non-zero values only in a neighborhood of $\tau < t$ we have

$$w(t) = \frac{1}{\eta_0} \int_{\tau-\epsilon}^{\tau+\epsilon} \mathsf{e}(t') \cdot \int_0^\infty \mathsf{M}(t'') \cdot \mathsf{e}'(t' - t'') \, \mathrm{d}t'' \, \mathrm{d}t' \geq 0$$

which we rewrite as

$$w(t) = \frac{1}{\eta_0} \int_{\tau-\epsilon}^{\tau+\epsilon} \mathsf{e}(t') \cdot \left(\mathsf{M}(0) \cdot \mathsf{e}(t) + \int_0^\infty \mathsf{M}'(t'') \cdot \mathsf{e}(t' - t'') \, \mathrm{d}t'' \right) \mathrm{d}t' \geq 0$$

Again, the first integral is of the order $O(\epsilon)$ and the second integral is of order $O(\epsilon^2)$ as $\tau \to 0$. The same argument as above implies that

$$\mathsf{a} \cdot (\mathsf{M}(0) \cdot \mathsf{a}) \geq 0$$

for all $\mathsf{a} \in \mathbb{R}^6$, *i.e.*, $\mathsf{M}(0)$ is a non-negative definite matrix.

If the early time behavior of the dispersion effects satisfy $\mathsf{M}(0) = \mathsf{0}$ (*e.g.*, a Lorentz model), then ($\mathsf{e}(t) = \mathsf{0}$)

$$\begin{aligned} w(t) &= \frac{1}{\eta_0}\int_{-\infty}^{t} \mathsf{e}(t')\cdot\int_{-\infty}^{t'} \mathsf{M}'(t'-t'')\cdot\mathsf{e}(t'')\,\mathrm{d}t''\,\mathrm{d}t' \\ &= \frac{1}{2\eta_0}\int_{-\infty}^{t} \mathsf{e}(t')\cdot\int_{-\infty}^{t} \Big(\mathsf{M}'(t'-t'') + \mathsf{M}'^{t}(t''-t')\Big)\cdot\mathsf{e}(t'')\,\mathrm{d}t''\,\mathrm{d}t' \geq 0 \end{aligned}$$

and we see that the function $K(t) = \mathsf{a}\cdot\big((\mathsf{M}'(t) + \mathsf{M}'^{t}(-t))\cdot\mathsf{a}\big)$ is a function of positive type for all vectors $\mathsf{a}\in\mathbb{R}^6$, see Appendix B.4.

Example 2.7
We illustrate the theory in this section with the Lorentz model which has a susceptibility function, see (2.12)

$$\chi(t) = H(t)\frac{\omega_{\mathrm{p}}^2}{\nu_0}\mathrm{e}^{-\frac{\nu t}{2}}\sin\nu_0 t$$

Consequently

$$\begin{cases} \boldsymbol{\epsilon}_\infty = \mathbf{I}_3 \\ \boldsymbol{\chi}(0) = \mathbf{0} \end{cases} \qquad \begin{cases} \boldsymbol{\chi}'(t) = H(t)\dfrac{\omega_{\mathrm{p}}^2}{\nu_0}\mathrm{e}^{-\frac{\nu t}{2}}\mathbf{I}_3\left(\nu_0\cos\nu_0 t - \dfrac{\nu}{2}\sin\nu_0 t\right) \\ \boldsymbol{\chi}'(0) = \omega_{\mathrm{p}}^2\mathbf{I}_3 \end{cases}$$

We conclude that $\boldsymbol{\chi}'(0)$ is a positive definite dyadic. Moreover, for each $\boldsymbol{a}\in\mathbb{R}^3$, the function

$$f(t) = \boldsymbol{a}\cdot\big((\boldsymbol{\chi}'(t)+\boldsymbol{\chi}^{t}(-t))\cdot\boldsymbol{a}\big) = \frac{\omega_{\mathrm{p}}^2}{\nu_0}\mathrm{e}^{-\frac{\nu|t|}{2}}|\boldsymbol{a}|^2\mathbf{I}_3\left(\nu_0\cos\nu_0|t| - \frac{\nu}{2}\sin\nu_0|t|\right)$$

is a function of positive type, see Appendix B.4. To see this, engage Bochner's theorem, Theorem B.4 on page 117, and prove that $f(t)$ has a positive Fourier transform. In fact,

$$\int_{-\infty}^{\infty} f(t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t = 2|\boldsymbol{a}|^2\frac{\omega_{\mathrm{p}}^2\omega^2\nu}{\left(\omega^2-\omega_0^2\right)^2+\omega^2\nu^2} > 0$$

■

Example 2.8
A more complex example is the model of the plasma in Example 2.5 on page 34. The constitutive relations satisfy

$$\begin{cases} \boldsymbol{\epsilon}_\infty = \mathbf{I}_3 \\ \boldsymbol{\chi}(0) = \mathbf{0} \end{cases} \qquad \begin{cases} \boldsymbol{\chi}'(t) = \omega_{\mathrm{p}}^2\mathrm{e}^{-\nu t}H(t)\left(\mathbf{I}_2\cos\omega_g t + \hat{\boldsymbol{z}}\hat{\boldsymbol{z}} - \mathbf{J}\sin\omega_g t\right) \\ \boldsymbol{\chi}'(0) = \omega_{\mathrm{p}}^2\mathbf{I}_3 \end{cases}$$

Note that $\boldsymbol{\chi}'(0)$ is a positive definite dyadic, and that for $\boldsymbol{a}\in\mathbb{R}^3$, the function

$$f(t) = \boldsymbol{a}\cdot\big((\boldsymbol{\chi}'(t)+\boldsymbol{\chi}^{t}(-t))\cdot\boldsymbol{a}\big) = \omega_{\mathrm{p}}^2\mathrm{e}^{-\nu|t|}\left((a_1^2+a_2^2)\cos\omega_g|t| + a_3^2\right)$$

is a function of positive type, due to Bochner's theorem, Theorem B.4 on page 117, since the Fourier transform of $f(t)$ is positive, *i.e.*, In fact,

$$\int_{-\infty}^{\infty} f(t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t = 2\omega_{\mathrm{p}}^2\left(\nu(a_1^2+a_2^2)\frac{\omega^2+\omega_g^2+\nu^2}{(\omega^2-\omega_g^2-\nu^2)^2+4\nu^2\omega^2} + a_3^2\frac{1}{\nu}\right) > 0$$

■

Problems for Chapter 2

2.1 A Debye medium with susceptibility function $\chi(t) = \alpha \exp\{-t/\tau\}$ is excited by an electric field $\boldsymbol{E} = \boldsymbol{E}_0[H(t) - H(t-T)]$ (abrupt on and off excitation), where $\boldsymbol{E}_0$ is a vector that is constant in time and $T > 0$. Determine the polarization $\boldsymbol{P}(t)$ in the material as a function of time t.

2.2 In an isotropic, weakly magnetic material an electromagnetic shock wave is propagating. The magnetic field of the wave is

$$\boldsymbol{H}(z,t) = \hat{\boldsymbol{x}} H_0 H(t - z/c_0)$$

where $H(t)$ is Heaviside's step function and H_0 is a real constant. The susceptibility function of the material is $\chi_{\mathrm{mm}}(t) = H(t)\alpha \mathrm{e}^{-\beta t}$ and its magnetic optical response is $\mu = 1$. Determine the magnetization $\boldsymbol{M}(z,t)$ in the material.

2.3 A plane interface separates vacuum from an homogeneous Lorentz material with negligible losses. The susceptibility function of the material is $\chi(t) = \alpha \sin \beta t$. At the interface there are no free charges. Close to the interface, the electric field in the Lorentz material is (E constant)

$$\boldsymbol{E}_2(t) = \begin{cases} \hat{\boldsymbol{\nu}} E & 0 < t < T \\ \boldsymbol{0} & \text{otherwise} \end{cases}$$

Determine the electric field in vacuum close to the interface, *i.e.*, determine $\boldsymbol{E}_1(t)$? Explain why $\boldsymbol{E}_1(t)$ and $\boldsymbol{E}_2(t)$ differ.

2.4 A plane interface ($z = 0$) separates vacuum and an homogeneous Lorentz material with negligible losses. The susceptibility function of the material is $\chi(t) = \alpha \sin \beta t$ and the optical response is $\epsilon = 1$. There are no free charges at the interface. A transient wave propagates parallel to the interface. The electric field of this wave is (E is a real constant and $\omega_0 > 0$)

$$\boldsymbol{E}(y,t) = \hat{\boldsymbol{x}} E H(t - y/c_0) \cos \omega_0 (t - y/c_0)$$

where $H(t)$ is Heaviside's stepfunction. Determine the polarization $\boldsymbol{P}(y,t)$ at the interface in the Lorentz material.

2.5 Denote the charge and the mass of the charge in a cold plasma by q, and m, respectively. The charges move freely in a static magnetic flux density $\boldsymbol{B}$, oriented along the z-axis, *i.e.*, $\boldsymbol{B} = B_0 \hat{\boldsymbol{z}}$, under the presence of a collision frequency ν. Make the following Ansatz of the constitutive relations:

$$\boldsymbol{J}(t) = \sigma \boldsymbol{E}(t) + \epsilon_0 \int_{-\infty}^{t} \boldsymbol{\Sigma}(t - t') \cdot \boldsymbol{E}(t')\, \mathrm{d}t'$$

or in Cartesian components

$$J_i(t) = \sigma E_i(t) + \epsilon_0 \sum_{j=1}^{3} \int_{-\infty}^{t} \Sigma_{ij}(t - t') E_j(t')\, \mathrm{d}t' \quad i = 1, 2, 3$$

Determine the constitutive relations of the plasma under the assumption that the static flux density is much stronger than the magnetic flux density generated by the charges.

Hint: Show that the current density $\boldsymbol{J}$ satisfies the following equation of motion (Lorentz' force):

$$\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t} + \nu \boldsymbol{J} + \omega_g \hat{\boldsymbol{z}} \times \boldsymbol{J} = \epsilon_0 \omega_\mathrm{p}^2 \boldsymbol{E}$$

where ω_g, the gyrotropic frequency, and ω_p, the plasma frequency of the material, are defined by

$$\omega_g = \frac{qB_0}{m}, \qquad \omega_\mathrm{p} = \sqrt{\frac{Nq^2}{m\epsilon_0}}$$

and N is the number of charges per unit of volume in the plasma. Then show that σ and $\mathbf{\Sigma}(t)$ satisfy the following system of differential equations:

$$\begin{cases} \sigma = 0 \\ \mathbf{\Sigma}(0) = \omega_\mathrm{p}^2 \mathbf{I}_3 \\ \dfrac{\mathrm{d}\mathbf{\Sigma}(t)}{\mathrm{d}t} + \nu \mathbf{\Sigma}(t) + \omega_g \hat{\boldsymbol{z}} \times \mathbf{\Sigma}(t) = \mathbf{0} \end{cases}$$

2.6 Use the result of Problem 2.5 to write the constitutive relations of the plasma in the following form:

$$\boldsymbol{D}(t) = \epsilon_0 \left\{ \boldsymbol{E}(t) + \int_{-\infty}^{t} \boldsymbol{\chi}(t - t') \cdot \boldsymbol{E}(t') \, \mathrm{d}t' \right\}$$

This gives the transformation from the conductivity model, given in Problem 2.5, to the dispersion model.

Chapter 3

Time harmonic fields and Fourier transform

Several important applications use time harmonic fields. In this section, we analyze the special simplifications time harmonic fields introduce.

We obtain the time harmonic case from the general results in the previous section by a Fourier[1] transform in the time variable of all fields (dyadic-valued, vector-valued, and scalar-valued fields). We investigate the consequences time harmonic fields have on the constitutive relations and we introduce the concept of active, passive and lossless media. Moreover, the concept of reciprocity is introduced, and we investigate the polarization state of a time harmonic field, which leads to concept of the polarization ellipse.

Time dependence $\mathrm{e}^{-\mathrm{i}\omega t}$ vs $\mathrm{e}^{\mathrm{i}\omega t}$ ($\mathrm{e}^{\mathrm{j}\omega t}$)

The are two sign conventions for the temporal (inverse) Fourier transform. There is the one we use in this textbook, *i.e.*, $\mathrm{e}^{-\mathrm{i}\omega t}$, which is used mostly by physicists. Electrical engineers often prefer the opposite sign in the exponential, *i.e.*, $\mathrm{e}^{\mathrm{i}\omega t}$ or $\mathrm{e}^{\mathrm{j}\omega t}$. The choice of sign is, of course, irrelevant in the computation of all physical quantities, but it leads to different signs in many of the complex quantities that are used in the calculations.

The choice of the electrical engineers is most appropriate when dealing with circuit applications where the dependence of the space variables is suppressed. However, using the $\mathrm{e}^{\mathrm{i}\omega t}$ time convention in wave propagation problems, like the scattering problems we are dealing within this textbook, leads to an extra minus signs in front of the spatial dependence, *e.g.*, an outgoing spherical wave would be $\mathrm{e}^{-\mathrm{i}kr}/kr$ with this time convention.

The Fourier transform in the time variable of a vector field, *e.g.*, the electric field $\boldsymbol{E}(\boldsymbol{r},t)$, is defined as

$$\boldsymbol{E}(\boldsymbol{r},\omega) = \int_{-\infty}^{\infty} \boldsymbol{E}(\boldsymbol{r},t)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t$$

with its inverse transform

$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \boldsymbol{E}(\boldsymbol{r},\omega)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega$$

Similarly, the Fourier transform for all other time dependent fields, dyadics, and scalars are defined. To avoid heavy notation, we use the same symbol for the physical

[1] Jean Baptiste Joseph Fourier (1768–1830), French mathematician and physicist.

field $\boldsymbol{E}(\boldsymbol{r},t)$, as for the Fourier transformed field $\boldsymbol{E}(\boldsymbol{r},\omega)$ — only the argument differs. Moreover, note that the Fourier transformed field no longer has the same unit as the time domain field, *e.g.*, the physical electric field $\boldsymbol{E}(\boldsymbol{r},t)$ has the unit V/m, but the Fourier transformed field $\boldsymbol{E}(\boldsymbol{r},\omega)$ has the unit Vs/m. In most cases the context suggests whether it is the physical field or the Fourier transformed field that is intended. When there is doubts which field that is intended, the time argument t or the angular frequency $\omega = 2\pi f$, where f is the frequency, is explicitly written out to distinguish the fields.

All physical quantities are real-valued, which imply constraints on the Fourier transform. The negative values of ω are related to the positive values of ω by a complex conjugate. To see this, we write down the criterion for the field $\boldsymbol{E}$ to be real

$$\int_{-\infty}^{\infty} \boldsymbol{E}(\boldsymbol{r},\omega)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega = \left\{\int_{-\infty}^{\infty} \boldsymbol{E}(\boldsymbol{r},\omega)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega\right\}^*$$

where the star (*) denotes the complex conjugate. For real ω, we have

$$\int_{-\infty}^{\infty} \boldsymbol{E}(\boldsymbol{r},\omega)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega = \int_{-\infty}^{\infty} \boldsymbol{E}^*(\boldsymbol{r},\omega)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega = \int_{-\infty}^{\infty} \boldsymbol{E}^*(\boldsymbol{r},-\omega)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega$$

where we in the last integral has made a change of variable $\omega \to -\omega$. Therefore, for real ω we have

$$\boldsymbol{E}(\boldsymbol{r},\omega) = \boldsymbol{E}^*(\boldsymbol{r},-\omega) \tag{3.1}$$

This shows that when the physical field is constructed from its Fourier transform, it suffices to integrate over the non-negative frequencies only. By a change of variable, $\omega \to -\omega$, and the use of the condition (3.1), we have

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{r},t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \boldsymbol{E}(\boldsymbol{r},\omega)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega\\ &= \frac{1}{2\pi}\int_{0}^{\infty} \left(\boldsymbol{E}(\boldsymbol{r},\omega)\mathrm{e}^{-\mathrm{i}\omega t} + \boldsymbol{E}(\boldsymbol{r},-\omega)\mathrm{e}^{\mathrm{i}\omega t}\right)\mathrm{d}\omega\\ &= \frac{1}{2\pi}\int_{0}^{\infty} \left(\boldsymbol{E}(\boldsymbol{r},\omega)\mathrm{e}^{-\mathrm{i}\omega t} + \boldsymbol{E}^*(\boldsymbol{r},\omega)\mathrm{e}^{\mathrm{i}\omega t}\right)\mathrm{d}\omega = \frac{1}{\pi}\,\mathrm{Re}\int_{0}^{\infty} \boldsymbol{E}(\boldsymbol{r},\omega)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega\end{aligned} \tag{3.2}$$

where $\mathrm{Re}\,z$ denotes the real part of the complex number z. A similar result holds for all other Fourier transformed fields that we are using. We also conclude that the real part of $\boldsymbol{E}(\boldsymbol{r},\omega)$ is an even function of ω and the imaginary part of $\boldsymbol{E}(\boldsymbol{r},\omega)$ is an odd function of ω.

Fields that are purely time harmonic are of special interests in many applications, see Table 3.1. If we concentrate on the time dependence, a purely time harmonic fields have time dependence of the form

$$\cos(\omega_0 t - \alpha)$$

Band	**Frequency**	**Wave length**	**Application**
ELF	< 3 KHz	> 100 km	
VLF	3–30 KHz	100–10 km	Navigation
LV	30–300 KHz	10–1 km	Navigation
MV	300–3000 KHz	1000–100 m	Radio
KV (HF)	3–30 MHz	100–10 m	Radio
VHF	30–300 MHz	10–1 m	FM, TV
UHF	300–1000 MHz	100–30 cm	Radar, TV, mobile communication
†[a]	1–30 GHz	30–1 cm	Radar, satellite communication
†[a]	30–300 GHz	10–1 mm	Radar
	$4.2–7.9 \cdot 10^{14}$ Hz	0.38–0.72 μm	Visible light

[a]See also Table 3.2.

Table 3.1: The spectrum of the electromagnetic waves.

Such fields are generated by the following Fourier transform:

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{r},\omega) &= \pi\Big\{\delta(\omega-\omega_0)\left(\hat{\boldsymbol{x}}E_x(\boldsymbol{r})+\hat{\boldsymbol{y}}E_y(\boldsymbol{r})+\hat{\boldsymbol{z}}E_z(\boldsymbol{r})\right)\\ &\qquad + \delta(\omega+\omega_0)\left(\hat{\boldsymbol{x}}E_x^*(\boldsymbol{r})+\hat{\boldsymbol{y}}E_y^*(\boldsymbol{r})+\hat{\boldsymbol{z}}E_z^*(\boldsymbol{r})\right)\Big\}\\ &= \pi\Big\{\delta(\omega-\omega_0)\left(\hat{\boldsymbol{x}}|E_x(\boldsymbol{r})|\mathrm{e}^{\mathrm{i}\alpha(\boldsymbol{r})}+\hat{\boldsymbol{y}}|E_y(\boldsymbol{r})|\mathrm{e}^{\mathrm{i}\beta(\boldsymbol{r})}+\hat{\boldsymbol{z}}|E_z(\boldsymbol{r})|\mathrm{e}^{\mathrm{i}\gamma(\boldsymbol{r})}\right)\\ &\qquad + \delta(\omega+\omega_0)\left(\hat{\boldsymbol{x}}|E_x(\boldsymbol{r})|\mathrm{e}^{-\mathrm{i}\alpha(\boldsymbol{r})}+\hat{\boldsymbol{y}}|E_y(\boldsymbol{r})|\mathrm{e}^{-\mathrm{i}\beta(\boldsymbol{r})}+\hat{\boldsymbol{z}}|E_z(\boldsymbol{r})|\mathrm{e}^{-\mathrm{i}\gamma(\boldsymbol{r})}\right)\Big\}\end{aligned}$$

where $\alpha(\boldsymbol{r})$, $\beta(\boldsymbol{r})$ and $\gamma(\boldsymbol{r})$ are the complex phase of the components, $\omega_0 \geq 0$, and where $\delta(\omega)$ denotes the delta function. Note that this Fourier transform satisfies $\boldsymbol{E}(\boldsymbol{r},\omega) = \boldsymbol{E}^*(\boldsymbol{r},-\omega)$, which is the criterion for a real-valued field. The inverse Fourier transform then gives

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{r},t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\boldsymbol{E}(\boldsymbol{r},\omega)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega\\ &= \big\{\hat{\boldsymbol{x}}|E_x(\boldsymbol{r})|\cos(\omega_0 t-\alpha(\boldsymbol{r}))+\hat{\boldsymbol{y}}|E_y(\boldsymbol{r})|\cos(\omega_0 t-\beta(\boldsymbol{r}))\\ &\qquad + \hat{\boldsymbol{z}}|E_z(\boldsymbol{r})|\cos(\omega_0 t-\gamma(\boldsymbol{r}))\big\}\end{aligned}$$

A simple way of obtaining purely time harmonic waves is to employ the following expression:

$$\boldsymbol{E}(\boldsymbol{r},t) = \mathrm{Re}\left\{\boldsymbol{E}(\boldsymbol{r},\omega)\mathrm{e}^{-\mathrm{i}\omega t}\right\} \tag{3.3}$$

where $\boldsymbol{E}(\boldsymbol{r},\omega)$ is a complex-valued vector. If we write $\boldsymbol{E}(\boldsymbol{r},\omega)$ as

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{r},\omega) &= \hat{\boldsymbol{x}}E_x(\boldsymbol{r},\omega)+\hat{\boldsymbol{y}}E_y(\boldsymbol{r},\omega)+\hat{\boldsymbol{z}}E_z(\boldsymbol{r},\omega)\\ &= \hat{\boldsymbol{x}}|E_x(\boldsymbol{r},\omega)|\mathrm{e}^{\mathrm{i}\alpha(\boldsymbol{r})}+\hat{\boldsymbol{y}}|E_y(\boldsymbol{r},\omega)|\mathrm{e}^{\mathrm{i}\beta(\boldsymbol{r})}+\hat{\boldsymbol{z}}|E_z(\boldsymbol{r},\omega)|\mathrm{e}^{\mathrm{i}\gamma(\boldsymbol{r})}\end{aligned}$$

we obtain the same result as in the expression above (without the index 0 on ω). This way of constructing purely time harmonic waves are convenient and often used.

Band	**Frequency** (GHz)
L	1–2
S	2–4
C	4–8
X	8–12
K_u	12–18
K	18–27
K_a	27–40
Millimeter band	40–300

Table 3.2: Table of radar band frequencies.

Note that the field $\boldsymbol{E}(\boldsymbol{r},\omega)$ has the same unit as the field $\boldsymbol{E}(\boldsymbol{r},t)$. This is in contrast to the Fourier transformation of the field above, but this difference seldom causes problems.

3.1 The Maxwell equations

As a first step in our analysis of time harmonic fields, we Fourier transform the Maxwell equations (1.1) and (1.2) ($\frac{\partial}{\partial t} \to -\mathrm{i}\omega$)

$$\nabla \times \boldsymbol{E}(\boldsymbol{r},\omega) = \mathrm{i}\omega \boldsymbol{B}(\boldsymbol{r},\omega) \tag{3.4}$$

$$\nabla \times \boldsymbol{H}(\boldsymbol{r},\omega) = \boldsymbol{J}(\boldsymbol{r},\omega) - \mathrm{i}\omega \boldsymbol{D}(\boldsymbol{r},\omega) \tag{3.5}$$

The explicit harmonic time dependence $\mathrm{e}^{-\mathrm{i}\omega t}$ has been suppressed from both sides of these equations, *i.e.*, the physical fields are

$$\boldsymbol{E}(\boldsymbol{r},t) = \mathrm{Re}\left\{\boldsymbol{E}(\boldsymbol{r},\omega)\mathrm{e}^{-\mathrm{i}\omega t}\right\}$$

This convention is applied to all purely time harmonic fields. Note that the electromagnetic fields $\boldsymbol{E}(\boldsymbol{r},\omega)$, $\boldsymbol{B}(\boldsymbol{r},\omega)$, $\boldsymbol{D}(\boldsymbol{r},\omega)$ and $\boldsymbol{H}(\boldsymbol{r},\omega)$, and the current density $\boldsymbol{J}(\boldsymbol{r},\omega)$ in general are complex-valued vector fields.

The continuity equation (1.4) is transformed in a similar way and we have

$$\nabla \cdot \boldsymbol{J}(\boldsymbol{r},\omega) - \mathrm{i}\omega\rho(\boldsymbol{r},\omega) = 0 \tag{3.6}$$

The remaining two equations from Section 1, (1.5) and (1.6), are transformed into

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r},\omega) = 0 \tag{3.7}$$

$$\nabla \cdot \boldsymbol{D}(\boldsymbol{r},\omega) = \rho(\boldsymbol{r},\omega) \tag{3.8}$$

These equations are consequences of (3.4) and (3.5), and the continuity equation (3.6) (*cf.* Section 1 on page 4). In fact, take the divergence of the Maxwell equations (3.4) and (3.5) and use (3.6), which gives ($\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$)

$$\begin{aligned}
&\mathrm{i}\omega\nabla \cdot \boldsymbol{B}(\boldsymbol{r},\omega) = 0\\
&\mathrm{i}\omega\nabla \cdot \boldsymbol{D}(\boldsymbol{r},\omega) = \nabla \cdot \boldsymbol{J}(\boldsymbol{r},\omega) = \mathrm{i}\omega\rho(\boldsymbol{r},\omega)
\end{aligned}$$

Division by iω (provided $\omega \neq 0$) then gives (3.7) and (3.8).

To summarize, in a source-free region the time-harmonic Maxwell equations are

$$\begin{cases} \nabla \times \boldsymbol{E}(\boldsymbol{r},\omega) = \mathrm{i}k_0\,(c_0\boldsymbol{B}(\boldsymbol{r},\omega)) \\ \nabla \times (\eta_0\boldsymbol{H}(\boldsymbol{r},\omega)) = -\mathrm{i}k_0\,(c_0\eta_0\boldsymbol{D}(\boldsymbol{r},\omega)) \end{cases} \tag{3.9}$$

where $\eta_0 = \sqrt{\mu_0/\epsilon_0}$ is the intrinsic wave impedance of vacuum, $c_0 = 1/\sqrt{\epsilon_0\mu_0}$ the speed of light in vacuum, and $k_0 = \omega/c_0$ is the wave number[2] in vacuum. In equation (3.9) all field quantities in parenthesis have the same units, *i.e.*, that of the electric field. This form is the standard form of the Maxwell equations that we use in this textbook.

3.2 Constitutive relations

The constitutive relations introduced in Section 2 for general time dependent fields contain temporal convolutions. In this section, we Fourier transform these relations and arrive at constitutive relations that hold for time harmonic fields.

In Section 2.3, the general Ansatz on the constitutive relations in bianisotropic media is, (2.13)

$$\begin{cases} \boldsymbol{D}(t) = \epsilon_0\Big\{\boldsymbol{\epsilon}\cdot\boldsymbol{E}(t) + \int_{-\infty}^{t} \boldsymbol{\chi}_{\mathrm{ee}}(t-t')\cdot\boldsymbol{E}(t')\,\mathrm{d}t' \\ \qquad\qquad + \eta_0\int_{-\infty}^{t} \boldsymbol{\chi}_{\mathrm{em}}(t-t')\cdot\boldsymbol{H}(t')\,\mathrm{d}t'\Big\} \\ \boldsymbol{B}(t) = \dfrac{1}{c_0}\Big\{\int_{-\infty}^{t} \boldsymbol{\chi}_{\mathrm{me}}(t-t')\cdot\boldsymbol{E}(t')\,\mathrm{d}t' \\ \qquad\qquad + \eta_0\boldsymbol{\mu}\cdot\boldsymbol{H}(t) + \eta_0\int_{-\infty}^{t} \boldsymbol{\chi}_{\mathrm{mm}}(t-t')\cdot\boldsymbol{H}(t')\,\mathrm{d}t'\Big\} \end{cases}$$

or in a six-vector notation of (2.16)

$$\mathsf{d}(t) = \mathsf{A}\cdot\mathsf{e}(t) + \int_{-\infty}^{t} \mathsf{M}(t-t')\cdot\mathsf{e}(t')\,\mathrm{d}t'$$

Since the Fourier transform of a convolution between two fields is a product of their Fourier transforms we get

$$\begin{cases} \boldsymbol{D}(\omega) = \epsilon_0\Big\{\Big(\boldsymbol{\epsilon} + \int_0^{\infty} \boldsymbol{\chi}_{\mathrm{ee}}(t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t\Big)\cdot\boldsymbol{E}(\omega) + \eta_0\int_0^{\infty} \boldsymbol{\chi}_{\mathrm{em}}(t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t\cdot\boldsymbol{H}(\omega)\Big\} \\ \boldsymbol{B}(\omega) = \dfrac{1}{c_0}\Big\{\int_0^{\infty} \boldsymbol{\chi}_{\mathrm{me}}(t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t\cdot\boldsymbol{E}(\omega) + \eta_0\Big(\boldsymbol{\mu} + \int_0^{\infty} \boldsymbol{\chi}_{\mathrm{mm}}(t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t\Big)\cdot\boldsymbol{H}(\omega)\Big\} \end{cases}$$

[2]More correctly, k_0 is the angular wave number in vacuum, and f/c_0 is the wave number in vacuum.

It is convenient to introduce the following notation:

$$\begin{cases} \boldsymbol{\epsilon}(\omega) = \boldsymbol{\epsilon} + \int_0^\infty \boldsymbol{\chi}_{\rm ee}(t) e^{i\omega t}\, dt \\ \boldsymbol{\mu}(\omega) = \boldsymbol{\mu} + \int_0^\infty \boldsymbol{\chi}_{\rm mm}(t) e^{i\omega t}\, dt \end{cases} \qquad \begin{cases} \boldsymbol{\xi}(\omega) = \int_0^\infty \boldsymbol{\chi}_{\rm em}(t) e^{i\omega t}\, dt \\ \boldsymbol{\zeta}(\omega) = \int_0^\infty \boldsymbol{\chi}_{\rm me}(t) e^{i\omega t}\, dt \end{cases} \tag{3.10}$$

with inverses

$$\begin{cases} \boldsymbol{\chi}_{\rm ee}(t) = \frac{1}{2\pi} \int_{-\infty}^\infty [\boldsymbol{\epsilon}(\omega) - \boldsymbol{\epsilon}]\, e^{-i\omega t}\, d\omega \\ \boldsymbol{\chi}_{\rm mm}(t) = \frac{1}{2\pi} \int_{-\infty}^\infty [\boldsymbol{\mu}(\omega) - \boldsymbol{\mu}]\, e^{-i\omega t}\, d\omega \end{cases} \qquad \begin{cases} \boldsymbol{\chi}_{\rm em}(t) = \frac{1}{2\pi} \int_{-\infty}^\infty \boldsymbol{\xi}(\omega)\, e^{-i\omega t}\, d\omega \\ \boldsymbol{\chi}_{\rm me}(t) = \frac{1}{2\pi} \int_{-\infty}^\infty \boldsymbol{\zeta}(\omega)\, e^{-i\omega t}\, d\omega \end{cases}$$

Note the difference between the optical responses $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$, which are constant, real-valued dyadics, and the Fourier transform of the generalized, frequency dependent, dyadic-valued susceptibility $\boldsymbol{\epsilon}(\omega)$ and $\boldsymbol{\mu}(\omega)$. Normally, the context reveals which of the quantities that is intended. In cases where misinterpretation might appear, we explicitly write out the argument.

The generalized susceptibility functions $\boldsymbol{\chi}_{\rm kk'}(t)$, k,k'= e,m, are real-valued, which imply that the complex-valued dyadics $\boldsymbol{\epsilon}(\omega)$, $\boldsymbol{\xi}(\omega)$, $\boldsymbol{\zeta}(\omega)$ and $\boldsymbol{\mu}(\omega)$ satisfy

$$\begin{cases} \boldsymbol{\epsilon}(\omega) = \boldsymbol{\epsilon}^*(-\omega) \\ \boldsymbol{\mu}(\omega) = \boldsymbol{\mu}^*(-\omega) \end{cases} \qquad \begin{cases} \boldsymbol{\xi}(\omega) = \boldsymbol{\xi}^*(-\omega) \\ \boldsymbol{\zeta}(\omega) = \boldsymbol{\zeta}^*(-\omega) \end{cases} \tag{3.11}$$

The (angular) frequency ω in the Fourier transforms in (3.10) is assumed to be real. Nothing prevents us from extending the variable ω into the upper half plane of the complex ω-plane. This is possible due to the fact that $\boldsymbol{\chi}_{\rm kk'}(t)$ are causal quantities, see also Appendix B.2. We adopt $\eta = \omega + i\varsigma$, where ω and ς are real, and $\varsigma \geq 0$. We extend the domain of definition in the Fourier transform to the upper half plane in the complex variable η, *i.e.*, the entries of the permittivity dyadic are

$$\epsilon_{ij}(\eta) = \epsilon_{ij} + \int_0^\infty \chi_{{\rm ee}ij}(t) e^{i\omega t - \varsigma t}\, dt, \quad \varsigma \geq 0$$

and similarly for the other dyadics in (3.10). In fact, the Fourier transform contains an extra exponentially decreasing function which ensures convergence. We have the identity ($-\eta^* = -\omega + i\varsigma$)

$$\epsilon_{ij}(-\eta^*) = \epsilon_{ij} + \int_0^\infty \chi_{{\rm ee}ij}(t) e^{-i\omega t - \varsigma t}\, dt = \epsilon_{ij}(\eta)^*, \quad \varsigma \geq 0 \tag{3.12}$$

This relation generalizes (3.11), which holds for real frequencies ω, to complex values of the frequency in the upper half plane. As a consequence

$$\epsilon_{ij}(i\varsigma) = \epsilon_{ij} + \int_0^\infty \chi_{{\rm ee}ij}(t) e^{-\varsigma t}\, dt = \epsilon_{ij}^*(i\varsigma)$$

is real.

The notation in (3.10) simplifies the constitutive relations for time harmonic fields (or the Fourier transformed fields). We have

$$\begin{cases} \boldsymbol{D} = \epsilon_0 \Big\{ \boldsymbol{\epsilon}(\omega) \cdot \boldsymbol{E}(\omega) + \eta_0 \boldsymbol{\xi}(\omega) \cdot \boldsymbol{H}(\omega) \Big\} \\ \boldsymbol{B} = \dfrac{1}{c_0} \Big\{ \boldsymbol{\zeta}(\omega) \cdot \boldsymbol{E}(\omega) + \eta_0 \boldsymbol{\mu}(\omega) \cdot \boldsymbol{H}(\omega) \Big\} \end{cases} \tag{3.13}$$

or in their Cartesian components

$$\begin{cases} D_i(\omega) = \epsilon_0 \displaystyle\sum_{j=1}^{3} \Big\{ \epsilon_{ij}(\omega) E_j(\omega) + \eta_0 \xi_{ij}(\omega) H_j(\omega) \Big\} \\ B_i(\omega) = \dfrac{1}{c_0} \displaystyle\sum_{j=1}^{3} \Big\{ \zeta_{ij}(\omega) E_j(\omega) + \eta_0 \mu_{ij}(\omega) H_j(\omega) \Big\} \end{cases}$$

In the six-vector notation we write the same relations in a more compact form as

$$\mathsf{d}(\omega) = \left(\mathsf{A} + \int_0^\infty \mathsf{M}(t) e^{i\omega t} \, dt \right) \cdot \mathsf{e}(\omega) = \mathsf{M}(\omega) \cdot \mathsf{e}(\omega)$$

where

$$\mathsf{e}(\omega) = \begin{pmatrix} \boldsymbol{E}(\omega) \\ \eta_0 \boldsymbol{H}(\omega) \end{pmatrix}, \qquad \mathsf{d}(\omega) = c_0 \begin{pmatrix} \eta_0 \boldsymbol{D}(\omega) \\ \boldsymbol{B}(\omega) \end{pmatrix} \tag{3.14}$$

and

$$\mathsf{M}(\omega) = \mathsf{A} + \int_0^\infty \mathsf{M}(t) e^{i\omega t} \, dt = \begin{pmatrix} \boldsymbol{\epsilon}(\omega) & \boldsymbol{\xi}(\omega) \\ \boldsymbol{\zeta}(\omega) & \boldsymbol{\mu}(\omega) \end{pmatrix} \tag{3.15}$$

satisfying $\mathsf{M}(\omega) = \mathsf{M}^*(-\omega)$.

The dyadics (or the corresponding matrix representations) $\boldsymbol{\epsilon}$, $\boldsymbol{\xi}$, $\boldsymbol{\zeta}$ and $\boldsymbol{\mu}$ are all dimensionless dyadics as a consequence of the normalization in (3.13). The dyadics $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ have separate names; they are called the permittivity dyadic, and the permeability dyadic, respectively.

The four dyadics $\boldsymbol{\epsilon}(\boldsymbol{r}, \omega)$, $\boldsymbol{\xi}(\boldsymbol{r}, \omega)$, $\boldsymbol{\zeta}(\boldsymbol{r}, \omega)$, and $\boldsymbol{\mu}(\boldsymbol{r}, \omega)$ depend in general of the spatial variables $\boldsymbol{r}$. For a homogeneous material, the constitutive dyadics are independent of $\boldsymbol{r}$. Notice that $\boldsymbol{\epsilon}$, $\boldsymbol{\xi}$, $\boldsymbol{\zeta}$, and $\boldsymbol{\mu}$ are generally still functions of the angular frequency ω due to (material) temporal dispersion.

3.2.1 Classifications

The classification of different material is made as in Table 3.3. This classification is analogous to the one we introduced for general time dependent fields, see Table 2.1.

Anisotropic materials, characterized by the dyadics $\boldsymbol{\epsilon}(\boldsymbol{r}, \omega)$ and $\boldsymbol{\mu}(\boldsymbol{r}, \omega)$, which in general contain nine independent parameters each. In an isotropic medium, $\boldsymbol{\epsilon}(\boldsymbol{r}, \omega)$ and $\boldsymbol{\mu}(\boldsymbol{r}, \omega)$ are proportional to the identity dyadic $\mathbf{I}_3$. In a biisotropic medium, which is the simplest complex material involving the cross-coupling terms $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$, all the constitutive dyadics are proportional to the identity dyadic $\mathbf{I}_3$.

Type	$\boldsymbol{\epsilon}, \boldsymbol{\mu}$	$\boldsymbol{\xi}, \boldsymbol{\zeta}$
Isotropic	Both $\sim \mathbf{I}_3$	Both $\mathbf{0}$
Anisotropic	Some not $\sim \mathbf{I}_3$	Both $\mathbf{0}$
Biisotropic	Both $\sim \mathbf{I}_3$	Both $\sim \mathbf{I}_3$
Bianisotropic	All other cases	

Table 3.3: Table of classification of materials w.r.t. their constitutive relations for time harmonic fields. The dyadic $\mathbf{I}_3$ denotes the unit dyadic in three dimensions.

Uniaxial	$\epsilon_1 = \epsilon_2 \neq \epsilon_3$
Biaxial	$\epsilon_1 \neq \epsilon_2 \neq \epsilon_3$

Table 3.4: Table of classification of anisotropic media.

The most complex material are modeled by (3.13), and the most simple material in this classification is the isotropic medium, which we analyzed in Section 2.1, in a general time dependent formulation. The constitutive relations for an isotropic medium in the frequency domain are

$$\begin{cases} \boldsymbol{D} = \epsilon_0 \epsilon \boldsymbol{E} \\ \boldsymbol{B} = \mu_0 \mu \boldsymbol{H} \end{cases} \tag{3.16}$$

The parameters ϵ and μ are the (relative) permittivity and permeability of the medium, respectively. The isotropic model is used frequently and is a good model for many insulation materials, *e.g.*, glass, china, and many plastic materials.

We also note that a material with a conductivity that satisfies Ohm's law, (2.4) on page 20, always can be included in the constitutive relations by redefining the permittivity[3] in (3.13).

$$\boldsymbol{\epsilon}_{\text{new}} = \boldsymbol{\epsilon}_{\text{old}} + \mathrm{i}\frac{\boldsymbol{\sigma}}{\omega\epsilon_0}$$

The right-hand side in Ampère's law (3.5) is

$$\boldsymbol{J} - \mathrm{i}\omega\boldsymbol{D} = \boldsymbol{\sigma}\boldsymbol{E} - \mathrm{i}\omega\epsilon_0\Big\{\boldsymbol{\epsilon}_{\text{old}} \cdot \boldsymbol{E} + \eta_0\boldsymbol{\xi} \cdot \boldsymbol{H}\Big\} = -\mathrm{i}\omega\epsilon_0\Big\{\boldsymbol{\epsilon}_{\text{new}} \cdot \boldsymbol{E} + \eta_0\boldsymbol{\xi} \cdot \boldsymbol{H}\Big\}$$

The anisotropic media can be classified further. The permittivity dyadic $\boldsymbol{\epsilon}$ is usually hermitian, *i.e.*,

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{\dagger}$$

which in the entries of the dyadic means $\epsilon_{ij}^{\dagger} = \epsilon_{ji}^{*}$. Furthermore, if the permittivity dyadic $\boldsymbol{\epsilon}$ is real, then there is a basis of real-valued orthonormal vectors such that the permittivity dyadic representation, $[\boldsymbol{\epsilon}]$, in this basis system is diagonal.

$$[\boldsymbol{\epsilon}] = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} \tag{3.17}$$

[3] This treatment is analogous to the one presented in Section 2 where the conductivity σ was included in a new susceptibility function $\chi_{\text{new}}(t)$, see page 22.

Type	**Diagonal elements**	**Crystal symmetry**
Isotropic	$\epsilon_1 = \epsilon_2 = \epsilon_3$	Cubic
Uniaxial	$\epsilon_1 = \epsilon_2 \neq \epsilon_3$	Tetragonal, Hexagonal
		Trigonal
Biaxial	$\epsilon_1 \neq \epsilon_2 \neq \epsilon_3$	Orthorhombic, Hexagonal
		Monoclinic, Triclinic

Table 3.5: Table of crystal symmetries and the values of ϵ_i.

	$\chi = 0$	$\chi \neq 0$
$\kappa = 0$	Isotropic	Chiral, reciprocal
$\kappa \neq 0$	Non-chiral, non-reciprocal	Chiral, non-reciprocal

Table 3.6: Classification of biisotropic materials. The parameters κ and χ are: $\xi = \kappa + \mathrm{i}\chi$ and $\zeta = \kappa - \mathrm{i}\chi$, *i.e.*, $\kappa = (\xi + \zeta)/2$ and $\chi = (\xi - \zeta)/2\mathrm{i}$.

Anisotropic media with a diagonal representation $[\boldsymbol{\epsilon}]$ can be classified as in Table 3.4. For uniaxial media the z-axis is a symmetry axis, which usually is called the optical axis of the material due to many applications at optical frequencies. Moreover, the uniaxial medium is positive (negative) uniaxial if $\epsilon_3 > \epsilon_1 = \epsilon_2$ ($\epsilon_3 < \epsilon_1 = \epsilon_2$).

In a medium that has a lattice structure, the symmetry of the crystal determines the values of ϵ_1, ϵ_2 and ϵ_3. The most important cases are summarized in Table 3.5.

Biisotropic media are classified w.r.t. their symmetry properties.[4] We classify these materials with the help of Table 3.6. From this table, we see that a reciprocal,[5] biisotropic material has the following constitutive relations

$$\begin{cases} \boldsymbol{D} = \epsilon_0 \Big\{ \epsilon(\omega)\boldsymbol{E}(\omega) + \mathrm{i}\eta_0\chi(\omega)\boldsymbol{H}(\omega) \Big\} \\ \boldsymbol{B} = \dfrac{1}{c_0} \Big\{ -\mathrm{i}\chi(\omega)\boldsymbol{E}(\omega) + \eta_0\mu(\omega)\boldsymbol{H}(\omega) \Big\} \end{cases} \tag{3.18}$$

These constitutive relations are often referred to as a chiral material.

[4]In the literature, the Fedorov model

$$\begin{cases} \boldsymbol{D} = \boldsymbol{\epsilon}_\mathrm{F} \cdot \{ \boldsymbol{E} + \boldsymbol{\alpha} \cdot (\nabla \times \boldsymbol{E}) \} \\ \boldsymbol{B} = \boldsymbol{\mu}_\mathrm{F} \cdot \{ \boldsymbol{H} + \boldsymbol{\beta} \cdot (\nabla \times \boldsymbol{H}) \} \end{cases}$$

is often used, which is related to our material parameters by

$$\begin{cases} \epsilon_0 \boldsymbol{\epsilon} = \boldsymbol{\epsilon}_\mathrm{F} \cdot \left(\mathbf{I}_3 - \omega^2 \boldsymbol{\alpha} \cdot \boldsymbol{\mu}_\mathrm{F} \cdot \boldsymbol{\beta} \cdot \boldsymbol{\epsilon}_\mathrm{F} \right)^{-1} \\ c_0^{-1} \boldsymbol{\xi} = \mathrm{i}\omega \boldsymbol{\epsilon}_\mathrm{F} \cdot \boldsymbol{\alpha} \cdot \left(\mathbf{I}_3 - \omega^2 \boldsymbol{\mu}_\mathrm{F} \cdot \boldsymbol{\beta} \cdot \boldsymbol{\epsilon}_\mathrm{F} \cdot \boldsymbol{\alpha} \right)^{-1} \cdot \boldsymbol{\mu}_\mathrm{F} \\ c_0^{-1} \boldsymbol{\zeta} = -\mathrm{i}\omega \left(\mathbf{I}_3 - \omega^2 \boldsymbol{\mu}_\mathrm{F} \cdot \boldsymbol{\beta} \cdot \boldsymbol{\epsilon}_\mathrm{F} \cdot \boldsymbol{\alpha} \right)^{-1} \cdot \boldsymbol{\mu}_\mathrm{F} \cdot \boldsymbol{\beta} \cdot \boldsymbol{\epsilon}_\mathrm{F} \\ \mu_0 \boldsymbol{\mu} = \left(\mathbf{I}_3 - \omega^2 \boldsymbol{\mu}_\mathrm{F} \cdot \boldsymbol{\beta} \cdot \boldsymbol{\epsilon}_\mathrm{F} \cdot \boldsymbol{\alpha} \right)^{-1} \cdot \boldsymbol{\mu}_\mathrm{F} \end{cases}$$

[5]The concept of reciprocity is introduced in Section 3.5.

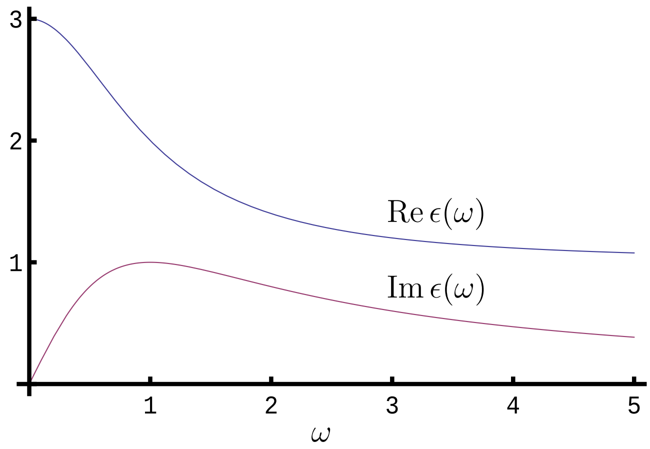


Figure 3.1: The permittivity $\epsilon(\omega)$ as a function of the angular frequency ω for Debye's model. The frequency scale is scaled by $1/\tau$ ($\alpha = 2/\tau$).

3.2.2 Examples

Two important examples of constitutive relations were introduced in Section 2 — Debye's and Lorentz' models. We are now prepared to transform these models by (3.10).

Debye's model: Debye's model has a susceptibility function, see (2.10)

$$\chi(t) = H(t)\alpha \mathrm{e}^{-t/\tau} \qquad \epsilon = 1$$

This susceptibility function is easily transformed to the frequency domain. The result is the permittivity dyadic, $\boldsymbol{\epsilon}(\omega) = \epsilon(\omega)\mathbf{I}_3$, where

$$\boxed{\epsilon(\omega) = \epsilon + \int_0^\infty \chi(t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t = 1 + \frac{\alpha\tau}{1 - i\omega\tau}} \tag{3.19}$$

with real and imaginary parts

$$\begin{cases} \operatorname{Re}\epsilon(\omega) = 1 + \dfrac{\alpha\tau}{1 + \omega^2\tau^2} = \dfrac{1 + \omega^2\tau^2 + \alpha\tau}{1 + \omega^2\tau^2} \\ \operatorname{Im}\epsilon(\omega) = \dfrac{\omega\alpha\tau^2}{1 + \omega^2\tau^2} \end{cases}$$

We note that $\epsilon(\omega = 0) = 1 + \alpha\tau$ and $\epsilon(\omega) = 1$ as $\omega \to \infty$. The typical behavior is depicted in Figure 3.1.

In Figure 3.2 experimental data of the permittivity for water are shown, and a fit to a Debye model is presented. The fit to Debye's model is augmented with an extra term of optical response to account for fast electronic processes in water. Debye's model is most conveniently written as

$$\epsilon(\omega) = \epsilon_\infty + \frac{\epsilon_\mathrm{s} - \epsilon_\infty}{1 - \mathrm{i}\omega\tau}$$

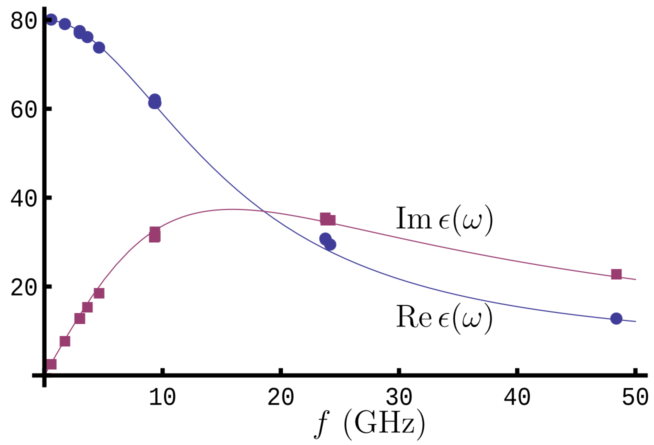


Figure 3.2: Experimental data of the permittivity $\epsilon(\omega)$ as a function of the angular frequency ω for water at 20° C and frequencies up to 50 GHz. (Data are obtained from [3, 7, 17].)

where ϵ_∞ is the permittivity for high frequencies and ϵ_{s} is the static value, $\omega = 0$. The real and imaginary parts of the permittivity with this notation are

$$\begin{cases} \mathrm{Re}\,\epsilon(\omega) = \epsilon_\infty + \dfrac{\epsilon_{\mathrm{s}} - \epsilon_\infty}{1+\omega^2\tau^2} = \dfrac{\epsilon_\infty\omega^2\tau^2 + \epsilon_{\mathrm{s}}}{1+\omega^2\tau^2} \\ \mathrm{Im}\,\epsilon(\omega) = \dfrac{\omega\tau\,(\epsilon_{\mathrm{s}} - \epsilon_\infty)}{1+\omega^2\tau^2} \end{cases} \tag{3.20}$$

Experimental values of the parameters ϵ_∞, ϵ_{s} and τ for water and ethanol are listed in Table 3.7.

In metals and water with salt contents, the permittivity is modified by an extra term of conductivity σ.

$$\boxed{\epsilon(\omega) = \epsilon_\infty + \frac{\epsilon_{\mathrm{s}} - \epsilon_\infty}{1 - \mathrm{i}\omega\tau} + \mathrm{i}\frac{\sigma}{\omega\epsilon_0}} \tag{3.21}$$

Experimental values of the parameters ϵ_∞, ϵ_{s}, τ, and σ for salt and fresh water at microwave frequencies and some metals at optical frequencies are listed in Table 3.7.

At higher frequencies the permittivity for water has a more complex behavior than modeled by Debye's model, due to other dominant more processes. The frequency behavior of the permittivity of water at higher frequencies are shown in Figure 3.3.

Example 3.1

From (3.20) we easily see that the real and the imaginary parts in Debye's model satisfy

$$\begin{aligned}\left(\mathrm{Re}\,\epsilon(\omega) - \frac{\epsilon_{\mathrm{s}} + \epsilon_\infty}{2}\right)^2 + (\mathrm{Im}\,\epsilon(\omega))^2 &= (\epsilon_{\mathrm{s}} - \epsilon_\infty)^2 \left\{\left(\frac{1}{1+\omega^2\tau^2} - \frac{1}{2}\right)^2 + \left(\frac{\omega\tau}{1+\omega^2\tau^2}\right)^2\right\} \\ &= \frac{(\epsilon_{\mathrm{s}} - \epsilon_\infty)^2}{4}\end{aligned}$$

Material	$\epsilon_{\rm s}$	ϵ_∞	τ (s)	σ (S/m)	**Range**
Water (20°)	80.0	5.27	$1.0 \cdot 10^{-11}$		0–50 GHz
Ethanol	25.1	4.4	$1.2 \cdot 10^{-10}$		
Salt water (20°)	80.0	5.27	$1.0 \cdot 10^{-11}$	3–5	0–50 GHz
Fresh water (20°)	80.0	5.27	$1.0 \cdot 10^{-11}$	10^{-3}	0–50 GHz
Gold	−15789	11.575	$8.71 \cdot 10^{-15}$	$1.6062 \cdot 10^{7}$	700–1200 nm
Silver	−9530.5	3.8344	$7.35 \cdot 10^{-15}$	$1.1486 \cdot 10^{7}$	450–1200 nm
Copper	−6672.7	12.076	$5.63 \cdot 10^{-15}$	$1.0513 \cdot 10^{7}$	550–850 nm
Platinum	−30.005	5.3741	$3.28 \cdot 10^{-16}$	$9.5505 \cdot 10^{5}$	400–1200 nm
Aluminum	−656.21	1.8614	$1.07 \cdot 10^{-15}$	$5.4455 \cdot 10^{6}$	200–700 nm

Table 3.7: Experimental values of the material parameters in Debye's model (and modified Debye's model) for water [3, 7, 17] and ethanol in the microwave domain. Data for some metals [9] in the optical domain are also presented. The range of validity is either given in frequency f or in wavelength $\lambda = c_0/f$.

which implies that as the angular frequency ω varies, a circle, centered at $(\epsilon_{\rm s} + \epsilon_\infty)/2$ and radius[6] $(\epsilon_{\rm s} - \epsilon_\infty)/2$, is traced out in the complex ϵ-plane, see Figure 3.4. The curve starts at $\epsilon = \epsilon_{\rm s}$ for $\omega = 0$ and ends at $\epsilon = \epsilon_\infty$ as $\omega \to \infty$. The maximum imaginary part $\epsilon = (\epsilon_{\rm s} + \epsilon_\infty)/2 + {\rm i}(\epsilon_{\rm s} - \epsilon_\infty)/2$ is obtained at $\omega = 1/\tau$. This circle in the complex ϵ-plane is the Cole-Cole plot. This feature can be used experimentally to verify that the constitutive relations is modeled by a Debye model, and the relevant parameters ϵ_∞, $\epsilon_{\rm s}$, and τ can be extracted from the plot. ∎

Example 3.2

In this example we investigate the temperature behavior of the permittivity[7] in a polar liquid, where we assume the dipoles are weakly interacting. The potential energy of the dipole in an electric field $\boldsymbol{E}$ depends on the orientation of the dipole. Provided the electric dipole moment of the constitutive parts of the material is $\boldsymbol{p}$, the potential energy U of each constitutive part of the material is

$$U = -\boldsymbol{p} \cdot \boldsymbol{E} = -pE \cos\theta$$

where the angle between the (permanent) electric dipole moment $\boldsymbol{p}$ and the electric field $\boldsymbol{E}$, directed along $\hat{\boldsymbol{z}}$, is denoted θ, and $p = |\boldsymbol{p}|$ and $E = |\boldsymbol{E}|$. The polarization $\boldsymbol{P}$ then is

$$\boldsymbol{P} = Np\hat{\boldsymbol{z}} <\cos\theta>$$

where N is the number of dipoles per unit volume, and the ensemble average is defined as

$$<f(U)>= \frac{\int f(U) {\rm e}^{-\beta U} \, {\rm d}U}{\int {\rm e}^{-\beta U} \, {\rm d}U}$$

where the integration is over all possible energies, and $\beta = 1/(k_{\rm B}T)$ ($k_{\rm B} = 1.38062 \cdot 10^{-23}$ J/K is the Boltzmann constant, and T is the absolute temperature). In our case,

[6] At end of Section 3.3, we prove that the quantity $\epsilon_{\rm s} - \epsilon_\infty$ always is positive.

[7] More precisely, we investigate the temperature behavior of the static value of the permittivity, $\epsilon_{\rm s} = \epsilon(\omega = 0)$.

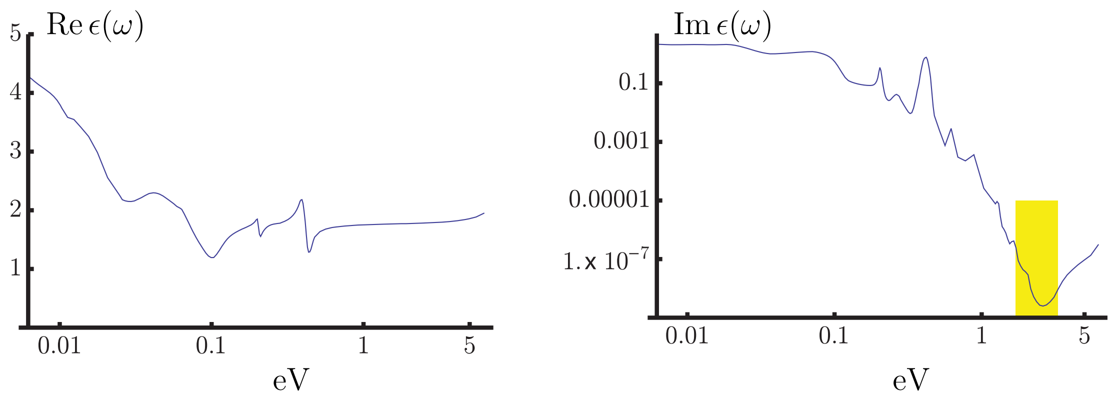


Figure 3.3: Real and imaginary parts of the permittivity for water as a function of frequency (electron volts). 1 eV corresponds to a frequency of $2.42 \cdot 10^{14}$ Hz or a wavelength of 1.24 μm. The frequency behavior at lower frequencies is shown in Figure 3.2. Note the low imaginary part of the permittivity at the optical window, 1.7–3.3 eV, which is marked with a yellow box. (Data are obtained from Hale and Querry, *Appl. Optics* **12**(3), 555 (1973) and Irvine and Pollack, *Icarus* **8**, 324 (1968).)

we get

$$<\cos\theta>=\frac{\int_{-1}^{1}\cos\theta e^{\beta pE\cos\theta}\ \mathrm{d}\cos\theta}{\int_{-1}^{1}e^{\beta pE\cos\theta}\ \mathrm{d}\cos\theta}$$

Introduce the dimensionless constant $x = pE\beta = pE/k_\mathrm{B}T$, and we get

$$<\cos\theta>=\frac{\int_{-1}^{1} te^{xt}\ \mathrm{d}t}{\int_{-1}^{1}e^{xt}\ \mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}x}\ln\int_{-1}^{1}e^{xt}\ \mathrm{d}t=\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(e^{x}-e^{-x}\right)-\frac{\mathrm{d}}{\mathrm{d}x}\ln x\equiv L(x)$$

where we have introduced the Langevin[8] function $L(x)$

$$L(x)=\coth x-\frac{1}{x}$$

which is depicted in Figure 3.5. For small arguments $x \ll 1$ we have

$$L(x)\approx\frac{x}{3}+O(x^2)$$

Finally, we get the average polarization

$$\boldsymbol{P}=Np\hat{\boldsymbol{z}}L(pE/k_\mathrm{B}T)$$

This relation shows that the polarization of the material, in general, depends nonlinearly on the electric field. However, for large temperatures $T \gg k_\mathrm{B}pE$, which is the most interesting case experimentally, we have a linear relation, *i.e.*,

$$\boldsymbol{P}=\frac{Np^2\boldsymbol{E}}{3k_\mathrm{B}T}$$

[8]Paul Langevin (1872–1946), French physicist.

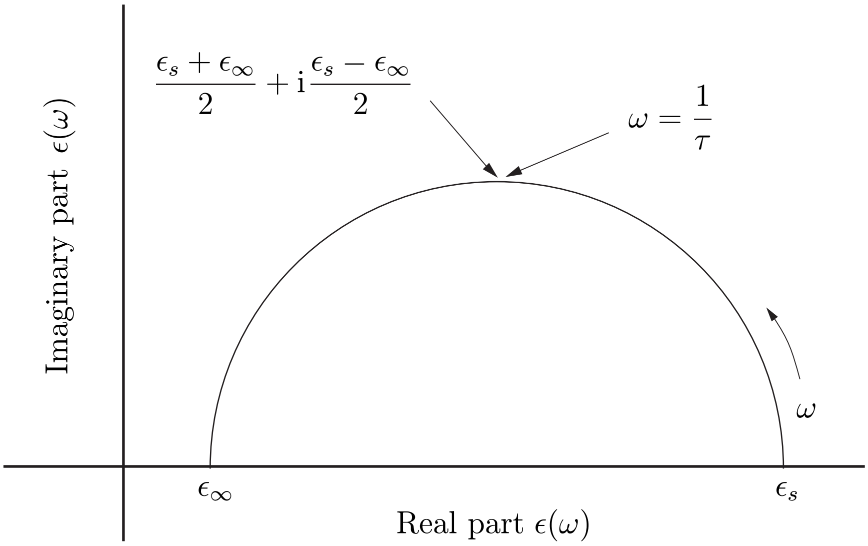


Figure 3.4: The Cole-Cole plot of the permittivity in Example 3.1 in Debye's model.

which implies that the static value of the permittivity $\epsilon_{\rm s}$, as a function of temperature T, is[9]

$$\epsilon_{\rm s}=1+\frac{Np^2}{3\epsilon_0 k_{\rm B}T}$$

■

Lorentz' model Lorentz' model, given by the susceptibility function, see (2.12)

$$\chi(t)=H(t)\frac{\omega_{\rm p}^2}{\nu_0}{\rm e}^{-\frac{\nu t}{2}}\sin\nu_0 t \qquad \epsilon=1$$

is transformed to the frequency domain by a Fourier transform, see (3.10). The result is the permittivity dyadic, $\boldsymbol{\epsilon}(\omega)=\epsilon(\omega)\mathbf{I}_3$, where

$$\boxed{\epsilon(\omega)=\epsilon+\int_0^\infty \chi(t){\rm e}^{{\rm i}\omega t}\,{\rm d}t=1-\frac{\omega_{\rm p}^2}{\omega^2-\omega_0^2+{\rm i}\omega\nu}} \tag{3.22}$$

[9]More precisely, $\boldsymbol{E}$ is the local field at the dipole, and we conclude that the electric polarizability α is

$$\alpha=\frac{p^2}{3\epsilon_0 k_{\rm B}T}$$

where the static permeability $\epsilon_{\rm s}$ and the α are related by Clausius-Mossotti's (Lorenz-Lorentz') law

$$\frac{\epsilon_{\rm s}-1}{\epsilon_{\rm s}+2}=\frac{N\alpha}{3}$$

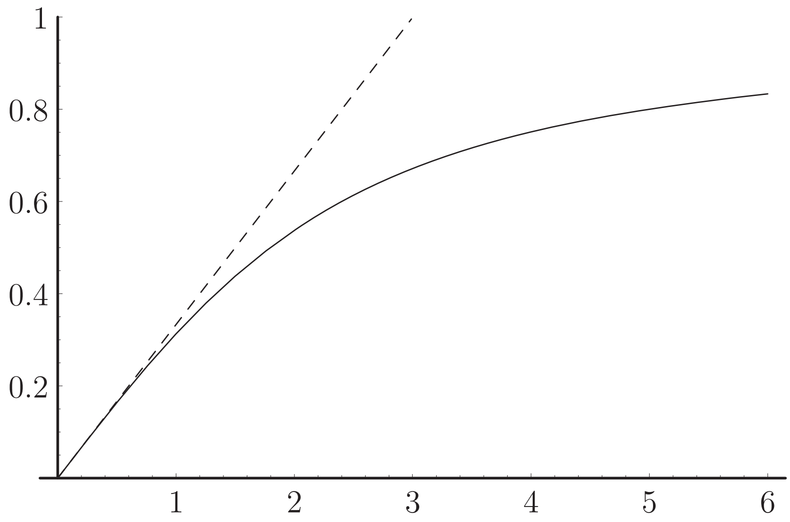


Figure 3.5: The Langevin function $L(x)$ (solid line), and the approximation $x/3$.

with real and imaginary parts

$$\begin{cases} \operatorname{Re}\epsilon(\omega) = 1 - \dfrac{\omega_{\mathrm{p}}^2(\omega^2-\omega_0^2)}{\left(\omega^2-\omega_0^2\right)^2+\omega^2\nu^2} = \dfrac{\left(\omega^2-\omega_0^2\right)^2+\omega^2\nu^2-\omega_{\mathrm{p}}^2(\omega^2-\omega_0^2)}{\left(\omega^2-\omega_0^2\right)^2+\omega^2\nu^2} \\ \operatorname{Im}\epsilon(\omega) = \dfrac{\omega_{\mathrm{p}}^2\omega\nu}{\left(\omega^2-\omega_0^2\right)^2+\omega^2\nu^2} \end{cases}$$

Notice that the real part can take negative values for certain combinations of the material parameters. The scalar function $\epsilon(\omega)$ as a function of frequency for a Lorentz' model is illustrated in Figure 3.6. From the expression above, we note that $\epsilon(\omega = 0) = 1 + \omega_{\mathrm{p}}^2/\omega_0^2$ and $\epsilon(\omega) = 1$ as $\omega \to \infty$.

Drude's model Drude's model[10], corresponds to the case of vanishing restoring force, *i.e.*, $\omega_0 \to 0$. The explicit form is

$$\boxed{\epsilon(\omega) = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2 + \mathrm{i}\omega\nu} = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega(\omega+\mathrm{i}\nu)}} \tag{3.23}$$

with real and imaginary parts

$$\begin{cases} \operatorname{Re}\epsilon(\omega) = 1 - \dfrac{\omega_{\mathrm{p}}^2}{\omega^2+\nu^2} = \dfrac{\omega^2+\nu^2-\omega_{\mathrm{p}}^2}{\omega^2+\nu^2} = -\dfrac{\omega_{\mathrm{p}}^2-\nu^2-\omega^2}{\omega^2+\nu^2} \\ \operatorname{Im}\epsilon(\omega) = \dfrac{\omega_{\mathrm{p}}^2\nu}{\omega(\omega^2+\nu^2)} \end{cases}$$

Notice that the real part of the permittivity assumes negative values for all frequencies $\omega < \sqrt{\omega_{\mathrm{p}}^2 - \nu^2}$, and that the imaginary part has a pole at $\omega = 0$. The

[10]Paul Karl Ludwig Drude (1863–1906), German physicist.

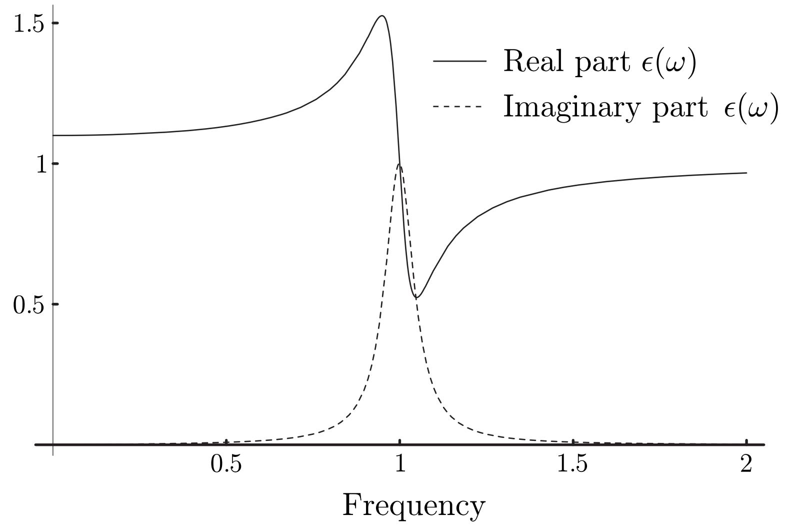


Figure 3.6: The permittivity $\epsilon(\omega)$ as a function of the angular frequency ω for Lorentz' model. The frequency scale is scaled by ω_0 ($\omega_\mathrm{p} = \sqrt{0.1}\omega_0$ and $\nu = 0.1\omega_0$).

scalar function $\epsilon(\omega)$ as a function of frequency for Drude's model is illustrated in Figure 3.7.

Drude's model is closely related to the modified Debye model in (3.21). To see this, write (3.23) as (we have added an optical response ϵ_∞ to Drude's model)

$$\epsilon(\omega) = \epsilon_\infty - \frac{\omega_\mathrm{p}^2/\nu^2}{1 - \mathrm{i}\omega/\nu} + \mathrm{i}\frac{\omega_\mathrm{p}^2/\nu}{\omega}$$

and we see that Drude's model is a special case of the modified Debye model with a conductivity term $\sigma = \epsilon_0\omega_\mathrm{p}^2/\nu$ and

$$\begin{cases} \epsilon_\mathrm{s} = \epsilon_\infty - \omega_\mathrm{p}^2/\nu^2 \\ \tau = 1/\nu \\ \sigma = \epsilon_0\omega_\mathrm{p}^2/\nu \end{cases}$$

Note that the modified Debye model has four free parameters (ϵ_∞, ϵ_s, τ, and σ), but Drude's model only has three (ϵ_∞, ω_p, and ν). The parameters in the modified Debye model therefore have to satisfy a constraint in order to be a Drude model. This constraint is

$$\epsilon_\mathrm{s} - \epsilon_\infty + \frac{\sigma\tau}{\epsilon_0} = 0$$

This constraint is satisfied for the metal data in Table 3.7. Both the modified Debye model and Drude's model have as a special case the standard conductivity model, *i.e.*,

$$\epsilon(\omega) = \widetilde{\epsilon}(\omega) + \mathrm{i}\frac{\sigma}{\omega\epsilon_0} \tag{3.24}$$

where $\widetilde{\epsilon}(\omega)$ is regular at the origin.

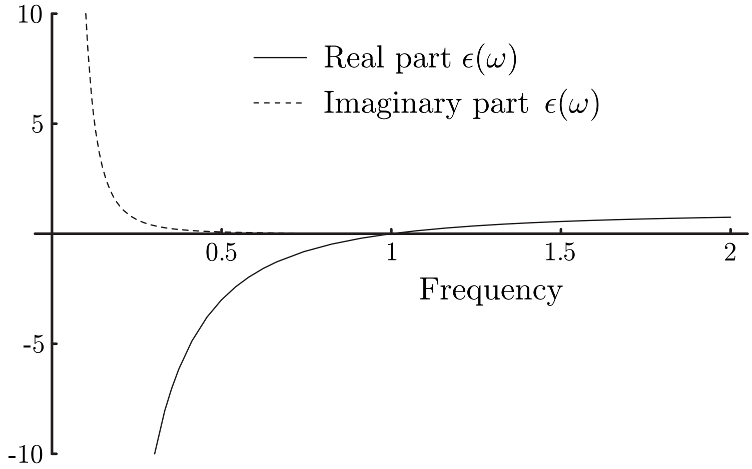


Figure 3.7: The permittivity $\epsilon(\omega)$ as a function of the angular frequency ω for Drude's model. The frequency scale is scaled by ω_0 ($\omega_{\rm p} = \omega_0$ and $\nu = 0.01\omega_0$).

Example 3.3
The plasma is an example of a gyrotropic material. The constitutive relations of a cold plasma are analyzed in Problem 3.5. The result is

$$\boldsymbol{\epsilon}(\omega) = \epsilon \mathbf{I}_2 - \mathrm{i}\mathbf{J}\epsilon_g + \hat{\boldsymbol{z}}\hat{\boldsymbol{z}}\epsilon_z$$

where

$$\begin{cases} \epsilon = 1 - \dfrac{\omega_{\rm p}^2(\omega + \mathrm{i}\nu)}{\omega\left(\omega^2 - \omega_g^2 - \nu^2 + 2\mathrm{i}\nu\omega\right)} \\ \epsilon_g = -\dfrac{\omega_{\rm p}^2\omega_g}{\omega\left(\omega^2 - \omega_g^2 - \nu^2 + 2\mathrm{i}\nu\omega\right)} \\ \epsilon_z = 1 - \dfrac{\omega_{\rm p}^2}{\omega(\omega + \mathrm{i}\nu)} \end{cases}$$

and where ν is the collision frequency, $\mathbf{I}_2$ the identity operator in the x-y-plane, and $\mathbf{J} = \hat{\boldsymbol{z}} \times \mathbf{I}_2$, which is a rotation of $\pi/2$ along the z-axis, and the gyrotropic frequency, ω_g, and, $\omega_{\rm p}$, the plasma frequency of the material, are defined by

$$\omega_g = \frac{qB_0}{m}, \qquad \omega_{\rm p} = \sqrt{\frac{Nq^2}{m\epsilon_0}}$$

and N is the number of charges per unit of volume in the plasma, and m and q are the mass and the charge of the charges, respectively.

In the limit of vanishing collision frequency, $\nu \to 0$, these expressions become

$$\begin{cases} \epsilon = 1 - \dfrac{\omega_{\rm p}^2}{\omega^2 - \omega_g^2} \\ \epsilon_g = -\dfrac{\omega_{\rm p}^2\omega_g}{\omega(\omega^2 - \omega_g^2)} \\ \epsilon_z = 1 - \dfrac{\omega_{\rm p}^2}{\omega^2} \end{cases}$$

The other limiting case, with vanishing gyrotropic frequency, $\omega_g \to 0$, leads to a recovery of Drude' model, (3.23), *i.e.*,

$$\begin{cases} \epsilon = \epsilon_z = 1 - \dfrac{\omega_{\mathrm{p}}^2}{\omega(\omega + \mathrm{i}\nu)} \\ \epsilon_g = 0 \end{cases}$$

∎

Example 3.4
Related to the Lorentz model is the Condon model for optical activity in Example 2.6 on page 35, where we derived the constitutive relations

$$\begin{cases} \chi_{\mathrm{ee}}(t) = H(t)\dfrac{\omega_{\mathrm{p}}^2}{\nu_0} e^{-\frac{\nu t}{2}} \sin \nu_0 t \\ \chi_{\mathrm{em}}(t) = H(t)\omega_{\mathrm{c}} e^{-\frac{\nu t}{2}} \left(\cos \nu_0 t - \dfrac{\nu}{2\nu_0} \sin \nu_0 t \right) \end{cases}$$

with Fourier transforms

$$\int_0^\infty \chi_{\mathrm{ee}}(t) \mathrm{e}^{\mathrm{i}\omega t} \, \mathrm{d}t = -\frac{\omega_{\mathrm{p}}^2}{\omega^2 - \omega_0^2 + \mathrm{i}\omega\nu}$$

and

$$\int_0^\infty \chi_{\mathrm{em}}(t) \mathrm{e}^{\mathrm{i}\omega t} \, \mathrm{d}t = \frac{\mathrm{i}\omega\omega_{\mathrm{c}}}{\omega^2 - \omega_0^2 + \mathrm{i}\omega\nu}$$

and we can identify

$$\begin{cases} \epsilon(\omega) = 1 + \displaystyle\int_0^\infty \chi_{\mathrm{ee}}(t) \mathrm{e}^{\mathrm{i}\omega t} \, \mathrm{d}t = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2 - \omega_0^2 + \mathrm{i}\omega\nu} \\ \xi(\omega) = \displaystyle\int_0^\infty \chi_{\mathrm{em}}(t) \mathrm{e}^{\mathrm{i}\omega t} \, \mathrm{d}t = \frac{\mathrm{i}\omega\omega_{\mathrm{c}}}{\omega^2 - \omega_0^2 + \mathrm{i}\omega\nu} \end{cases}$$

This analysis determines $\epsilon(\omega)$ and $\xi(\omega)$. If the material is non-magnetic $\mu = 1$, and if the material also is reciprocal, see Section 3.5 on page 73 and Table 3.6 on page 51, we have also $\zeta(\omega) = -\xi(\omega)$. Notice that $\xi(\omega) \to 0$ as $\omega \to 0$, and no biisotropic effects remain for static fields. ∎

3.3 Poynting's theorem, active, passive and lossless media

In Section 1 we derived Poynting's theorem, see (1.12) on page 10.

$$\nabla \cdot \boldsymbol{S}(t) + \boldsymbol{H}(t) \cdot \frac{\partial \boldsymbol{B}(t)}{\partial t} + \boldsymbol{E}(t) \cdot \frac{\partial \boldsymbol{D}(t)}{\partial t} + \boldsymbol{E}(t) \cdot \boldsymbol{J}(t) = 0$$

The equation describes conservation of power and contains products of two fields. For a product of time harmonic fields, the most pertinent quantity is the time average over one period.[11] We denote the time average as $<\cdot>$ and for Poynting's theorem we obtain

$$<\nabla\cdot\boldsymbol{S}(t)> + <\boldsymbol{H}(t)\cdot\frac{\partial\boldsymbol{B}(t)}{\partial t}> + <\boldsymbol{E}(t)\cdot\frac{\partial\boldsymbol{D}(t)}{\partial t}> + <\boldsymbol{E}(t)\cdot\boldsymbol{J}(t)>= 0$$

The different terms in this quantity after a time average are

$$<\boldsymbol{S}(t)>= \frac{1}{2}\operatorname{Re}\left\{\boldsymbol{E}(\omega)\times\boldsymbol{H}^*(\omega)\right\} \tag{3.25}$$

and

$$\begin{aligned}
<\boldsymbol{H}(t)\cdot\frac{\partial\boldsymbol{B}(t)}{\partial t}>&= \frac{1}{2}\operatorname{Re}\left\{\mathrm{i}\omega\boldsymbol{H}(\omega)\cdot\boldsymbol{B}^*(\omega)\right\}\\
<\boldsymbol{E}(t)\cdot\frac{\partial\boldsymbol{D}(t)}{\partial t}>&= \frac{1}{2}\operatorname{Re}\left\{\mathrm{i}\omega\boldsymbol{E}(\omega)\cdot\boldsymbol{D}^*(\omega)\right\}\\
<\boldsymbol{E}(t)\cdot\boldsymbol{J}(t)>&= \frac{1}{2}\operatorname{Re}\left\{\boldsymbol{E}(\omega)\cdot\boldsymbol{J}^*(\omega)\right\}
\end{aligned}$$

Poynting's theorem (balance of power) for time harmonic fields, averaged over a period, becomes (Notice that the time average and the differentiation w.r.t. space commute, *i.e.*, $<\nabla\cdot\boldsymbol{S}(t)>= \nabla\cdot<\boldsymbol{S}(t)>$):

$$\begin{aligned}
\nabla\cdot<\boldsymbol{S}(t)> &+ \frac{1}{2}\operatorname{Re}\left\{\mathrm{i}\omega\left[\boldsymbol{H}(\omega)\cdot\boldsymbol{B}^*(\omega)+\boldsymbol{E}(\omega)\cdot\boldsymbol{D}^*(\omega)\right]\right\}\\
&+\frac{1}{2}\operatorname{Re}\left\{\boldsymbol{E}(\omega)\cdot\boldsymbol{J}^*(\omega)\right\} = 0
\end{aligned} \tag{3.26}$$

Of special interest is the case without currents[12] $\boldsymbol{J}=\boldsymbol{0}$. Poynting's theorem is then simplified to

$$\begin{aligned}
\nabla\cdot<\boldsymbol{S}(t)> &= -\frac{1}{2}\operatorname{Re}\left\{\mathrm{i}\omega\left[\boldsymbol{H}(\omega)\cdot\boldsymbol{B}^*(\omega)+\boldsymbol{E}(\omega)\cdot\boldsymbol{D}^*(\omega)\right]\right\}\\
&= -\frac{\mathrm{i}\omega}{4}\Big\{\boldsymbol{H}(\omega)\cdot\boldsymbol{B}^*(\omega)-\boldsymbol{H}^*(\omega)\cdot\boldsymbol{B}(\omega)\\
&\qquad +\boldsymbol{E}(\omega)\cdot\boldsymbol{D}^*(\omega)-\boldsymbol{E}(\omega)^*\cdot\boldsymbol{D}(\omega)\Big\}
\end{aligned}$$

where we used $\operatorname{Re} z = (z+z^*)/2$.

[11]The time average of a product of two time harmonic fields $f_1(t)$ and $f_2(t)$ is easily obtained by an average over one period $T=2\pi/\omega$.

$$\begin{aligned}
<f_1(t)f_2(t)> &= \frac{1}{T}\int_0^T f_1(t)f_2(t)\,\mathrm{d}t = \frac{1}{T}\int_0^T \operatorname{Re}\left\{f_1(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\right\}\operatorname{Re}\left\{f_2(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\right\}\,\mathrm{d}t\\
&= \frac{1}{4T}\int_0^T \left\{f_1(\omega)f_2(\omega)\mathrm{e}^{-2\mathrm{i}\omega t}+f_1^*(\omega)f_2^*(\omega)\mathrm{e}^{2\mathrm{i}\omega t}+f_1(\omega)f_2^*(\omega)+f_1^*(\omega)f_2(\omega)\right\}\,\mathrm{d}t\\
&= \frac{1}{4}\left\{f_1(\omega)f_2^*(\omega)+f_1^*(\omega)f_2(\omega)\right\} = \frac{1}{2}\operatorname{Re}\left\{f_1(\omega)f_2^*(\omega)\right\}
\end{aligned}$$

[12]Conducting currents can, as we have seen, be included in the permittivity dyadic $\boldsymbol{\epsilon}$.

Enter the constitutive relations from equation (3.13). The divergence of Poynting's vector is then expressed in the fields $\boldsymbol{E}$ and $\boldsymbol{H}$.

$$\begin{aligned}\nabla\cdot <\boldsymbol{S}(t)> &= -\frac{\mathrm{i}\omega\epsilon_0}{4}\Big\{\eta_0\boldsymbol{H}\cdot\big(\boldsymbol{\zeta}^*\cdot\boldsymbol{E}^*+\eta_0\boldsymbol{\mu}^*\cdot\boldsymbol{H}^*\big)-\eta_0\boldsymbol{H}^*\cdot\big(\boldsymbol{\zeta}\cdot\boldsymbol{E}+\eta_0\boldsymbol{\mu}\cdot\boldsymbol{H}\big)\\ &\qquad +\boldsymbol{E}\cdot\big(\boldsymbol{\epsilon}^*\cdot\boldsymbol{E}^*+\eta_0\boldsymbol{\xi}^*\cdot\boldsymbol{H}^*\big)-\boldsymbol{E}^*\cdot\big(\boldsymbol{\epsilon}\cdot\boldsymbol{E}+\eta_0\boldsymbol{\xi}\cdot\boldsymbol{H}\big)\Big\}\\ &= \frac{\mathrm{i}\omega\epsilon_0}{4}\Big\{\boldsymbol{E}^*\cdot\big(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}^\dagger\big)\cdot\boldsymbol{E}+\eta_0^2\boldsymbol{H}^*\cdot\big(\boldsymbol{\mu}-\boldsymbol{\mu}^\dagger\big)\cdot\boldsymbol{H}\\ &\qquad +\eta_0\boldsymbol{E}^*\cdot\big(\boldsymbol{\xi}-\boldsymbol{\zeta}^\dagger\big)\cdot\boldsymbol{H}+\eta_0\boldsymbol{H}^*\cdot\big(\boldsymbol{\zeta}-\boldsymbol{\xi}^\dagger\big)\cdot\boldsymbol{E}\Big\}\end{aligned}$$

where the dagger † denotes the Hermitian transpose of the dyadic, see Appendix C, and we have also used

$$\boldsymbol{a}\cdot\mathbf{A}^*\cdot\boldsymbol{b}^*=\boldsymbol{b}^*\cdot\mathbf{A}^\dagger\cdot\boldsymbol{a}$$

It is often convenient to use a combined matrix and dyadic notation. The divergence of Poynting's vector can then be written in the following compact form:

$$\nabla\cdot <\boldsymbol{S}(t)>=\frac{\mathrm{i}\omega\epsilon_0}{4}\begin{pmatrix}\boldsymbol{E}\\ \eta_0\boldsymbol{H}\end{pmatrix}^\dagger\cdot\begin{pmatrix}\boldsymbol{\epsilon}-\boldsymbol{\epsilon}^\dagger & \boldsymbol{\xi}-\boldsymbol{\zeta}^\dagger\\ \boldsymbol{\zeta}-\boldsymbol{\xi}^\dagger & \boldsymbol{\mu}-\boldsymbol{\mu}^\dagger\end{pmatrix}\cdot\begin{pmatrix}\boldsymbol{E}\\ \eta_0\boldsymbol{H}\end{pmatrix} \tag{3.27}$$

In the six-dimensional formulation of (3.14) and (3.15), we have

$$\nabla\cdot <\boldsymbol{S}(t)>=\frac{\mathrm{i}\omega\epsilon_0}{4}\mathsf{e}^\dagger(\omega)\cdot\big(\mathsf{M}(\omega)-\mathsf{M}^\dagger(\omega)\big)\cdot\mathsf{e}(\omega)=-\frac{\omega\epsilon_0}{2}\mathsf{e}^\dagger(\omega)\cdot\operatorname{Im}\mathsf{M}(\omega)\cdot\mathsf{e}(\omega)$$

where the imaginary part of the dyadic M is defined as

$$\operatorname{Im}\mathsf{M}=\mathsf{M}_\mathrm{i}=\frac{1}{2\mathrm{i}}\left(\mathsf{M}-\mathsf{M}^\dagger\right)$$

Note that the imaginary part M_i is an Hermitian dyadic, see also Appendix A.2.

The quantity $-\nabla\cdot <\boldsymbol{S}(t)>$ gives a measure of the average power the electromagnetic field delivers to the material per unit volume. This quantity can be used to classify the material as active, passive, or lossless depending on the sign of this quantity. The following definitions are introduced for a fixed (angular) frequency $\omega\neq 0$:

Passive material if	$\nabla\cdot <\boldsymbol{S}(t)> < 0$	
Active material if	$\nabla\cdot <\boldsymbol{S}(t)> > 0$	for all fields $\{\boldsymbol{E},\boldsymbol{H}\}\neq\{\mathbf{0},\mathbf{0}\}$
Lossless material if	$\nabla\cdot <\boldsymbol{S}(t)> = 0$	

These definitions have the following physical implications. Integrate $\nabla\cdot <\boldsymbol{S}(t)>$ over a volume V bounded by S (outward directed normal $\hat{\boldsymbol{\nu}}$). The divergence theorem gives

$$\text{Passive material} \qquad \iint_S <\boldsymbol{S}(t)>\cdot\hat{\boldsymbol{\nu}}\ \mathrm{d}S < 0$$

$$\text{Active material} \qquad \iint_S <\boldsymbol{S}(t)>\cdot\hat{\boldsymbol{\nu}}\ \mathrm{d}S > 0$$

$$\text{Lossless material} \qquad \iint_S <\boldsymbol{S}(t)>\cdot\hat{\boldsymbol{\nu}}\ \mathrm{d}S = 0$$

These definitions imply that for a passive material in V, the outward radiated power is always negative, $\iint_S <\boldsymbol{S}(t)>\cdot\hat{\boldsymbol{\nu}}\ \mathrm{d}S < 0$, but for an active material this is positive, due to creation of electromagnetic energy (by non-electromagnetic sources in V). In a lossless material the outward radiated power averaged over a period is always balanced by the power radiated inward through the surface during one period. Notice that this classification holds for a fixed frequency. A material can be passive for one frequency, active or lossless for another. However, a material cannot be lossless for all frequencies.

3.3.1 Lossless material

In a lossless material, the dyadics $\boldsymbol{\epsilon}$, $\boldsymbol{\xi}$, $\boldsymbol{\zeta}$, and $\boldsymbol{\mu}$ in (3.13) have to satisfy some conditions. From (3.27) we see that

$$\begin{cases} \boldsymbol{\epsilon} = \boldsymbol{\epsilon}^\dagger \\ \boldsymbol{\mu} = \boldsymbol{\mu}^\dagger \\ \boldsymbol{\xi} = \boldsymbol{\zeta}^\dagger \end{cases} \tag{3.28}$$

since the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ can be chosen arbitrary. In an isotropic material we immediately see that ϵ and μ in (3.16) have to be real for a lossless material.[13] Moreover, for lossless, anisotropic material with real permittivity matrix

$$\epsilon_{ij} = \epsilon_{ji}$$

i.e., $[\boldsymbol{\epsilon}]$ is a symmetric matrix, which can be diagonalized (with real coordinate axes) and a classification as in Table 3.4.

In the six-dimensional formulation of (3.14) and (3.15), the lossless materials satisfy

$$\operatorname{Im} \mathsf{M} = 0$$

[13] An isotropic material with dispersion cannot be lossless for all frequencies, since then, see (3.10)

$$0 = \int_0^\infty \chi(t)\sin\omega t\ \mathrm{d}t = \frac{1}{2\mathrm{i}}\Big\{\int_0^\infty \chi(t)\mathrm{e}^{\mathrm{i}\omega t}\ \mathrm{d}t - \int_{-\infty}^0 \chi(-t)\mathrm{e}^{\mathrm{i}\omega t}\ \mathrm{d}t\Big\} \text{ for all } \omega$$

which implies that the function $H(t)\chi(t) - H(-t)\chi(-t) = 0$, and $\chi(t)$ is identically zero, which is a contradiction.

3.3.2 Passive material

We now proceed by investigating the consequences on the the constitutive relations for a passive material.

In a passive material the constitutive relations satisfy, see (3.27)

$$\begin{pmatrix} \boldsymbol{E} \\ \eta_0 \boldsymbol{H} \end{pmatrix}^\dagger \cdot \begin{pmatrix} \mathrm{i}\omega\left(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}^\dagger\right) & \mathrm{i}\omega\left(\boldsymbol{\xi}-\boldsymbol{\zeta}^\dagger\right) \\ \mathrm{i}\omega\left(\boldsymbol{\zeta}-\boldsymbol{\xi}^\dagger\right) & \mathrm{i}\omega\left(\boldsymbol{\mu}-\boldsymbol{\mu}^\dagger\right) \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{E} \\ \eta_0 \boldsymbol{H} \end{pmatrix} < 0$$

for all (non-static) fields $\{\boldsymbol{E},\boldsymbol{H}\} \neq \{\mathbf{0},\mathbf{0}\}$, *i.e.*, the Hermitian dyadic

$$2\omega \operatorname{Im} \mathsf{M} = \begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\beta} \\ \boldsymbol{\gamma} & \boldsymbol{\delta} \end{pmatrix} = \begin{pmatrix} -\mathrm{i}\omega\left(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}^\dagger\right) & -\mathrm{i}\omega\left(\boldsymbol{\xi}-\boldsymbol{\zeta}^\dagger\right) \\ -\mathrm{i}\omega\left(\boldsymbol{\zeta}-\boldsymbol{\xi}^\dagger\right) & -\mathrm{i}\omega\left(\boldsymbol{\mu}-\boldsymbol{\mu}^\dagger\right) \end{pmatrix} > 0$$

is a positive definite six-dimensional dyadic.[14] Here, we have employed the six-dimensional formulation of (3.14) and (3.15), and we for convenience have introduced the notations

$$\begin{cases} \boldsymbol{\alpha} = -\mathrm{i}\omega\left(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}^\dagger\right) = 2\omega \operatorname{Im}\boldsymbol{\epsilon} \\ \boldsymbol{\beta} = -\mathrm{i}\omega\left(\boldsymbol{\xi}-\boldsymbol{\zeta}^\dagger\right) \end{cases} \qquad \begin{cases} \boldsymbol{\gamma} = -\mathrm{i}\omega\left(\boldsymbol{\zeta}-\boldsymbol{\xi}^\dagger\right) \\ \boldsymbol{\delta} = -\mathrm{i}\omega\left(\boldsymbol{\mu}-\boldsymbol{\mu}^\dagger\right) = 2\omega \operatorname{Im}\boldsymbol{\mu} \end{cases}$$

Note that the $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$ are Hermitian dyadics, and $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ are related, *i.e.*,

$$\begin{cases} \boldsymbol{\alpha} = \boldsymbol{\alpha}^\dagger \\ \boldsymbol{\delta} = \boldsymbol{\delta}^\dagger \end{cases} \qquad \begin{cases} \boldsymbol{\beta} = \boldsymbol{\gamma}^\dagger \\ \boldsymbol{\gamma} = \boldsymbol{\beta}^\dagger \end{cases}$$

In a passive material, we thus have

$$\boldsymbol{E}^* \cdot \boldsymbol{\alpha} \cdot \boldsymbol{E} + \boldsymbol{E}^* \cdot \boldsymbol{\beta} \cdot \eta_0 \boldsymbol{H} + \eta_0 \boldsymbol{H}^* \cdot \boldsymbol{\gamma} \cdot \boldsymbol{E} + \eta_0 \boldsymbol{H}^* \cdot \boldsymbol{\delta} \cdot \eta_0 \boldsymbol{H} > 0 \tag{3.29}$$

for all (non-static) fields $\{\boldsymbol{E},\boldsymbol{H}\} \neq \{\mathbf{0},\mathbf{0}\}$. Applied to the fields $\{\boldsymbol{E},\mathbf{0}\}$ and $\{\mathbf{0},\boldsymbol{H}\}$ implies

$$\boldsymbol{\alpha} > 0, \qquad \boldsymbol{\delta} > 0$$

i.e., the three-dimensional dyadics $\boldsymbol{\alpha} = -\mathrm{i}\omega\left(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}^\dagger\right)$ and $\boldsymbol{\delta} = -\mathrm{i}\omega\left(\boldsymbol{\mu}-\boldsymbol{\mu}^\dagger\right)$ are positive definite dyadics. Specifically, the diagonal entries are positive for all frequencies.

Introduce the square root $\sqrt{\boldsymbol{\alpha}}$ of the positive definite dyadic $\boldsymbol{\alpha}$, *i.e.*,

$$\boldsymbol{\alpha} = \sqrt{\boldsymbol{\alpha}} \cdot \sqrt{\boldsymbol{\alpha}}$$

where $\sqrt{\boldsymbol{\alpha}}$ is a Hermitian dyadic. This dyadic is also positive definite. From the identity and $\boldsymbol{\beta}^\dagger = \boldsymbol{\gamma}$

$$\begin{aligned} &\left|\sqrt{\boldsymbol{\alpha}} \cdot \boldsymbol{E} + \sqrt{\boldsymbol{\alpha}}^{-1} \cdot \boldsymbol{\beta} \cdot \eta_0 \boldsymbol{H}\right|^2 \\ =&\left(\boldsymbol{E}^* \cdot \sqrt{\boldsymbol{\alpha}} + \eta_0 \boldsymbol{H}^* \cdot \boldsymbol{\gamma} \cdot \sqrt{\boldsymbol{\alpha}}^{-1}\right) \cdot \left(\sqrt{\boldsymbol{\alpha}} \cdot \boldsymbol{E} + \sqrt{\boldsymbol{\alpha}}^{-1} \cdot \boldsymbol{\beta} \cdot \eta_0 \boldsymbol{H}\right) \\ =&\boldsymbol{E}^* \cdot \boldsymbol{\alpha} \cdot \boldsymbol{E} + \boldsymbol{E}^* \cdot \boldsymbol{\beta} \cdot \eta_0 \boldsymbol{H} + \eta_0 \boldsymbol{H}^* \cdot \boldsymbol{\gamma} \cdot \boldsymbol{E} + \eta_0 \boldsymbol{H}^* \cdot \boldsymbol{\gamma} \cdot \boldsymbol{\alpha}^{-1} \cdot \boldsymbol{\beta} \cdot \eta_0 \boldsymbol{H} \end{aligned}$$

[14] As a consequence, all eigenvalues of the dyadic are positive.

we rewrite the condition (3.29) as

$$\left|\sqrt{\boldsymbol{\alpha}}\cdot\boldsymbol{E}+\sqrt{\boldsymbol{\alpha}}^{-1}\cdot\boldsymbol{\beta}\cdot\eta_0\boldsymbol{H}\right|^2+\eta_0\boldsymbol{H}^*\cdot\left(\boldsymbol{\delta}-\boldsymbol{\gamma}\cdot\boldsymbol{\alpha}^{-1}\cdot\boldsymbol{\beta}\right)\cdot\eta_0\boldsymbol{H}>0$$

This inequality implies that

$$\boldsymbol{\delta}-\boldsymbol{\gamma}\cdot\boldsymbol{\alpha}^{-1}\cdot\boldsymbol{\beta}>0$$

i.e., the dyadic $\boldsymbol{\delta}-\boldsymbol{\gamma}\cdot\boldsymbol{\alpha}^{-1}\cdot\boldsymbol{\beta}$ is positive definite for all frequencies. In terms of the material dyadics $\boldsymbol{\epsilon}$, $\boldsymbol{\mu}$, $\boldsymbol{\xi}$, and $\boldsymbol{\zeta}$, this condition becomes

$$\boldsymbol{\delta}-\boldsymbol{\gamma}\cdot\boldsymbol{\alpha}^{-1}\cdot\boldsymbol{\beta}=2\omega\,\mathrm{Im}\,\boldsymbol{\mu}+\frac{1}{2}\omega\left(\boldsymbol{\zeta}-\boldsymbol{\xi}^{\dagger}\right)\cdot\left(\mathrm{Im}\,\boldsymbol{\epsilon}\right)^{-1}\cdot\left(\boldsymbol{\xi}-\boldsymbol{\zeta}^{\dagger}\right)>0$$

In summary, the condition for a passive material for a bianisotropic material is

$$\begin{cases}\omega\,\mathrm{Im}\,\boldsymbol{\epsilon}>0\\ \omega\,\mathrm{Im}\,\boldsymbol{\mu}>0\\ \omega\left\{4\,\mathrm{Im}\,\boldsymbol{\mu}-\left(\boldsymbol{\xi}-\boldsymbol{\zeta}^{\dagger}\right)^{\dagger}\cdot\left(\mathrm{Im}\,\boldsymbol{\epsilon}\right)^{-1}\cdot\left(\boldsymbol{\xi}-\boldsymbol{\zeta}^{\dagger}\right)\right\}>0\end{cases}\tag{3.30}$$

Example 3.5

In an isotropic material, *i.e.*,

$$\begin{cases}\boldsymbol{\epsilon}=\epsilon\mathbf{I}\\ \boldsymbol{\mu}=\mu\mathbf{I}\end{cases}\qquad\begin{cases}\boldsymbol{\xi}=\mathbf{0}\\ \boldsymbol{\zeta}=\mathbf{0}\end{cases}$$

the passive condition in (3.30)implies

$$\begin{cases}\omega\,\mathrm{Im}\,\epsilon>0\\ \omega\,\mathrm{Im}\,\mu>0\end{cases}$$

This condition is true for the models by Debye and Lorentz. ∎

Example 3.6

In a biisotropic material, the constitutive relations are

$$\begin{cases}\boldsymbol{\epsilon}=\epsilon\mathbf{I}\\ \boldsymbol{\mu}=\mu\mathbf{I}\end{cases}\qquad\begin{cases}\boldsymbol{\xi}=\xi\mathbf{I}=(\kappa+\mathrm{i}\chi)\mathbf{I}\\ \boldsymbol{\zeta}=\zeta\mathbf{I}=(\kappa-\mathrm{i}\chi)\mathbf{I}\end{cases}$$

where we for convenience have introduced two new, complex-valued parameters, κ and χ, the reciprocity and the chirality parameters, respectively. These are expressed in the previous ones as

$$\begin{cases}\xi=\kappa+\mathrm{i}\chi\\ \zeta=\kappa-\mathrm{i}\chi\end{cases}\qquad\begin{cases}\kappa=\dfrac{\zeta+\xi}{2}\\ \chi=\mathrm{i}\dfrac{\zeta-\xi}{2}\end{cases}$$

The specific condition on the material parameters in a biisotropic material to be passive is, see (3.30) ($\omega > 0$)

$$\begin{cases} \operatorname{Im}\epsilon > 0 \\ \operatorname{Im}\mu > 0 \\ |\xi - \zeta^*|^2 < 4\operatorname{Im}\epsilon\operatorname{Im}\mu \end{cases}$$

Specifically, if the material is reciprocal ($\xi = -\zeta$) we have

$$|\operatorname{Re}\xi| = |\operatorname{Re}\zeta| < \sqrt{\operatorname{Im}\epsilon\operatorname{Im}\mu}$$

An immediate consequence of this is that passive, reciprocal, biisotropic material cannot have vanishing imaginary parts of μ or ϵ. In particular, adopting the Condon model in Example 3.4, and Lorentz models for ϵ and μ, *i.e.*,

$$\begin{cases} \epsilon(\omega) = 1 - \dfrac{\omega_{\text{pe}}^2}{\omega^2 - \omega_{0\text{e}}^2 + \mathrm{i}\omega\nu_{\text{e}}} \\ \mu(\omega) = 1 - \dfrac{\omega_{\text{pm}}^2}{\omega^2 - \omega_{0\text{m}}^2 + \mathrm{i}\omega\nu_{\text{m}}} \\ \xi(\omega) = -\zeta(\omega) = \dfrac{\mathrm{i}\omega\omega_{\text{c}}}{\omega^2 - \omega_{0\text{ce}}^2 + \mathrm{i}\omega\nu_{\text{c}}} \end{cases} \qquad \begin{cases} \operatorname{Im}\epsilon(\omega) = \dfrac{\omega\nu_{\text{e}}\omega_{\text{pe}}^2}{\left(\omega^2 - \omega_{0\text{e}}^2\right)^2 + \omega^2\nu_{\text{e}}^2} \\ \operatorname{Im}\mu(\omega) = \dfrac{\omega\nu_{\text{m}}\omega_{\text{pm}}^2}{\left(\omega^2 - \omega_{0\text{m}}^2\right)^2 + \omega^2\nu_{\text{m}}^2} \\ \operatorname{Re}\xi(\omega) = \dfrac{\omega^2\nu_{\text{c}}\omega_{\text{c}}}{\left(\omega^2 - \omega_{0\text{c}}^2\right)^2 + \omega^2\nu_{\text{c}}^2} \end{cases}$$

We see that this model always becomes active at high frequencies, and, therefore, Condon's model has its main potential as a model of chiral material and optic activity at low frequencies. There are many ways to remedy this deficiency. A slight modification of the model above, *e.g.*,

$$\begin{cases} \epsilon(\omega) = 1 - \dfrac{\omega_{\text{pe}}^2}{\omega^2 - \omega_0^2 + \mathrm{i}\omega\nu_{\text{e}}} + \dfrac{\alpha\tau}{1 - \mathrm{i}\omega\tau} \\ \mu(\omega) = 1 - \dfrac{\omega_{\text{pm}}^2}{\omega^2 - \omega_0^2 + \mathrm{i}\omega\nu_{\text{m}}} \\ \xi(\omega) = -\zeta(\omega) = \dfrac{\mathrm{i}\omega\omega_{\text{c}}}{\omega^2 - \omega_0^2 + \mathrm{i}\omega\nu_{\text{c}}} \end{cases}$$

makes the model passive again, since

$$\begin{cases} \operatorname{Im}\epsilon(\omega) = \dfrac{\omega\nu_{\text{e}}\omega_{\text{pe}}^2}{\left(\omega^2 - \omega_0^2\right)^2 + \omega^2\nu_{\text{e}}^2} + \dfrac{\omega\alpha\tau^2}{1 + \omega^2\tau^2} \\ \operatorname{Im}\mu(\omega) = \dfrac{\omega\nu_{\text{m}}\omega_{\text{pm}}^2}{\left(\omega^2 - \omega_0^2\right)^2 + \omega^2\nu_{\text{m}}^2} \\ \operatorname{Re}\xi(\omega) = \dfrac{\omega^2\omega_{\text{c}}\nu_{\text{c}}}{\left(\omega^2 - \omega_0^2\right)^2 + \omega^2\nu_{\text{c}}^2} \end{cases}$$

and to leading power in ω, $\operatorname{Im}\epsilon\operatorname{Im}\mu - (\operatorname{Re}\xi)^2$ behaves as $\omega \to \infty$

$$\frac{\alpha}{\omega}\frac{\nu_{\text{m}}\omega_{\text{pm}}^2}{\omega^3} - \frac{\omega_{\text{c}}^2\nu_{\text{c}}^2}{\omega^4}$$

which is positive provided $\alpha\nu_{\text{m}}\omega_{\text{pm}}^2 > \omega_{\text{c}}^2\nu_{\text{c}}^2$. ∎

Example 3.7
The plasma model in Example 3.3 shows that

$$\boldsymbol{\epsilon}(\omega) = \epsilon \mathbf{I}_2 - \mathrm{i}\mathbf{J}\epsilon_g + \hat{\boldsymbol{z}}\hat{\boldsymbol{z}}\epsilon_z$$

where

$$\begin{cases} \epsilon = 1 - \dfrac{\omega_\mathrm{p}^2(\omega + \mathrm{i}\nu)}{\omega\left(\omega^2 - \omega_g^2 - \nu^2 + 2\mathrm{i}\nu\omega\right)} \\ \epsilon_g = -\dfrac{\omega_\mathrm{p}^2\omega_g}{\omega\left(\omega^2 - \omega_g^2 - \nu^2 + 2\mathrm{i}\nu\omega\right)} \\ \epsilon_z = 1 - \dfrac{\omega_\mathrm{p}^2}{\omega(\omega + \mathrm{i}\nu)} \end{cases}$$

To investigate whether this model is passive or not, form

$$\omega \operatorname{Im} \boldsymbol{\epsilon} = \mathbf{I}_2\omega \operatorname{Im} \epsilon - \mathrm{i}\mathbf{J}\omega \operatorname{Im} \epsilon_g + \hat{\boldsymbol{z}}\hat{\boldsymbol{z}}\omega \operatorname{Im} \epsilon_z$$

The eigenvalues of $\omega \operatorname{Im} \boldsymbol{\epsilon}(\omega)$ are $\lambda = \omega \operatorname{Im}(\epsilon \pm \epsilon_g), \omega \operatorname{Im} \epsilon_z$. Therefore, the plasma model is a passive material, since

$$\omega \operatorname{Im} \epsilon_z = \frac{\omega_\mathrm{p}^2\nu}{\omega^2 + \nu^2} > 0$$

and

$$\omega \operatorname{Im}(\epsilon \pm \epsilon_g) = -\operatorname{Im} \frac{\omega_\mathrm{p}^2(\omega \pm \omega_g + \mathrm{i}\nu)}{\omega^2 - \omega_g^2 - \nu^2 + 2\mathrm{i}\nu\omega} = \frac{\omega_\mathrm{p}^2((\omega \pm \omega_g)2\nu\omega - \nu(\omega^2 - \omega_g^2 - \nu^2))}{(\omega^2 - \omega_g^2 - \nu^2)^2 + 4\nu^2\omega^2}$$

or

$$\omega \operatorname{Im}(\epsilon \pm \epsilon_g) = \omega_\mathrm{p}^2\nu \frac{(\omega \pm \omega_g)^2 + \nu^2}{(\omega^2 - \omega_g^2 - \nu^2)^2 + 4\nu^2\omega^2} > 0$$

■

3.4 Sum rules for the constitutive relations

The Hilbert transform, see Appendix B, can be used to relate specific frequency values of the material dyadics $\boldsymbol{\epsilon}$, $\boldsymbol{\mu}$, $\boldsymbol{\xi}$, and $\boldsymbol{\zeta}$ to their values along the entire frequency band. In particular, the static behavior is related to an integral over all frequencies. In this section, we exploit these integral relations — sum rules — in more detail. In particular, we prove that the eigenvalues of the static permittivity $\boldsymbol{\epsilon}(\omega)$ in a passive material without a static conductivity $\boldsymbol{\sigma}(\omega)$ are real, and that they always exceed the eigenvalues of the optical response $\boldsymbol{\epsilon}_\infty$ of the material, *i.e.*, $\boldsymbol{\epsilon}(0) - \boldsymbol{\epsilon}_\infty > 0$. This result, for an isotropic material, is also given in [16, p. 59] and in Example **??**.**??**, Footnote **??** on page **??**.

The Fourier transform of any real-valued, which vanishes for negative arguments (causal quantity) and which is square-integrable on the positive real line, satisfies Plemelj's formulas, see Titchmash's theorem B.1 in Appendix B.2. From (B.3) on page 111, we have

$$\begin{cases} \operatorname{Re} f(\omega) = \dfrac{2}{\pi} P \displaystyle\int_0^\infty \frac{\omega' \operatorname{Im} f(\omega')}{\omega'^2 - \omega^2} \, \mathrm{d}\omega' \\ \operatorname{Im} f(\omega) = -\dfrac{2\omega}{\pi} P \displaystyle\int_0^\infty \frac{\operatorname{Re} f(\omega')}{\omega'^2 - \omega^2} \, \mathrm{d}\omega' \end{cases}$$

Moreover, the extended function $f(\eta = f(\omega + \mathrm{i}\varsigma)$ is analytic in the upper half η-plane, $\varsigma > 0$, and satisfies it $f(\eta) = (f(-\eta^*))^*$.

We apply this result to the entries of the permittivity dyadic $\boldsymbol{\epsilon}(\omega)$ as a function of ω. Specifically, from Section 3.2 we have that each Cartesian component of the permittivity dyadic $\epsilon_{ij}(\omega) = \hat{\boldsymbol{e}}_i \cdot \boldsymbol{\epsilon}(\omega) \cdot \hat{\boldsymbol{e}}_j$, $i, j = 1, 2, 3$, is the Fourier transform of a causal quantity, so one of the prerequisite for Titchmash's theorem is satisfied, and, moreover, from (3.12) we have that $\epsilon_{ij}(-\eta^*) = \epsilon_{ij}(\eta)^*$. We assume that the permittivity at high frequencies approaches the optical response,[15] $\boldsymbol{\epsilon}_\infty$, *i.e.*, $\boldsymbol{\epsilon}(\eta) \to \boldsymbol{\epsilon}_\infty$ as $\eta \to \infty$ in the upper complex plane of η. Plemelj's formulas applied to the scalar function $f(\eta) = \hat{\boldsymbol{e}}_i \cdot (\boldsymbol{\epsilon}(\eta) - \boldsymbol{\epsilon}_\infty) \cdot \hat{\boldsymbol{e}}_j$, $i, j = 1, 2, 3$ imply

$$\begin{cases} \operatorname{Re}\left(\hat{\boldsymbol{e}}_i \cdot (\boldsymbol{\epsilon}(\omega) - \boldsymbol{\epsilon}_\infty) \cdot \hat{\boldsymbol{e}}_j\right) = \dfrac{2}{\pi} P \displaystyle\int_0^\infty \frac{\omega' \operatorname{Im}\left(\hat{\boldsymbol{e}}_i \cdot \boldsymbol{\epsilon}(\omega') \cdot \hat{\boldsymbol{e}}_j\right)}{\omega'^2 - \omega^2} \, \mathrm{d}\omega' \\ \operatorname{Im}\left(\hat{\boldsymbol{e}}_i \cdot \boldsymbol{\epsilon}(\omega) \cdot \hat{\boldsymbol{e}}_j\right) = -\dfrac{2\omega}{\pi} P \displaystyle\int_0^\infty \frac{\operatorname{Re}\left(\hat{\boldsymbol{e}}_i \cdot (\boldsymbol{\epsilon}(\omega') - \boldsymbol{\epsilon}_\infty) \cdot \hat{\boldsymbol{e}}_j\right)}{\omega'^2 - \omega^2} \, \mathrm{d}\omega' \end{cases}$$

where we have assumed that $f(\omega)$ is square integrable. These are the famous Kramers-Kronig relations. In particular, all components $\operatorname{Im}\left(\hat{\boldsymbol{e}}_i \cdot \boldsymbol{\epsilon}(0) \cdot \hat{\boldsymbol{e}}_j\right) = 0$, as seen by the second identity, so $\boldsymbol{\epsilon}(0)$ is a real-valued dyadic. A more compact form of these relations is

$$\begin{cases} \operatorname{Re} \boldsymbol{\epsilon}(\omega) = \boldsymbol{\epsilon}_\infty + \dfrac{2}{\pi} P \displaystyle\int_0^\infty \frac{\omega' \operatorname{Im} \boldsymbol{\epsilon}(\omega')}{\omega'^2 - \omega^2} \, \mathrm{d}\omega' \\ \operatorname{Im} \boldsymbol{\epsilon}(\omega) = -\dfrac{2\omega}{\pi} P \displaystyle\int_0^\infty \frac{\operatorname{Re} \boldsymbol{\epsilon}(\omega') - \boldsymbol{\epsilon}_\infty}{\omega'^2 - \omega^2} \, \mathrm{d}\omega' \end{cases} \tag{3.31}$$

Example 3.8

For an isotropic material, $\boldsymbol{\epsilon}(\omega) = \epsilon(\omega)\mathbf{I}_3$, the result in (3.31) is more well known [16].

$$\begin{cases} \operatorname{Re} \epsilon(\omega) = \epsilon_\infty + \dfrac{2}{\pi} P \displaystyle\int_0^\infty \frac{\omega' \operatorname{Im} \epsilon(\omega')}{\omega'^2 - \omega^2} \, \mathrm{d}\omega' \\ \operatorname{Im} \epsilon(\omega) = -\dfrac{2\omega}{\pi} P \displaystyle\int_0^\infty \frac{\operatorname{Re} \epsilon(\omega') - \epsilon_\infty}{\omega'^2 - \omega^2} \, \mathrm{d}\omega' \end{cases}$$

■

Take the limit $\omega \to 0$ in the first relation in (3.31), and we get

$$\boldsymbol{\epsilon}(0) = \operatorname{Re} \boldsymbol{\epsilon}(0) = \boldsymbol{\epsilon}_\infty + \frac{2}{\pi} P \int_0^\infty \frac{\operatorname{Im} \boldsymbol{\epsilon}(\omega)}{\omega} \, \mathrm{d}\omega \tag{3.32}$$

[15] For a passive material, this dyadic is a real-valued, symmetric, positive definite dyadic, see Section 2.4, and in most models simply $\boldsymbol{\epsilon}_\infty = \mathbf{I}$.

For a passive material, we get from (3.30) on page 65 that

$$\boldsymbol{x}^* \cdot (\boldsymbol{\epsilon}(0) - \boldsymbol{\epsilon}_\infty) \cdot \boldsymbol{x} + \frac{2}{\pi} P \int_0^\infty \frac{\boldsymbol{x}^* \cdot \operatorname{Im} \boldsymbol{\epsilon}(\omega) \cdot \boldsymbol{x}}{\omega} \, \mathrm{d}\omega > 0, \quad \text{for all } \boldsymbol{x} \in \mathbb{C}^3$$

and we conclude that $\boldsymbol{\epsilon}(0) - \boldsymbol{\epsilon}_\infty$ is a positive definite dyadic. In particular, the diagonal elements are

$$\epsilon_{ii}(0) > \hat{\boldsymbol{e}}_i \cdot \boldsymbol{\epsilon}_\infty \cdot \hat{\boldsymbol{e}}_i, \quad i = 1, 2, 3$$

i.e., the diagonal entries are larger than the diagonal entries of the optical response. More generally, the same statement holds for the eigenvalues of $\boldsymbol{\epsilon}(0) - \boldsymbol{\epsilon}_\infty$.

We obtain a more systematic analysis of the sum rules if we assume the permittivity dyadic to have the following explicit expansions at high and low frequencies:

$$\boldsymbol{\epsilon}(\omega) = \begin{cases} \boldsymbol{\epsilon}_\infty + \mathrm{i}\omega^{-1}\boldsymbol{\epsilon}_{-1} + \omega^{-2}\boldsymbol{\epsilon}_{-2} + \ldots, & |\omega| \to \infty \\ \boldsymbol{\epsilon}_0 + \mathrm{i}\omega\boldsymbol{\epsilon}_1 + \omega^2\boldsymbol{\epsilon}_2 + \ldots, & \omega \to 0 \end{cases}$$

The dyadics in this expansion all are real-valued, *i.e.*, $\operatorname{Im} \boldsymbol{\epsilon}_i = \mathbf{0}$, $i = 0, \pm 1, \pm 2, \ldots$, independent of ω due to $\boldsymbol{\epsilon}(-\eta^*) = \boldsymbol{\epsilon}(\eta)^*$. This assumption is consistent with all models of the permittivity presented in this chapter. Provided the integrals converge, we get the following sum rules:

$$\begin{cases} \boldsymbol{\epsilon}_0 = \boldsymbol{\epsilon}_\infty + \dfrac{2}{\pi} P \displaystyle\int_0^\infty \frac{\operatorname{Im} \boldsymbol{\epsilon}(\omega)}{\omega} \, \mathrm{d}\omega \\ \boldsymbol{\epsilon}_{-1} = \dfrac{2}{\pi} P \displaystyle\int_0^\infty \operatorname{Re} \boldsymbol{\epsilon}(\omega) - \boldsymbol{\epsilon}_\infty \, \mathrm{d}\omega \\ \boldsymbol{\epsilon}_{-2} = -\dfrac{2}{\pi} P \displaystyle\int_0^\infty \omega \operatorname{Im} \boldsymbol{\epsilon}(\omega) \, \mathrm{d}\omega \end{cases} \tag{3.33}$$

The first sum rule is identical to (3.32), and the second and second are obtained from (3.31) after multiplication of ω and ω^2, respectively, and then taking the limit $\omega \to \infty$.

Example 3.9

We illustrate the sum rules by the model of Debye, see (3.19). The real and imaginary parts of the permittivity in this model are

$$\begin{cases} \operatorname{Re} \epsilon(\omega) - 1 = \dfrac{\alpha\tau}{1 + \omega^2\tau^2} \\ \operatorname{Im} \epsilon(\omega) = \dfrac{\omega\alpha\tau^2}{1 + \omega^2\tau^2} \end{cases}$$

with $\boldsymbol{\epsilon}_\infty = \mathbf{I}$, $\boldsymbol{\epsilon}_0 = (1 + \alpha\tau)\mathbf{I}$, $\boldsymbol{\epsilon}_{-1} = \alpha\mathbf{I}$, $\boldsymbol{\epsilon}_{-2} = \alpha/\tau\mathbf{I}$. The first two sum rules in (3.33) are equivalent to the integral identity (principal value integral not necessary)

$$\int_0^\infty \frac{\tau}{1 + \omega^2\tau^2} \, \mathrm{d}\omega = \frac{\pi}{2}$$

■

Example 3.10
For an isotropic material (3.32) simplifies to

$$\epsilon(0) = \epsilon_\infty + \frac{2}{\pi} P \int_0^\infty \frac{\operatorname{Im}\epsilon(\omega)}{\omega}\, \mathrm{d}\omega$$

If we apply this identity to the Lorentz model in (3.22), we obtain a relation between the quotient $\omega_\mathrm{p}/\omega_0$ and the losses of the material, *i.e.*,

$$\frac{\omega_\mathrm{p}^2}{\omega_0^2} = \frac{2}{\pi} P \int_0^\infty \frac{\operatorname{Im}\epsilon(\omega)}{\omega}\, \mathrm{d}\omega$$

So, the knowledge of the losses in the material gives information of the oscillator strength. If there are several processes present, the generalization is

$$\sum_{i=1}^{M} \frac{\omega_{\mathrm{p}i}^2}{\omega_{0i}^2} = \frac{2}{\pi} P \int_0^\infty \frac{\operatorname{Im}\epsilon(\omega)}{\omega}\, \mathrm{d}\omega$$

■

If the material has a static conductivity $\boldsymbol{\sigma}$, we have to modify the analysis above. The permittivity is now

$$\boldsymbol{\epsilon}(\omega) = \boldsymbol{\epsilon}_\mathrm{reg}(\omega) + \mathrm{i}\frac{\boldsymbol{\sigma}}{\omega\epsilon_0}$$

where $\boldsymbol{\epsilon}_\mathrm{reg}(\omega)$ has no singularity at $\omega = 0$. We use Plemelj's formulas on the function $f(\eta) = \hat{\boldsymbol{e}}_i \cdot (\boldsymbol{\epsilon}(\eta) - \boldsymbol{\epsilon}_\infty - \mathrm{i}\boldsymbol{\sigma}/\eta\epsilon_0) \cdot \hat{\boldsymbol{e}}_j = \hat{\boldsymbol{e}}_i \cdot (\boldsymbol{\epsilon}_\mathrm{reg}(\eta) - \boldsymbol{\epsilon}_\infty) \cdot \hat{\boldsymbol{e}}_j$, $i, j = 1, 2, 3$, where we as above assume that $f(\omega)$ is square integrable. In a dyadic-valued notation, the result is

$$\begin{cases} \operatorname{Re}\boldsymbol{\epsilon}_\mathrm{reg}(\omega) = \boldsymbol{\epsilon}_\infty + \dfrac{2}{\pi} P \displaystyle\int_0^\infty \frac{\omega' \operatorname{Im}\boldsymbol{\epsilon}_\mathrm{reg}(\omega')}{\omega'^2 - \omega^2}\, \mathrm{d}\omega' \\ \operatorname{Im}\boldsymbol{\epsilon}_\mathrm{reg}(\omega) = -\dfrac{2\omega}{\pi} P \displaystyle\int_0^\infty \frac{\operatorname{Re}(\boldsymbol{\epsilon}_\mathrm{reg}(\omega) - \boldsymbol{\epsilon}_\infty)}{\omega'^2 - \omega^2}\, \mathrm{d}\omega' \end{cases} \tag{3.34}$$

Note that with a conductivity term, the passive material condition does **not**, in general, imply that $\omega \operatorname{Im}\boldsymbol{\epsilon}_\mathrm{reg}(\omega) > 0$ for all ω, only that $\omega \operatorname{Im}\boldsymbol{\epsilon}(\omega) = \omega \operatorname{Im}\boldsymbol{\epsilon}_\mathrm{reg}(\omega) + \operatorname{Re}\boldsymbol{\sigma}/\epsilon_0 > 0$, see the analysis in Section 3.3, and *e.g.*, Example 3.11. Therefore, we cannot any longer make the conclusion that $\boldsymbol{\epsilon}_\mathrm{reg}(0)$ has diagonal entries larger than the diagonal entries of the optical response.

Example 3.11
The model by Drude, see (3.23), illustrates the theory presented in this section. The real and imaginary parts of the permittivity in this model are

$$\begin{cases} \operatorname{Re}\epsilon(\omega) - 1 = -\dfrac{\omega_\mathrm{p}^2}{\omega^2 + \nu^2} \\ \operatorname{Im}\epsilon(\omega) - \dfrac{\omega_\mathrm{p}^2}{\omega\nu} = -\dfrac{\omega_\mathrm{p}^2\omega}{(\omega^2 + \nu^2)\nu} \end{cases}$$

The identities in (3.34) for this model are the integral identities

$$\begin{cases} \dfrac{\nu}{\omega^2+\nu^2} = \dfrac{2}{\pi} P \displaystyle\int_0^\infty \dfrac{\omega'^2}{(\omega'^2-\omega^2)(\omega'^2+\nu^2)}\,\mathrm{d}\omega' \\ -\dfrac{1}{\omega^2+\nu^2} = \dfrac{2}{\pi} P \displaystyle\int_0^\infty \dfrac{\nu}{(\omega'^2-\omega^2)(\omega'^2+\nu^2)}\,\mathrm{d}\omega' \end{cases}$$

or equivalently

$$\begin{cases} P \displaystyle\int_0^\infty \dfrac{t^2}{(t^2-x^2)(1+t^2)}\,\mathrm{d}t = \dfrac{\pi}{2}\dfrac{1}{1+x^2} \\ P \displaystyle\int_0^\infty \dfrac{1}{(t^2-x^2)(1+t^2)}\,\mathrm{d}t = -\dfrac{\pi}{2}\dfrac{1}{1+x^2} \end{cases}$$

which is a generalization to the integral in Example 3.9. ∎

Example 3.12
The plasma model in Example 3.3 on page 59 illustrates a more complex situation. The constitutive relations of the plasma are

$$\boldsymbol{\epsilon}(\omega) = \epsilon \mathbf{I}_2 - \mathrm{i}\mathbf{J}\epsilon_g + \hat{\boldsymbol{z}}\hat{\boldsymbol{z}}\epsilon_z$$

where

$$\begin{cases} \epsilon = 1 - \dfrac{\omega_\mathrm{p}^2(\omega+\mathrm{i}\nu)}{\omega\left(\omega^2-\omega_g^2-\nu^2+2\mathrm{i}\nu\omega\right)} \\ \epsilon_g = -\dfrac{\omega_\mathrm{p}^2\omega_g}{\omega\left(\omega^2-\omega_g^2-\nu^2+2\mathrm{i}\nu\omega\right)} \\ \epsilon_z = 1 - \dfrac{\omega_\mathrm{p}^2}{\omega(\omega+\mathrm{i}\nu)} \end{cases}$$

The real and imaginary parts of these components are:

$$\begin{cases} \operatorname{Re}\epsilon = 1 - \omega_\mathrm{p}^2 \dfrac{\omega^2-\omega_g^2+\nu^2}{(\omega^2-\omega_g^2-\nu^2)^2+4\nu^2\omega^2} \\ \operatorname{Im}\epsilon = \omega_\mathrm{p}^2\nu \dfrac{\omega^2+\omega_g^2+\nu^2}{\omega\left((\omega^2-\omega_g^2-\nu^2)^2+4\nu^2\omega^2\right)} \\ \qquad = \dfrac{\omega_\mathrm{p}^2\nu}{\omega\left(\omega_g^2+\nu^2\right)} + \omega_\mathrm{p}^2\nu\omega \dfrac{3\omega_g^2-\nu^2-\omega^2}{\left(\omega_g^2+\nu^2\right)\left((\omega^2-\omega_g^2-\nu^2)^2+4\nu^2\omega^2\right)} \end{cases}$$

$$\begin{cases} \operatorname{Re}\epsilon_g = -\omega_\mathrm{p}^2\omega_g \dfrac{\omega^2-\omega_g^2-\nu^2}{\omega\left((\omega^2-\omega_g^2-\nu^2)^2+4\nu^2\omega^2\right)} \\ \qquad = \dfrac{\omega_\mathrm{p}^2\omega_g}{\omega\left(\omega_g^2+\nu^2\right)} + \omega_\mathrm{p}^2\omega_g\omega \dfrac{\omega_g^2-3\nu^2-\omega^2}{\left(\omega_g^2+\nu^2\right)\left((\omega^2-\omega_g^2-\nu^2)^2+4\nu^2\omega^2\right)} \\ \operatorname{Im}\epsilon_g = \omega_\mathrm{p}^2\nu \dfrac{2\omega_g}{(\omega^2-\omega_g^2-\nu^2)^2+4\nu^2\omega^2} \end{cases}$$

and

$$\begin{cases} \operatorname{Re}\epsilon_z = 1 - \dfrac{\omega_\mathrm{p}^2}{\omega^2+\nu^2} \\ \operatorname{Im}\epsilon_z = \dfrac{\omega_\mathrm{p}^2\nu}{\omega(\omega^2+\nu^2)} \end{cases}$$

From (3.34), we get

$$\begin{cases} -\dfrac{\omega^2-\omega_g^2+\nu^2}{(\omega^2-\omega_g^2-\nu^2)^2+4\nu^2\omega^2} = \dfrac{2}{\pi}P\displaystyle\int_0^\infty \dfrac{\nu\omega'^2\frac{3\omega_g^2-\nu^2-\omega'^2}{(\omega_g^2+\nu^2)\left((\omega'^2-\omega_g^2-\nu^2)^2+4\nu^2\omega'^2\right)}}{\omega'^2-\omega^2}\,\mathrm{d}\omega' \\ \nu\dfrac{3\omega_g^2-\nu^2-\omega^2}{(\omega_g^2+\nu^2)\left((\omega^2-\omega_g^2-\nu^2)^2+4\nu^2\omega^2\right)} = -\dfrac{2}{\pi}P\displaystyle\int_0^\infty \dfrac{\frac{\omega'^2-\omega_g^2+\nu^2}{(\omega'^2-\omega_g^2-\nu^2)^2+4\nu^2\omega^2}}{\omega'^2-\omega^2}\,\mathrm{d}\omega' \end{cases}$$

and

$$\begin{cases} -\nu\dfrac{2}{(\omega^2-\omega_g^2-\nu^2)^2+4\nu^2\omega^2} = \dfrac{2}{\pi}P\displaystyle\int_0^\infty \dfrac{\omega'^2\frac{\omega_g^2-3\nu^2-\omega'^2}{(\omega_g^2+\nu^2)\left((\omega'^2-\omega_g^2-\nu^2)^2+4\nu^2\omega^2\right)}}{\omega'^2-\omega^2}\,\mathrm{d}\omega' \\ \dfrac{\omega_g^2-3\nu^2-\omega^2}{(\omega_g^2+\nu^2)\left((\omega^2-\omega_g^2-\nu^2)^2+4\nu^2\omega^2\right)} = \dfrac{2}{\pi}P\displaystyle\int_0^\infty \dfrac{\nu\frac{2}{(\omega'^2-\omega_g^2-\nu^2)^2+4\nu^2\omega'^2}}{\omega'^2-\omega^2}\,\mathrm{d}\omega' \end{cases}$$

The z-component leads to the same result as in Example 3.11. ∎

The analysis above, was explicitly performed for the permittivity $\boldsymbol{\epsilon}(\omega)$, *i.e.*, utilizing Plemelj's formula on $f(\eta) = \hat{\boldsymbol{e}}_i \cdot (\boldsymbol{\epsilon}(\eta) - \boldsymbol{\epsilon}_\infty) \cdot \hat{\boldsymbol{e}}_j$, $i,j = 1,2,3$. Several identities were obtained, see (3.33). Similar identities can be obtained for the other material dyadic $\boldsymbol{\mu}(\omega)$, and combinations of $\boldsymbol{\xi}(\omega)$, and $\boldsymbol{\zeta}(\omega)$.

Example 3.13

We exemplify the more general sum rules by a simple identity for chiral materials, see also [6]. The relevant constitutive relations are, see (3.18) on page 51

$$\begin{cases} \boldsymbol{D} = \epsilon_0\Big\{\epsilon(\omega)\boldsymbol{E}(\omega) + \mathrm{i}\eta_0\chi(\omega)\boldsymbol{H}(\omega)\Big\} \\ \boldsymbol{B} = \dfrac{1}{c_0}\Big\{-\mathrm{i}\chi(\omega)\boldsymbol{E}(\omega) + \eta_0\mu(\omega)\boldsymbol{H}(\omega)\Big\} \end{cases}$$

and the chirality parameter satisfy

$$\begin{cases} \operatorname{Im}\chi(\omega) = -\dfrac{2}{\pi}P\displaystyle\int_0^\infty \dfrac{\omega'\operatorname{Re}\chi(\omega')}{\omega'^2-\omega^2}\,\mathrm{d}\omega' \\ \operatorname{Re}\chi(\omega) = \dfrac{2\omega}{\pi}P\displaystyle\int_0^\infty \dfrac{\operatorname{Im}\chi(\omega')}{\omega'^2-\omega^2}\,\mathrm{d}\omega' \end{cases}$$

where we assumed the chirality parameter lacks optical response. ∎

Moreover, since due to (2.19), the inverse of $\boldsymbol{\epsilon}(\omega)$, $\boldsymbol{\epsilon}^{-1}(\omega)$, is the Fourier transform of a causal quantity. Therefore, utilizing Plemelj's formula on $f(\eta) = \hat{\boldsymbol{e}}_i \cdot (\boldsymbol{\epsilon}^{-1}(\eta) - \boldsymbol{\epsilon}_\infty^{-1}) \cdot \hat{\boldsymbol{e}}_j$, $i,j = 1,2,3$, we get, *cf.* (3.31)

$$\begin{cases} \operatorname{Re}\boldsymbol{\epsilon}^{-1}(\omega) = \boldsymbol{\epsilon}_\infty^{-1} + \dfrac{2}{\pi}P\displaystyle\int_0^\infty \dfrac{\omega'\operatorname{Im}\boldsymbol{\epsilon}^{-1}(\omega')}{\omega'^2-\omega^2}\,\mathrm{d}\omega' \\ \operatorname{Im}\boldsymbol{\epsilon}^{-1}(\omega) = -\dfrac{2\omega}{\pi}P\displaystyle\int_0^\infty \dfrac{\operatorname{Re}\boldsymbol{\epsilon}^{-1}(\omega') - \boldsymbol{\epsilon}_\infty^{-1}}{\omega'^2-\omega^2}\,\mathrm{d}\omega' \end{cases}$$

with the following explicit expansions at high and low frequencies in terms of the expansion dyadics $\boldsymbol{\epsilon}_i$, $i = 0, \pm 1, \pm 2, \ldots$ above:

$$\boldsymbol{\epsilon}^{-1}(\omega) = \begin{cases} \boldsymbol{\epsilon}_\infty^{-1} \cdot \{\boldsymbol{\epsilon}_\infty - \mathrm{i}\omega^{-1}\boldsymbol{\epsilon}_{-1} - \omega^{-2}\left(\boldsymbol{\epsilon}_{-1} \cdot \boldsymbol{\epsilon}_\infty^{-1} \cdot \boldsymbol{\epsilon}_{-1} + \boldsymbol{\epsilon}_{-2}\right) + \ldots\} \cdot \boldsymbol{\epsilon}_\infty^{-1} \\ \boldsymbol{\epsilon}_0^{-1} \cdot \left\{\boldsymbol{\epsilon}_0 - \mathrm{i}\omega\boldsymbol{\epsilon}_1 - \omega^2\left(\boldsymbol{\epsilon}_1 \cdot \boldsymbol{\epsilon}_0^{-1} \cdot \boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_2\right) + \ldots\right\} \cdot \boldsymbol{\epsilon}_0^{-1} \end{cases}$$

Provided the integrals converge, we get the following analogous sum rules:

$$\begin{cases} \boldsymbol{\epsilon}_0^{-1} = \boldsymbol{\epsilon}_\infty^{-1} + \dfrac{2}{\pi} P \displaystyle\int_0^\infty \dfrac{\operatorname{Im} \boldsymbol{\epsilon}^{-1}(\omega)}{\omega} \, \mathrm{d}\omega \\ \boldsymbol{\epsilon}_\infty^{-1} \cdot \boldsymbol{\epsilon}_{-1} \cdot \boldsymbol{\epsilon}_\infty^{-1} = -\dfrac{2}{\pi} P \displaystyle\int_0^\infty \operatorname{Re} \boldsymbol{\epsilon}^{-1}(\omega) - \boldsymbol{\epsilon}_\infty^{-1} \, \mathrm{d}\omega \\ \boldsymbol{\epsilon}_\infty^{-1} \cdot \left(\boldsymbol{\epsilon}_{-1} \cdot \boldsymbol{\epsilon}_\infty^{-1} \cdot \boldsymbol{\epsilon}_{-1} + \boldsymbol{\epsilon}_{-2}\right) \cdot \boldsymbol{\epsilon}_\infty^{-1} = \dfrac{2}{\pi} P \displaystyle\int_0^\infty \omega \operatorname{Im} \boldsymbol{\epsilon}^{-1}(\omega) \, \mathrm{d}\omega \end{cases} \tag{3.35}$$

Example 3.14
Again, we illustrate these sum rules by the model of Debye. The real and imaginary parts of the permittivity in this model are

$$\frac{1}{\epsilon(\omega)} = \frac{1 - \mathrm{i}\omega\tau}{\alpha\tau + 1 - \mathrm{i}\omega\tau} = 1 - \frac{\alpha\tau}{\alpha\tau + 1 - \mathrm{i}\omega\tau}$$

with real and imaginary parts

$$\begin{cases} \operatorname{Re} \dfrac{1}{\epsilon(\omega)} - 1 = -\dfrac{\alpha\tau(1 + \alpha\tau)}{(1 + \alpha\tau)^2 + \omega^2\tau^2} \\ \operatorname{Im} \dfrac{1}{\epsilon(\omega)} = -\dfrac{\omega\alpha\tau^2}{(1 + \alpha\tau)^2 + \omega^2\tau^2} \end{cases}$$

This implies with, as in Example 3.9, $\boldsymbol{\epsilon}_\infty = \mathbf{I}$, $\boldsymbol{\epsilon}_0 = (1 + \alpha\tau)\mathbf{I}$, $\boldsymbol{\epsilon}_{-1} = \alpha\mathbf{I}$, $\boldsymbol{\epsilon}_{-2} = \alpha/\tau\mathbf{I}$. The first two sum rules in (3.35) are equivalent to the integral identity (principal value integral not necessary)

$$\int_0^\infty \frac{\tau}{(1 + \alpha\tau)^2 + \omega^2\tau^2} \, \mathrm{d}\omega = \frac{\pi}{2} \frac{1}{1 + \alpha\tau}$$

■

3.5 Reciprocity

In Section 3.3 we studied the power dissipation in a material. Specifically, we found that the quantity $\nabla\cdot < \boldsymbol{S}(\boldsymbol{r}, t) >$ was useful in classifying different materials. We introduced the notion of active, passive and lossless materials, depending on whether this quantity was positive, negative or zero, respectively. This notion was local in space, *i.e.*, it holds in a specific point in space.

In this section we introduce a new concept for classifying materials — the reciprocity property. The reciprocity concept compares the effects on the material from two different source configurations. The first set of sources, denoted $\boldsymbol{J}^{\mathrm{a}}$, give rise to

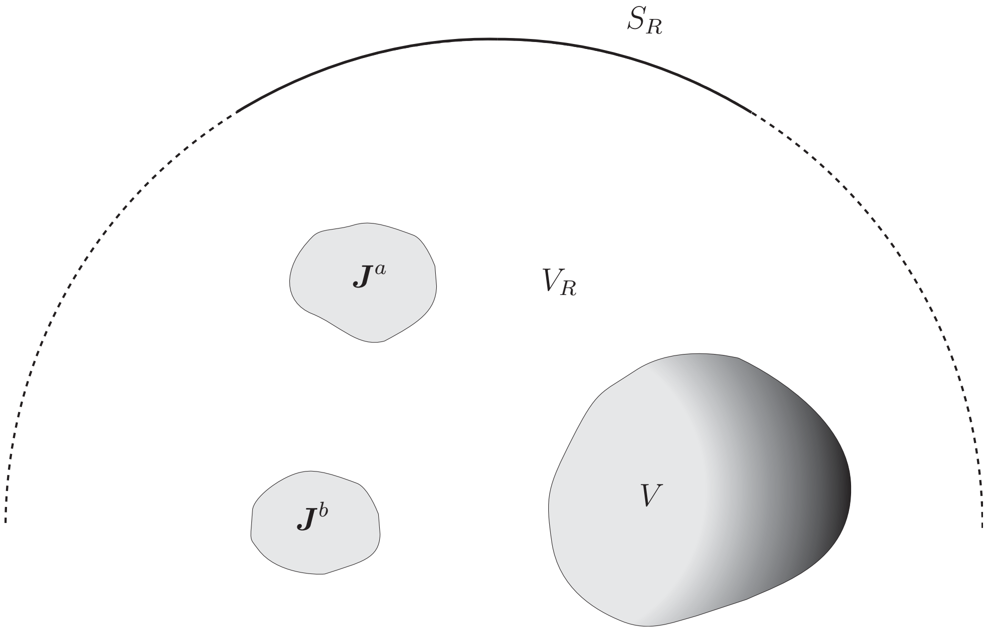


Figure 3.8: The material in the volume V is excited by two different sources $\boldsymbol{J}^{\mathrm{a}}$ and $\boldsymbol{J}^{\mathrm{b}}$, respectively.

an electromagnetic field, which we denote by the superscript a, *i.e.*, the fields are $\boldsymbol{E}^{\mathrm{a}}$, $\boldsymbol{H}^{\mathrm{a}}$, $\boldsymbol{D}^{\mathrm{a}}$ and $\boldsymbol{B}^{\mathrm{a}}$. The second set of sources, which we denote by the superscript b, has sources and fields $\boldsymbol{J}^{\mathrm{b}}$, $\boldsymbol{E}^{\mathrm{b}}$, $\boldsymbol{H}^{\mathrm{b}}$, $\boldsymbol{D}^{\mathrm{b}}$ and $\boldsymbol{B}^{\mathrm{b}}$. All sources are assumed to be located in a finite region of space outside or inside the material, which is confined to the volume V. Outside these sources and the volume V we assume vacuum. The geometry is illustrated in Figure 3.8.

In analogy with the dissipation concept in Section 3.3, we make the definition of reciprocity as a local property of the material. To this end, the medium is reciprocal $\boldsymbol{r}$ if

$$\boldsymbol{E}^{\mathrm{a}} \cdot \boldsymbol{D}^{\mathrm{b}} - \boldsymbol{E}^{\mathrm{b}} \cdot \boldsymbol{D}^{\mathrm{a}} = \boldsymbol{H}^{\mathrm{a}} \cdot \boldsymbol{B}^{\mathrm{b}} - \boldsymbol{H}^{\mathrm{b}} \cdot \boldsymbol{B}^{\mathrm{a}} \tag{3.36}$$

for all accessible fields at this point. Equivalently, in a reciprocal material the quantity $\boldsymbol{E}^{\mathrm{a}} \cdot \boldsymbol{D}^{\mathrm{b}} + \boldsymbol{B}^{\mathrm{a}} \cdot \boldsymbol{H}^{\mathrm{b}}$ is invariant under the change a $\leftrightarrow$ b.

Before investigating the consequences of this definition on the constitutive relations, we analyze the physical background to this definition. We assume the material in V is reciprocal and the surrounding region is vacuous, and integrate (3.36) over the a volume V_R, a large ball of radius R, containing the material V and the sources $\boldsymbol{J}^{\mathrm{a}}$ and $\boldsymbol{J}^{\mathrm{b}}$, see Figure 3.8. We get

$$\mathrm{i}\omega \iiint\limits_{V_R} \left\{ \boldsymbol{E}^{\mathrm{a}} \cdot \boldsymbol{D}^{\mathrm{b}} - \boldsymbol{E}^{\mathrm{b}} \cdot \boldsymbol{D}^{\mathrm{a}} - \boldsymbol{H}^{\mathrm{a}} \cdot \boldsymbol{B}^{\mathrm{b}} + \boldsymbol{H}^{\mathrm{b}} \cdot \boldsymbol{B}^{\mathrm{a}} \right\} \mathrm{d}v = 0$$

Now use the Maxwell equation $\mathrm{i}\omega \boldsymbol{D} = \boldsymbol{J} - \nabla \times \boldsymbol{H}$ and $\mathrm{i}\omega \boldsymbol{B} = \nabla \times \boldsymbol{E}$ and the differentiation rule $\nabla \cdot (\boldsymbol{a} \times \boldsymbol{b}) = (\nabla \times \boldsymbol{a}) \cdot \boldsymbol{b} - \boldsymbol{a} \cdot (\nabla \times \boldsymbol{b})$ to rewrite the volume integral as

$$\begin{aligned}
0 &= \mathrm{i}\omega \iiint\limits_{V_R} \left\{ \boldsymbol{E}^{\mathrm{a}} \cdot \boldsymbol{D}^{\mathrm{b}} - \boldsymbol{E}^{\mathrm{b}} \cdot \boldsymbol{D}^{\mathrm{a}} - \boldsymbol{H}^{\mathrm{a}} \cdot \boldsymbol{B}^{\mathrm{b}} + \boldsymbol{H}^{\mathrm{b}} \cdot \boldsymbol{B}^{\mathrm{a}} \right\} \,\mathrm{d}v \\
&= \iiint\limits_{V_R} \left\{ \boldsymbol{E}^{\mathrm{a}} \cdot \left(\boldsymbol{J}^{\mathrm{b}} - \nabla \times \boldsymbol{H}^{\mathrm{b}} \right) - \{\mathrm{a} \leftrightarrow \mathrm{b}\} - \boldsymbol{H}^{\mathrm{a}} \cdot \left(\nabla \times \boldsymbol{E}^{\mathrm{b}} \right) + \{\mathrm{a} \leftrightarrow \mathrm{b}\} \right\} \,\mathrm{d}v \\
&= \iiint\limits_{V_R} \left\{ \boldsymbol{E}^{\mathrm{a}} \cdot \boldsymbol{J}^{\mathrm{b}} - \boldsymbol{E}^{\mathrm{b}} \cdot \boldsymbol{J}^{\mathrm{a}} \right\} \,\mathrm{d}v + \iiint\limits_{V_R} \nabla \cdot \left(\boldsymbol{E}^{\mathrm{a}} \times \boldsymbol{H}^{\mathrm{b}} - \boldsymbol{E}^{\mathrm{b}} \times \boldsymbol{H}^{\mathrm{a}} \right) \,\mathrm{d}v
\end{aligned}$$

The domain of integration in first integral in the last equality is only over the finite region of the sources. This integral is therefore independent of the radius R of the volume V_R. The divergence theorem converts the last integral to a surface integral over the bounding surface S_R of the volume V_R. Since the first integral is independent of the radius of the volume V_R, the surface integral is also independent of R. In the limit as this radius goes to infinity, *i.e.*, $V_R \to \mathbb{R}^3$, it is possible to show that the surface integral is zero.[16] For a reciprocal material, we have proved

$$\iiint \boldsymbol{E}^{\mathrm{a}} \cdot \boldsymbol{J}^{\mathrm{b}} \,\mathrm{d}v = \iiint \boldsymbol{E}^{\mathrm{b}} \cdot \boldsymbol{J}^{\mathrm{a}} \,\mathrm{d}v$$

If this condition is not satisfied for a set of sources or fields we have a non-reciprocal material somewhere inside V. This condition is a measure of the difference between the sources in "a" ($\boldsymbol{J}^{\mathrm{a}}$) and measuring at "b" ($\boldsymbol{E}^{\mathrm{b}}$) and vice versa. Very simplified, the reciprocity concept implies that if sources and receivers change places, the integral above remains the same.

We now continue to investigate the effects of reciprocity on the constitutive relations. Introduce the constitutive relations (3.13) in the definition of reciprocity, (3.36). We get

$$\begin{aligned}
&\boldsymbol{E}^{\mathrm{a}} \cdot \left(\boldsymbol{\epsilon} \cdot \boldsymbol{E}^{\mathrm{b}} + \eta_0 \boldsymbol{\xi} \cdot \boldsymbol{H}^{\mathrm{b}} \right) - \boldsymbol{E}^{\mathrm{b}} \cdot \left(\boldsymbol{\epsilon} \cdot \boldsymbol{E}^{\mathrm{a}} + \eta_0 \boldsymbol{\xi} \cdot \boldsymbol{H}^{\mathrm{a}} \right) \\
&- \eta_0 \boldsymbol{H}^{\mathrm{a}} \cdot \left(\boldsymbol{\zeta} \cdot \boldsymbol{E}^{\mathrm{b}} + \eta_0 \boldsymbol{\mu} \cdot \boldsymbol{H}^{\mathrm{b}} \right) + \eta_0 \boldsymbol{H}^{\mathrm{b}} \cdot \left(\boldsymbol{\zeta} \cdot \boldsymbol{E}^{\mathrm{a}} + \eta_0 \boldsymbol{\mu} \cdot \boldsymbol{H}^{\mathrm{a}} \right) = 0
\end{aligned}$$

Using

$$\boldsymbol{a} \cdot \mathrm{A} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \mathrm{A}^t \cdot \boldsymbol{a}$$

where ^{t} denotes the transposed dyadic, we can simplify the reciprocity definition.

$$\begin{aligned}
&\boldsymbol{E}^{\mathrm{a}} \cdot \left(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^t \right) \cdot \boldsymbol{E}^{\mathrm{b}} - \eta_0^2 \boldsymbol{H}^{\mathrm{a}} \cdot \left(\boldsymbol{\mu} - \boldsymbol{\mu}^t \right) \cdot \boldsymbol{H}^{\mathrm{b}} \\
&+ \eta_0 \boldsymbol{E}^{\mathrm{a}} \cdot \left(\boldsymbol{\xi} + \boldsymbol{\zeta}^t \right) \cdot \boldsymbol{H}^{\mathrm{b}} - \eta_0 \boldsymbol{H}^{\mathrm{a}} \cdot \left(\boldsymbol{\zeta} + \boldsymbol{\xi}^t \right) \cdot \boldsymbol{E}^{\mathrm{b}} = 0
\end{aligned}$$

---

[16] We have assumed that the region outside the material in the volume V is vacuum. At large distances the fields satisfy the radiation condition ($\hat{\boldsymbol{r}} = \boldsymbol{r}/|\boldsymbol{r}|$, $\eta_0 = \sqrt{\mu_0/\epsilon_0}$, $k_0 = \omega/c_0$)

$$(\hat{\boldsymbol{r}} \times \boldsymbol{E}(\boldsymbol{r})) - \eta_0 \eta \boldsymbol{H}(\boldsymbol{r}) = o((k_0 r)^{-1}) \quad \text{or} \quad \eta_0 \eta \left(\hat{\boldsymbol{r}} \times \boldsymbol{H}(\boldsymbol{r}) \right) + \boldsymbol{E}(\boldsymbol{r}) = o((k_0 r)^{-1}) \quad \text{as } r \to \infty$$

These conditions are discussed and analyzed further in Section **??**, and they imply the statement.

Since the fields are arbitrary, we obtain the following conditions on the constitutive relations for a reciprocal material:

$$\begin{cases} \boldsymbol{\epsilon} = \boldsymbol{\epsilon}^t \\ \boldsymbol{\mu} = \boldsymbol{\mu}^t \\ \boldsymbol{\xi} = -\boldsymbol{\zeta}^t \end{cases} \tag{3.37}$$

An immediate consequence of this result is that all isotropic materials are reciprocal. If the dielectric dyadic is real, there are a set of (real) coordinate axes that diagonalizes the dyadic and the classification in the Table 3.4 on page 50 holds. This situation for *e.g.*, a lossless, reciprocal material.

3.6 Ellipse of polarization

A time harmonic field can be described in geometrical terms. All time harmonic fields oscillate in a fixed plane and the field follows the trace of an ellipse in this plane. The presentation in this section is coordinate-free, which is advantageous since the analysis can be made without referring to any specific coordinate system.

We consider the time harmonic field $\boldsymbol{E}(t)$ (all dependence on the space coordinates $\boldsymbol{r}$ is suppressed in this section) at a fixed point in space. The time dependence of the field is

$$\boldsymbol{E}(t) = \operatorname{Re}\left\{\boldsymbol{E}_0 \mathrm{e}^{-\mathrm{i}\omega t}\right\} \tag{3.38}$$

where $\boldsymbol{E}_0$ is a constant complex vector (can depend on, *e.g.*, ω and $\boldsymbol{r}$), which Cartesian components are

$$\boldsymbol{E}_0 = \hat{\boldsymbol{x}} E_{0x} + \hat{\boldsymbol{y}} E_{0y} + \hat{\boldsymbol{z}} E_{0z} = \hat{\boldsymbol{x}} |E_{0x}| \mathrm{e}^{\mathrm{i}\alpha} + \hat{\boldsymbol{y}} |E_{0y}| \mathrm{e}^{\mathrm{i}\beta} + \hat{\boldsymbol{z}} |E_{0z}| \mathrm{e}^{\mathrm{i}\gamma}$$

and α, β and γ are the phase of the components, respectively.

First we observe that the vector $\boldsymbol{E}(t)$ in (3.38) for all times lies in a fixed plane in space. To see this, we express the complex vector $\boldsymbol{E}_0$ in its real and imaginary parts, $\boldsymbol{E}_{0\mathrm{r}}$ and $\boldsymbol{E}_{0\mathrm{i}}$, respectively.

$$\boldsymbol{E}_0 = \boldsymbol{E}_{0\mathrm{r}} + \mathrm{i}\boldsymbol{E}_{0\mathrm{i}}$$

The real vectors $\boldsymbol{E}_{0\mathrm{r}}$ and $\boldsymbol{E}_{0\mathrm{i}}$ are fixes in time, and their explicit Cartesian components are

$$\begin{aligned} \boldsymbol{E}_{0\mathrm{r}} &= \hat{\boldsymbol{x}} |E_{0x}| \cos\alpha + \hat{\boldsymbol{y}} |E_{0y}| \cos\beta + \hat{\boldsymbol{z}} |E_{0z}| \cos\gamma \\ \boldsymbol{E}_{0\mathrm{i}} &= \hat{\boldsymbol{x}} |E_{0x}| \sin\alpha + \hat{\boldsymbol{y}} |E_{0y}| \sin\beta + \hat{\boldsymbol{z}} |E_{0z}| \sin\gamma \end{aligned}$$

The vector $\boldsymbol{E}(t)$ in (3.38) is now rewritten as

$$\boldsymbol{E}(t) = \operatorname{Re}\left\{(\boldsymbol{E}_{0\mathrm{r}} + \mathrm{i}\boldsymbol{E}_{0\mathrm{i}})\, \mathrm{e}^{-\mathrm{i}\omega t}\right\} = \boldsymbol{E}_{0\mathrm{r}} \cos\omega t + \boldsymbol{E}_{0\mathrm{i}} \sin\omega t \tag{3.39}$$

from which we conclude that the vector $\boldsymbol{E}(t)$ lies in the plane spanned by the real vectors $\boldsymbol{E}_{0\mathrm{r}}$ and $\boldsymbol{E}_{0\mathrm{i}}$ for all times t. The normal to this plane is

$$\hat{\boldsymbol{\nu}} = \pm \frac{\boldsymbol{E}_{0\mathrm{r}} \times \boldsymbol{E}_{0\mathrm{i}}}{|\boldsymbol{E}_{0\mathrm{r}} \times \boldsymbol{E}_{0\mathrm{i}}|}$$

provided that $\boldsymbol{E}_{0\mathrm{r}} \times \boldsymbol{E}_{0\mathrm{i}} \neq \mathbf{0}$. In the case $\boldsymbol{E}_{0\mathrm{r}} \times \boldsymbol{E}_{0\mathrm{i}} = \mathbf{0}$, *i.e.*, the two real vectors $\boldsymbol{E}_{0\mathrm{r}}$ and $\boldsymbol{E}_{0\mathrm{i}}$ are parallel, the field $\boldsymbol{E}$ oscillates along a fixed line in space, and no plane can be defined.

In general, the real vectors $\boldsymbol{E}_{0\mathrm{r}}$ and $\boldsymbol{E}_{0\mathrm{i}}$, which span the plane in which the vector $\boldsymbol{E}(t)$ oscillates, are not orthogonal. However, it is convenient to use orthogonal vectors. To this end, we introduce two new orthogonal vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, which are linear combinations of the vectors $\boldsymbol{E}_{0\mathrm{r}}$ and $\boldsymbol{E}_{0\mathrm{i}}$. Let

$$\begin{cases} \boldsymbol{a} = \boldsymbol{E}_{0\mathrm{r}} \cos\vartheta + \boldsymbol{E}_{0\mathrm{i}} \sin\vartheta \\ \boldsymbol{b} = -\boldsymbol{E}_{0\mathrm{r}} \sin\vartheta + \boldsymbol{E}_{0\mathrm{i}} \cos\vartheta \end{cases} \tag{3.40}$$

where the angle $\vartheta \in [-\pi/4, \pi/4]$ is defined as

$$\tan 2\vartheta = \frac{2\boldsymbol{E}_{0\mathrm{r}} \cdot \boldsymbol{E}_{0\mathrm{i}}}{|\boldsymbol{E}_{0\mathrm{r}}|^2 - |\boldsymbol{E}_{0\mathrm{i}}|^2}$$

By this construction $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal, since

$$\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{b} &= (\boldsymbol{E}_{0\mathrm{r}} \cos\vartheta + \boldsymbol{E}_{0\mathrm{i}} \sin\vartheta) \cdot (-\boldsymbol{E}_{0\mathrm{r}} \sin\vartheta + \boldsymbol{E}_{0\mathrm{i}} \cos\vartheta) \\ &= -\left(|\boldsymbol{E}_{0\mathrm{r}}|^2 - |\boldsymbol{E}_{0\mathrm{i}}|^2\right) \sin\vartheta \cos\vartheta + \boldsymbol{E}_{0\mathrm{r}} \cdot \boldsymbol{E}_{0\mathrm{i}} \left(\cos^2\vartheta - \sin^2\vartheta\right) \\ &= -\frac{1}{2}\left(|\boldsymbol{E}_{0\mathrm{r}}|^2 - |\boldsymbol{E}_{0\mathrm{i}}|^2\right) \sin 2\vartheta + \boldsymbol{E}_{0\mathrm{r}} \cdot \boldsymbol{E}_{0\mathrm{i}} \cos 2\vartheta = 0 \end{aligned}$$

by the definition of the angle ϑ.

The vectors $\boldsymbol{E}_{0\mathrm{r}}$ and $\boldsymbol{E}_{0\mathrm{i}}$ can be expressed in the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. The result is

$$\begin{cases} \boldsymbol{E}_{0\mathrm{r}} = \boldsymbol{a} \cos\vartheta - \boldsymbol{b} \sin\vartheta \\ \boldsymbol{E}_{0\mathrm{i}} = \boldsymbol{a} \sin\vartheta + \boldsymbol{b} \cos\vartheta \end{cases}$$

i.e.,

$$\boldsymbol{E}_0 = \boldsymbol{E}_{0\mathrm{r}} + \mathrm{i}\boldsymbol{E}_{0\mathrm{i}} = (\boldsymbol{a} \cos\vartheta - \boldsymbol{b} \sin\vartheta) + \mathrm{i}\,(\boldsymbol{a} \sin\vartheta + \boldsymbol{b} \cos\vartheta) = \mathrm{e}^{\mathrm{i}\vartheta}(\boldsymbol{a} + \mathrm{i}\boldsymbol{b}) \tag{3.41}$$

This representation also implies a simple form of the magnitude of the complex vector $\boldsymbol{E}_0$, *i.e.*,

$$|\boldsymbol{E}_0|^2 = \boldsymbol{E}_0 \cdot \boldsymbol{E}_0^* = (\boldsymbol{a} + \mathrm{i}\boldsymbol{b}) \cdot (\boldsymbol{a} - \mathrm{i}\boldsymbol{b}) = a^2 + b^2$$

Inserting in (3.39) we get the physical field, *i.e.*,

$$\begin{aligned} \boldsymbol{E}(t) &= \boldsymbol{E}_{0\mathrm{r}} \cos\omega t + \boldsymbol{E}_{0\mathrm{i}} \sin\omega t \\ &= (\boldsymbol{a} \cos\vartheta - \boldsymbol{b} \sin\vartheta) \cos\omega t + (\boldsymbol{a} \sin\vartheta + \boldsymbol{b} \cos\vartheta) \sin\omega t \\ &= \boldsymbol{a} \cos(\omega t - \vartheta) + \boldsymbol{b} \sin(\omega t - \vartheta) \end{aligned} \tag{3.42}$$

The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be used as a basis in an orthogonal coordinate system in the plane where the field $\boldsymbol{E}$ oscillates. From a comparison with the equation of the ellipse in the xy-planet (half axes a and b along the x- and the y-axes, respectively)

$$\begin{cases} x = a\cos\phi \\ y = b\sin\phi \end{cases}$$

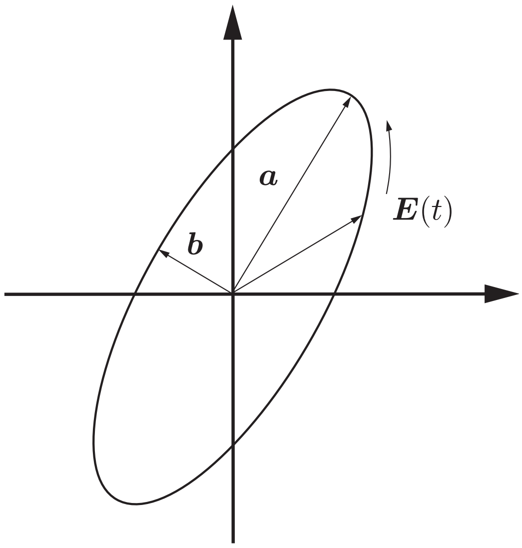


Figure 3.9: The ellipse of polarization and its half axes $\boldsymbol{a}$ and $\boldsymbol{b}$.

and (3.42), we conclude that the field $\boldsymbol{E}$ traces an ellipse in the plane spanned by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ and that these vectors are the half axes (both the direction and size) of the ellipse, see Figure 3.9. From (3.42) we also see that the field $\boldsymbol{E}$ is directed along the half axis $\boldsymbol{a}$ when $\omega t = \vartheta + 2n\pi$, and that the field $\boldsymbol{E}$ is directed along the other half axis $\boldsymbol{b}$ when $\omega t = \vartheta + \pi/2 + 2n\pi$. The angle ϑ is the parameter that marks where on the ellipse the field $\boldsymbol{E}$ is directed at $t = 0$, *i.e.*,

$$\boldsymbol{E}(t = 0) = \boldsymbol{a} \cos\vartheta - \boldsymbol{b} \sin\vartheta$$

and the vector $\boldsymbol{E}$ moves along the ellipse in a direction from $\boldsymbol{a}$ to $\boldsymbol{b}$ (shortest way). The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ describes the polarization state[17] of the field $\boldsymbol{E}$ completely, except for the phase angle ϑ.

We are now classifying the polarization state of the time harmonic field $\boldsymbol{E}(t)$. This field can either be rotating along the elliptic curve in a clockwise or a counterclockwise direction. Without a preferred direction in space, the direction of rotation is a relative concept — depending on which side of the plane we observe the oscillations. From the direction of the power flow of the electromagnetic field at the point of observation, $<\boldsymbol{S}(t)>$, we define a preferred direction in space. Let $\hat{\boldsymbol{e}}$ be the normal to the plane of polarization, such that $<\boldsymbol{S}(t)>\cdot\hat{\boldsymbol{e}} > 0$. We use this unit vector $\hat{\boldsymbol{e}}$ as a reference direction.

The polarization of the field is now classified according to the sign of the component of $\mathrm{i}\boldsymbol{E}_0 \times \boldsymbol{E}_0^* = 2\boldsymbol{E}_{0\mathrm{r}} \times \boldsymbol{E}_{0\mathrm{i}} = 2\boldsymbol{a} \times \boldsymbol{b}$ on $\hat{\boldsymbol{e}}$, see Table 3.8. The field vector either rotates counterclockwise (right-handed elliptic polarization) or clockwise (left-handed elliptic polarization) in the $\boldsymbol{a}$-$\boldsymbol{b}$-plane, see Figure 3.10, if we assume that the unit vector $\hat{\boldsymbol{e}}$ is directed **towards** the observer, and that the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ have

[17]Do not mix the concept of polarization of the material, $\boldsymbol{P}$, with the polarization of a vector field.

$\mathrm{i}\hat{\boldsymbol{e}} \cdot (\boldsymbol{E}_0 \times \boldsymbol{E}_0^*)$	**Polarization**
$= 0$	Linear polarization
> 0	Right-handed elliptic polarization
< 0	Left-handed elliptic polarization

Table 3.8: Table of the state of polarization of a time harmonic field.

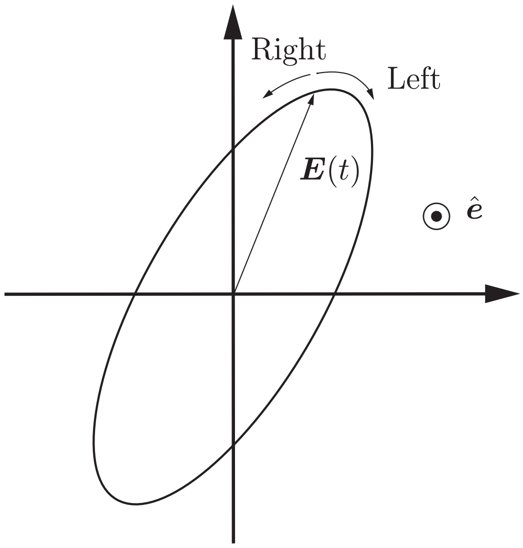


Figure 3.10: The ellipse of polarization and the definition of right- and left-handed polarization. The unit vector $\hat{\boldsymbol{e}}$ perpendicular to the plane in which the field vector $\boldsymbol{E}(t)$ oscillates and satisfies $<\boldsymbol{S}(t)>\cdot\hat{\boldsymbol{e}} > 0$. It is assumed that the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ have the position depicted in Figure 3.9.

the position depicted in Figure 3.9.[18]

The degenerated case, when the vectors $\boldsymbol{E}_{0\mathrm{r}}$ and $\boldsymbol{E}_{0\mathrm{i}}$ are parallel, implies that the field vector moves along a line through the origin — therefore the notion linear polarization. The linear polarization is characterized by $\boldsymbol{E}_0 \times \boldsymbol{E}_0^* = \boldsymbol{0}$. The case of a linear polarization can be viewed as a special case of an elliptic polarization, where one of the half axes is zero.

One special case of elliptic polarization is particularly important. This occurs when the half axes of the ellipse, $\boldsymbol{a}$ and $\boldsymbol{b}$, have the same length, and the ellipse is a circle. We then have circular polarization. Whether the polarization is circular or not is decided by testing if $\boldsymbol{E}_0 \cdot \boldsymbol{E}_0 = 0$. To see this, we use (3.41) and the orthogonality between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and we get

$$\boldsymbol{E}_0 \cdot \boldsymbol{E}_0 = \mathrm{e}^{2\mathrm{i}\vartheta} (\boldsymbol{a} + \mathrm{i}\boldsymbol{b}) \cdot (\boldsymbol{a} + \mathrm{i}\boldsymbol{b}) = \mathrm{e}^{2\mathrm{i}\vartheta} \left(|\boldsymbol{a}|^2 - |\boldsymbol{b}|^2\right)$$

The ellipse of polarization is therefore a circle, $a = |\boldsymbol{a}| = |\boldsymbol{b}| = b$, if and only if

[18] In the literature there are also occur the opposite definition of right- and left-handed elliptic polarization. Examples with the opposite definition are: [10], [23], and [25]. In this book, we are using the same definition as, *e.g.*, [2], [5], [13], and [14]. Our definition also coincides with the IEEE-standard.

Π	**Polarization**	χ	$\hat{\boldsymbol{e}}\cdot(\boldsymbol{a}\times\boldsymbol{b})$
-1	Circular polarization LCP	$\pi/4$	-1
<0	Left-handed elliptic polarization	$(0,\pi/2)$	-1
0	Linear polarization LP	$0,\pi/2$	± 1
<0	Right-handed elliptic polarization	$(0,\pi/2)$	1
1	Circular polarization RCP	$\pi/4$	1

Table 3.9: Table of the polarization state Π of a time harmonic field.

$\boldsymbol{E}_0\cdot\boldsymbol{E}_0 = 0$. The direction of rotation is determined by the sign of the quantity $\mathrm{i}\hat{\boldsymbol{e}}\cdot(\boldsymbol{E}_0\times\boldsymbol{E}_0^*)$. Right (left) circular polarization is abbreviated RCP (LCP).

Another, more convenient, way of determining whether the polarization is circular or not is to study the quantity, use (3.41)

$$\mathrm{i}\hat{\boldsymbol{e}}\cdot(\hat{\boldsymbol{p}}_\mathrm{e}\times\hat{\boldsymbol{p}}_\mathrm{e}^*) = \frac{2\hat{\boldsymbol{e}}\cdot(\boldsymbol{a}\times\boldsymbol{b})}{|\boldsymbol{a}|^2+|\boldsymbol{b}|^2} = \frac{\pm 2ab}{a^2+b^2} = \pm\left(1-\frac{(a-b)^2}{a^2+b^2}\right) \tag{3.43}$$

where $\hat{\boldsymbol{p}}_\mathrm{e} = \boldsymbol{E}_0/|\boldsymbol{E}_0|$. If this quantity is ± 1, we have RCP (upper sign) or LCP (lower sign). It is therefore convenient to define a polarization state quantity Π as

$$\Pi = \mathrm{i}\hat{\boldsymbol{e}}\cdot(\hat{\boldsymbol{p}}_\mathrm{e}\times\hat{\boldsymbol{p}}_\mathrm{e}^*)$$

This quantity is always in the interval $[-1,1]$. $\Pi=-1$ corresponds to LCP, $\Pi=0$ corresponds to LP, and $\Pi=1$ corresponds to RCP. We summarize these observations in Table 3.9.

In terms of the notation above, a general polarization state is given by

$$\hat{\boldsymbol{p}}_\mathrm{e} = \mathrm{e}^{\mathrm{i}\vartheta}\frac{(\boldsymbol{a}+\mathrm{i}\boldsymbol{b})}{\sqrt{a^2+b^2}}$$

Define the angle χ by

$$\tan\chi = \frac{b}{a},\quad \chi\in[0,\pi/2] \quad\Rightarrow\quad \begin{cases} \cos\chi = \dfrac{a}{\sqrt{a^2+b^2}} \\ \sin\chi = \dfrac{b}{\sqrt{a^2+b^2}} \end{cases}$$

and in terms of the natural orthonormal basis $\{\hat{\boldsymbol{a}},\hat{\boldsymbol{b}}\}$ aligned along the two half axis of the polarization ellipse, we get

$$\hat{\boldsymbol{p}}_\mathrm{e} = \mathrm{e}^{\mathrm{i}\vartheta}\frac{(\boldsymbol{a}+\mathrm{i}\boldsymbol{b})}{\sqrt{a^2+b^2}} = \mathrm{e}^{\mathrm{i}\vartheta}\left(\hat{\boldsymbol{a}}\cos\chi + \mathrm{i}\hat{\boldsymbol{b}}\sin\chi\right) \tag{3.44}$$

and also the polarization state

$$\Pi = \mathrm{i}\hat{\boldsymbol{e}}\cdot(\hat{\boldsymbol{p}}_\mathrm{e}\times\hat{\boldsymbol{p}}_\mathrm{e}^*) = 2\hat{\boldsymbol{e}}\cdot(\hat{\boldsymbol{a}}\times\hat{\boldsymbol{b}})\sin\chi\cos\chi = \hat{\boldsymbol{e}}\cdot(\hat{\boldsymbol{a}}\times\hat{\boldsymbol{b}})\sin 2\chi$$

The canonical form of a RCP field is

$$\boldsymbol{E}_0 = E_0\left(\hat{\boldsymbol{a}}+\mathrm{i}\hat{\boldsymbol{b}}\right)$$

and the canonical form of a LCP field is

$$\boldsymbol{E}_0 = E_0\left(\hat{\boldsymbol{a}} - \mathrm{i}\hat{\boldsymbol{b}}\right)$$

if $\{\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}, \hat{\boldsymbol{e}}\}$ forms a right-handed orthonormal basis.

Example 3.15
The most general harmonic field in the $\hat{\boldsymbol{e}}_1$-$\hat{\boldsymbol{e}}_2$-plane has the form (we assume $\{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}\}$ forms a right-handed orthonormal basis)

$$\boldsymbol{E}(t) = \hat{\boldsymbol{e}}_1 A\cos(\omega t - \alpha) + \hat{\boldsymbol{e}}_2 B\cos(\omega t - \beta)$$

where $A \geq 0$, $B \geq 0$, and α and β are real angles. The corresponding complex vector $\boldsymbol{E}_0$ is

$$\begin{cases} \boldsymbol{E}(t) = \mathrm{Re}\left\{\boldsymbol{E}_0 \mathrm{e}^{-\mathrm{i}\omega t}\right\} \\ \boldsymbol{E}_0 = A\mathrm{e}^{\mathrm{i}\alpha}\hat{\boldsymbol{e}}_1 + B\mathrm{e}^{\mathrm{i}\beta}\hat{\boldsymbol{e}}_2 \end{cases} \tag{3.45}$$

which implies

$$\mathrm{i}\boldsymbol{E}_0 \times \boldsymbol{E}_0^* = \mathrm{i}AB\mathrm{e}^{\mathrm{i}(\alpha-\beta)}\hat{\boldsymbol{e}}_3 - \mathrm{i}AB\mathrm{e}^{-\mathrm{i}(\alpha-\beta)}\hat{\boldsymbol{e}}_3 = -2AB\hat{\boldsymbol{e}}_3\sin(\alpha-\beta)$$

From this we conclude that the field is

Left-handed polarization if $0 < \alpha - \beta < \pi$
Right-handed polarization if $\pi < \alpha - \beta < 2\pi$
Linear polarization if $\alpha = \beta$ or $\alpha = \beta + \pi$

where the inequalities are interpreted mod 2π.

The real and imaginary part of $\boldsymbol{E}_0$ are

$$\begin{cases} \boldsymbol{E}_{0\mathrm{r}} = \hat{\boldsymbol{e}}_1 A\cos\alpha + \hat{\boldsymbol{e}}_2 B\cos\beta \\ \boldsymbol{E}_{0\mathrm{i}} = \hat{\boldsymbol{e}}_1 A\sin\alpha + \hat{\boldsymbol{e}}_2 B\sin\beta \end{cases}$$

From equation (3.40) we have

$$\begin{cases} \boldsymbol{a} = (\hat{\boldsymbol{e}}_1 A\cos\alpha + \hat{\boldsymbol{e}}_2 B\cos\beta)\cos\vartheta + (\hat{\boldsymbol{e}}_1 A\sin\alpha + \hat{\boldsymbol{e}}_2 B\sin\beta)\sin\vartheta \\ \boldsymbol{b} = -(\hat{\boldsymbol{e}}_1 A\cos\alpha + \hat{\boldsymbol{e}}_2 B\cos\beta)\sin\vartheta + (\hat{\boldsymbol{e}}_1 A\sin\alpha + \hat{\boldsymbol{e}}_2 B\sin\beta)\cos\vartheta \end{cases}$$

where the angle ϑ is determined by

$$\tan 2\vartheta = \frac{A^2\sin 2\alpha + B^2\sin 2\beta}{A^2\cos 2\alpha + B^2\cos 2\beta} \tag{3.46}$$

which implies that the half axes of the ellipse are

$$\begin{cases} \boldsymbol{a} = A\hat{\boldsymbol{e}}_1\cos(\vartheta-\alpha) + B\hat{\boldsymbol{e}}_2\cos(\vartheta-\beta) \\ \boldsymbol{b} = -A\hat{\boldsymbol{e}}_1\sin(\vartheta-\alpha) - B\hat{\boldsymbol{e}}_2\sin(\vartheta-\beta) \end{cases}$$

The length of the half axes are

$$\begin{cases} a = \sqrt{A^2\cos^2(\vartheta-\alpha) + B^2\cos^2(\vartheta-\beta)} \\ b = \sqrt{A^2\sin^2(\vartheta-\alpha) + B^2\sin^2(\vartheta-\beta)} \end{cases} \tag{3.47}$$

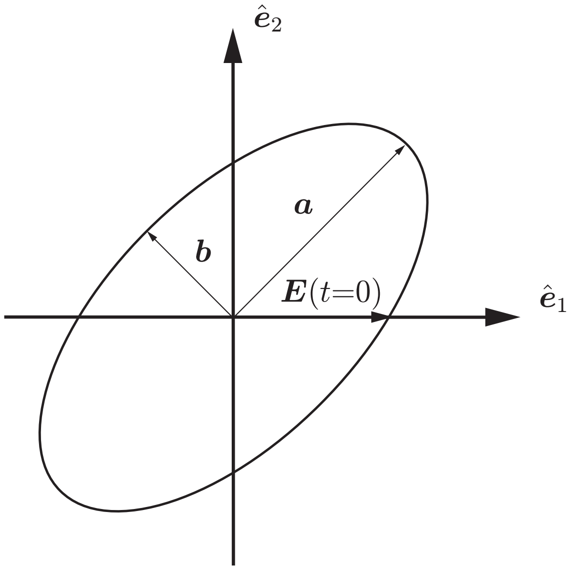


Figure 3.11: Ellipse of polarization in Example 3.16.

and the angles ϕ_a and ϕ_b between the $\hat{\boldsymbol{e}}_1$-axis and the half axes $\boldsymbol{a}$ and $\boldsymbol{b}$ are determined by

$$\begin{cases} \tan\phi_a = \dfrac{B\cos(\vartheta-\beta)}{A\cos(\vartheta-\alpha)} \\ \tan(\phi_b-\pi) = \dfrac{B\sin(\vartheta-\beta)}{A\sin(\vartheta-\alpha)} \end{cases} \tag{3.48}$$

respectively. ∎

Example 3.16
Construct the harmonic field, oscillating in the $\hat{\boldsymbol{e}}_1$-$\hat{\boldsymbol{e}}_2$-plane, satisfying the following specification (see also Figure 3.11):

- The field is at time $t = 0$ polarized along the $\hat{\boldsymbol{e}}_1$-axis and strength E (a given real constant), *i.e.*, $\boldsymbol{E}(t = 0) = \hat{\boldsymbol{e}}_1 E$.
- The quotient between the axes of the ellipse is $\varepsilon = b/a$. The axis $\boldsymbol{a}$, with the length a, is located in the first quadrant, and the angle between $\boldsymbol{a}$ and the $\hat{\boldsymbol{e}}_1$-axis is ϕ.
- The field has right-handed elliptic polarization ($<\boldsymbol{S}(t)>$ is assumed to be directed along $\hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{e}}_1 \times \hat{\boldsymbol{e}}_2$).

We assume $\{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}\}$ forms a right-handed orthonormal basis. Determine the real constants E_1, E_2, α, and β in the expression

$$\boldsymbol{E}(t) = \hat{\boldsymbol{e}}_1 E_1 \cos(\omega t - \alpha) + \hat{\boldsymbol{e}}_2 E_2 \cos(\omega t - \beta)$$

i.e., determine the amplitude and the phase of the $\hat{\boldsymbol{e}}_1$- and $\hat{\boldsymbol{e}}_2$-components.

Solution: Introduce the half axes of the ellipse.

$$\begin{cases} \boldsymbol{a} = \dfrac{a}{\sqrt{1+\tan^2\phi}}\,(\hat{\boldsymbol{e}}_1 + \hat{\boldsymbol{e}}_2\tan\phi) \\ \boldsymbol{b} = \dfrac{a\varepsilon}{\sqrt{1+\tan^2\phi}}\,(-\hat{\boldsymbol{e}}_1\tan\phi + \hat{\boldsymbol{e}}_2) \end{cases}$$

which implies that the length of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are a and b, respectively, and, moreover, this choice gives a right-handed elliptic polarization of the field since

$$(\boldsymbol{a}\times\boldsymbol{b})\cdot\hat{\boldsymbol{e}}_3 = a^2\varepsilon > 0$$

Now determine the angle ϑ in the expression $\boldsymbol{E}_0 = \mathrm{e}^{\mathrm{i}\vartheta}(\boldsymbol{a}+\mathrm{i}\boldsymbol{b})$.

$$\boldsymbol{E}(t) = \boldsymbol{E}_{0\mathrm{r}}\cos\omega t + \boldsymbol{E}_{0\mathrm{i}}\sin\omega t = \boldsymbol{a}\cos(\omega t-\vartheta) + \boldsymbol{b}\sin(\omega t-\vartheta)$$

At time $t=0$ we have

$$\boldsymbol{E}(0) = \boldsymbol{E}_{0\mathrm{r}} = \boldsymbol{a}\cos\vartheta - \boldsymbol{b}\sin\vartheta = E\hat{\boldsymbol{e}}_1$$

i.e., the components satisfy

$$\begin{cases} \dfrac{a}{\sqrt{1+\tan^2\phi}}(\cos\vartheta + \varepsilon\tan\phi\sin\vartheta) = E \\ \dfrac{a}{\sqrt{1+\tan^2\phi}}(\tan\phi\cos\vartheta - \varepsilon\sin\vartheta) = 0 \end{cases}$$

with solution

$$\begin{cases} \dfrac{a}{\sqrt{1+\tan^2\phi}}\cos\vartheta = \dfrac{E}{1+\tan^2\phi} \\ \dfrac{a}{\sqrt{1+\tan^2\phi}}\sin\vartheta = \dfrac{E\tan\phi}{\varepsilon\,(1+\tan^2\phi)} \end{cases}$$

and we have

$$\frac{a}{\sqrt{1+\tan^2\phi}}\mathrm{e}^{\mathrm{i}\vartheta} = \frac{E\,(\varepsilon+\mathrm{i}\tan\phi)}{\varepsilon\,(1+\tan^2\phi)}$$

We get

$$\begin{aligned} \boldsymbol{E}_0 = \mathrm{e}^{\mathrm{i}\vartheta}\,(\boldsymbol{a}+\mathrm{i}\boldsymbol{b}) &= \frac{E\,(\varepsilon+\mathrm{i}\tan\phi)}{\varepsilon\,(1+\tan^2\phi)}\left\{(\hat{\boldsymbol{e}}_1+\hat{\boldsymbol{e}}_2\tan\phi) + \mathrm{i}\varepsilon\,(-\hat{\boldsymbol{e}}_1\tan\phi+\hat{\boldsymbol{e}}_2)\right\} \\ &= E\left\{\hat{\boldsymbol{e}}_1\left(1+\mathrm{i}\frac{\tan\phi}{1+\tan^2\phi}\frac{1-\varepsilon^2}{\varepsilon}\right) + \mathrm{i}\hat{\boldsymbol{e}}_2\frac{\varepsilon^2+\tan^2\phi}{\varepsilon\,(1+\tan^2\phi)}\right\} \\ &= E\left\{\hat{\boldsymbol{e}}_1\left(1+\mathrm{i}\frac{1-\varepsilon^2}{2\varepsilon}\sin 2\phi\right) + \mathrm{i}\hat{\boldsymbol{e}}_2\left(\varepsilon\cos^2\phi + \frac{1}{\varepsilon}\sin^2\phi\right)\right\} \end{aligned}$$

From this expression of $\boldsymbol{E}_0$ we can identify the amplitudes A and$B$ and the phases α and β in

$$\boldsymbol{E}(t) = \hat{\boldsymbol{e}}_1 A\cos(\omega t-\alpha) + \hat{\boldsymbol{e}}_2 B\cos(\omega t-\beta)$$

by rewriting the complex vector $\boldsymbol{E}_0$ in polar form

$$\boldsymbol{E}_0 = \hat{\boldsymbol{e}}_1 A\mathrm{e}^{\mathrm{i}\alpha} + \hat{\boldsymbol{e}}_2 B\mathrm{e}^{\mathrm{i}\beta}$$

i.e.,

$$\begin{cases} A = \sqrt{1 + \dfrac{(1-\varepsilon^2)^2}{4\varepsilon^2}\sin^2 2\phi} \\ B = \varepsilon\cos^2\phi + \dfrac{1}{\varepsilon}\sin^2\phi \end{cases} \qquad \begin{cases} \alpha = \arctan\left(\dfrac{1-\varepsilon^2}{2\varepsilon}\sin 2\phi\right) \\ \beta = \dfrac{\pi}{2} \end{cases}$$

$\blacksquare$

Problems for Chapter 3

3.1 Find two complex vectors, $\boldsymbol{A}$ and $\boldsymbol{B}$, such that $\boldsymbol{A}\cdot\boldsymbol{B} = 0$ and

$$\begin{aligned} \boldsymbol{A}'\cdot\boldsymbol{B}' &\neq 0 \\ \boldsymbol{A}''\cdot\boldsymbol{B}'' &\neq 0 \end{aligned}$$

where $\boldsymbol{A}'$ and $\boldsymbol{B}'$ are the real parts of the vectors, respectively, and where the imaginary parts are denoted $\boldsymbol{A}''$ and $\boldsymbol{B}''$, respectively.

3.2 For real vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ we have

$$\boldsymbol{B}\cdot(\boldsymbol{B}\times\boldsymbol{A}) = 0$$

Prove that this equality also holds for arbitrary complex vectors $\boldsymbol{A}$ and $\boldsymbol{B}$.

3.3 In some applications a generalized Debye's model is used. The susceptibility function is

$$\chi(t) = (1+\beta t)\mathrm{e}^{-\alpha t}$$

The real constant α is assumed to be positive. What conditions must the real constant β satisfy such that $\chi(t)$ is a model of a passive material?

3.4 Consider the susceptibility function

$$\chi(t) = \mathrm{e}^{-\alpha t}\cos\beta t$$

The real constant α is assumed to be positive. For what values of the real constant β is $\chi(t)$ a model of a passive material?

3.5 Determine the constitutive relations in the frequency domain for a plasma. Use the result in Problem 2.6 to find the permittivity.

3.6 In a ferrite the magnetization $\boldsymbol{M}$ is determined by

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{M} = g\mu_0\boldsymbol{M}\times\boldsymbol{H}$$

where g is the *gyromagnetic quotient*, which for electrons is $g = -e/m \approx -1.7588\cdot 10^{11}$ C/kg. Let

$$\begin{cases} \boldsymbol{H} = \hat{\boldsymbol{z}}H_0 + \boldsymbol{H}_1 \\ \boldsymbol{M} = \hat{\boldsymbol{z}}M_0 + \boldsymbol{M}_1 \\ \boldsymbol{B} = \hat{\boldsymbol{z}}B_0 + \boldsymbol{B}_1 \end{cases}$$

where $\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M})$, $B_0 = \mu_0(H_0 + M_0)$, and where

$$\begin{cases} |\boldsymbol{H}_1| \ll H_0 \\ |\boldsymbol{M}_1| \ll M_0 \\ |\boldsymbol{B}_1| \ll B_0 \end{cases}$$

Determine, using the linearized equations in $\boldsymbol{H}_1$, $\boldsymbol{M}_1$ and $\boldsymbol{B}_1$, the constitutive relations in the frequency domain, *i.e.*, find $\boldsymbol{\mu}$ in

$$\boldsymbol{B}_1 = \mu_0 \boldsymbol{\mu} \cdot \boldsymbol{H}_1$$

3.7 A simple model for a superconducting material is the "two-fluid-model". In this model we assume that the conduction electrons are in a state of superconduction where they can move freely without friction, and, moreover, assume that the remaining part of the electron are "normal" conduction electrons that are affected by a friction term. The density of the charges for the two different states are denoted N_s and N_n, respectively. The equations of dynamics of the velocities of the charges in the superconducting and the "normal" state, $\boldsymbol{v}_s$ and $\boldsymbol{v}_n$, respectively, are assumed to be

$$\begin{cases} m\dfrac{\mathrm{d}\boldsymbol{v}_s}{\mathrm{d}t} = -e\boldsymbol{E} \\ m\dfrac{\mathrm{d}\boldsymbol{v}_n}{\mathrm{d}t} + m\nu\boldsymbol{v}_n = -e\boldsymbol{E} \end{cases}$$

where m and $-e$ are the mass and the charge, respectively, of the electron and ν is the collision frequency in the "normal" state. Determine the permittivity $\epsilon(\omega)$ for this model.

3.8 Determine the state of polarization in the following cases (a and b are real, positive constants, and α is a real constant):

a) $\boldsymbol{E}(t) = \hat{\boldsymbol{e}}_1 a \cos(\omega t + \alpha) + \hat{\boldsymbol{e}}_2 b \cos(\omega t + \alpha)$

b) $\boldsymbol{E}(t) = a\left(\hat{\boldsymbol{e}}_1 \cos(\omega t + \alpha) + \hat{\boldsymbol{e}}_2 \sin(\omega t + \alpha)\right)$

c) $\boldsymbol{E}(t) = a\left(\hat{\boldsymbol{e}}_1 \cos(\omega t + \alpha) - \hat{\boldsymbol{e}}_2 \sin(\omega t + \alpha)\right)$

d) $\boldsymbol{E}(t) = \hat{\boldsymbol{e}}_1 a \cos \omega t + \hat{\boldsymbol{e}}_2 b \sin \omega t$

e) $\boldsymbol{E}(t) = a\left(\hat{\boldsymbol{e}}_1 \cos \omega t + \hat{\boldsymbol{e}}_2 \cos(\omega t - \pi/4)\right)$

$<\boldsymbol{S}(t)>$ is assumed directed along $\hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{e}}_1 \times \hat{\boldsymbol{e}}_2$, and $\hat{\boldsymbol{e}}_1 \perp \hat{\boldsymbol{e}}_2$.

3.9 a) Show that an arbitrary elliptic polarized wave can be decomposed in a superposition of a LCP and a RCP wave.

b) Let $\boldsymbol{E}_0$ be a superposition of a LCP and a RCP wave, *i.e.*,

$$\boldsymbol{E}_0 = a\boldsymbol{E}_+ + b\boldsymbol{E}_-$$

where $\boldsymbol{E}_\pm = \hat{\boldsymbol{e}}_1 \pm \mathrm{i}\hat{\boldsymbol{e}}_2$. What conditions do the complex numbers a and b satisfy in order to the wave to be linearly polarized ($<\boldsymbol{S}(t)>$ is assumed directed along $\hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{e}}_1 \times \hat{\boldsymbol{e}}_2$).

3.10 Show that a circular polarized wave satisfies

$$\boldsymbol{E}_0 = \pm \mathrm{i}\hat{\boldsymbol{e}} \times \boldsymbol{E}_0$$

where the upper (lower) sign holds for RCP (LCP).

3.11 A plane interface, $z = 0$, separates vacuum from an homogeneous ferrite. At the interface there are no surface currents and the ferrite is assumed to have a static magnetization $\boldsymbol{M} = \hat{\boldsymbol{z}} M_0$. The magnetic flux density in vacuum close to the interface is linearly polarized (B_0 real constant, B komplex constant)

$$\boldsymbol{B}_1(t) = \hat{\boldsymbol{z}} B_0 + \boldsymbol{B}_v(t) = \hat{\boldsymbol{z}} B_0 + \operatorname{Re}\left\{\hat{\boldsymbol{x}} B \mathrm{e}^{-\mathrm{i}\omega t}\right\}$$

and the fields in the ferrite, close to the interface, are

$$\begin{cases} \boldsymbol{H}_2(t) = \hat{\boldsymbol{z}} H_0 + \boldsymbol{H}_f(t) \\ \boldsymbol{B}_2(t) = \hat{\boldsymbol{z}} B_0 + \boldsymbol{B}_f(t) \end{cases}$$

The time harmonic fields $\boldsymbol{B}_v(t)$, $\boldsymbol{B}_f(t)$, and $\boldsymbol{H}_f(t)$ are assumed to be small compared to the corresponding static fields. The constants H_0 and M_0 are assumed positive and related to B_0 by

$$B_0 = \mu_0 (H_0 + M_0)$$

such that

$$\begin{cases} \omega_0 = -g\mu_0 H_0 \\ \omega_m = -g\mu_0 M_0 \end{cases}$$

are positive frequencies for electrons ($g \approx -1.7588 \cdot 10^{11}$ C/kg). Use the constitutive relations in Problem 3.6 and determine at what frequency $\omega > 0$ the magnetic flux density $\boldsymbol{B}_f(t)$ close to the interface in the ferrite is left circular polarized ($<\boldsymbol{S}(t)>$ is assumed directed along the $\hat{\boldsymbol{z}}$-direction), *i.e.*,

$$\boldsymbol{B}_f(\omega) = B_f(\hat{\boldsymbol{x}} - \mathrm{i}\hat{\boldsymbol{y}})/\sqrt{2}$$

Chapter 4

Coherence and degree of polarization

In Chapters 1 and 2 the focus was on general time varying fields, and in Section 3 we investigated the special case of time harmonic variations, *i.e.*, fields oscillating with a fixed angular frequency ω. The time harmonic wave has infinite extent in both space and time. Moreover, in Section 3.6 we showed that all time harmonic fields oscillate in a fixed plane and that the field traces an elliptic curve in this plane, *i.e.*, the ellipse of polarization. However, all physical fields have a finite extent in time, and therefore they are not strictly monochromatic, but a mixture of different frequencies. Moreover, different states of polarization are often occurring in applications. This situation is particularly frequently occurring in natural sources, *e.g.*, sources of radio astronomer waves from the sun. In this section analyze fields of this kind.

A general transient field is a superposition of time harmonic fields of different frequencies, see (3.2) on page 44.

$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{1}{\pi}\,\mathrm{Re}\int_0^\infty \boldsymbol{E}(\boldsymbol{r},\omega)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega$$

The fields in this section are assumed to be almost time harmonic, which implies that the temporal spectrum of the field, $\boldsymbol{E}(\boldsymbol{r},\omega)$, has a well defined average frequency $\bar{\omega} > 0$ and half width $\Delta\omega > 0$, and the bandwidth is small. As a consequence, we assume that the spectrum satisfies

$$\boldsymbol{E}(\boldsymbol{r},\omega) \approx \boldsymbol{0}, \qquad \text{for } |\omega - \bar{\omega}| \geq \frac{\Delta\omega}{2}$$

and that $\Delta\omega/\bar{\omega} \ll 1$. An example of a wave and its corresponding frequency spectrum is depicted in Figure 4.1. The width of frequency, $\Delta\omega > 0$, introduces two new concepts, time of coherence τ and length of coherence l, defined by

$$\tau = \frac{2\pi}{\Delta\omega}, \qquad l = \frac{2\pi c_0}{\Delta\omega} = c_0\tau$$

The length of coherence, l, is of the the same order of magnitude as the length of the wave.

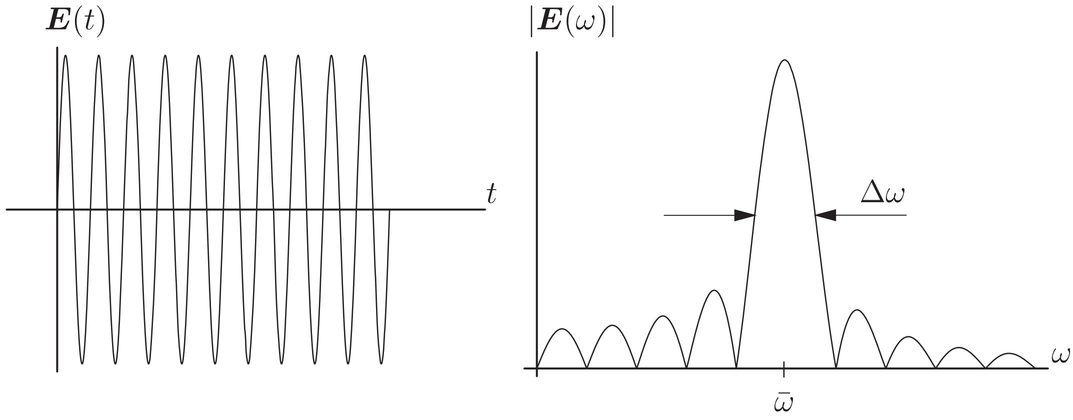


Figure 4.1: An example of a wave and the corresponding frequency spectrum and half width $\Delta\omega$.

We make a model of non-monochromatic waves, and introduce a field defined by,[1] *cf.* (3.2) and (3.3)

$$\boldsymbol{E}(t) = \mathrm{Re}\left\{\boldsymbol{E}_0(t)\mathrm{e}^{-\mathrm{i}\bar{\omega}t}\right\}$$

where the complex-valued, time dependent vector $\boldsymbol{E}_0(t)$ is defined by

$$\boldsymbol{E}_0(t) = \mathrm{e}^{\mathrm{i}\bar{\omega}t}\frac{1}{\pi}\int_0^\infty \boldsymbol{E}(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\ \mathrm{d}\omega$$

We call this field a quasi-monochromatic field provided the assumptions for $\boldsymbol{E}(\omega)$ above hold. If the vector $\boldsymbol{E}_0(t)$ is independent of time, we have the previous case and a completely time harmonic field, see Section 3.

The time variations of the complex-valued field $\boldsymbol{E}_0(t)$ are slow compared to an time interval of the order of $2\pi/\bar{\omega}$. To see this, we make a Fourier transform of the vector $\boldsymbol{E}_0(t)$. We have

$$\int_{-\infty}^\infty \boldsymbol{E}_0(t)\mathrm{e}^{\mathrm{i}\omega t}\ \mathrm{d}t = \frac{1}{\pi}\int_{-\infty}^\infty\int_0^\infty \boldsymbol{E}(\omega')\mathrm{e}^{\mathrm{i}(\omega+\bar{\omega}-\omega')t}\ \mathrm{d}\omega'\ \mathrm{d}t = 2\boldsymbol{E}(\omega+\bar{\omega})$$

The assumptions on the spectrum of the field imply that the Fourier transform of $\boldsymbol{E}_0(t)$ is negligible outside a frequency interval $[-\Delta\omega/2, \Delta\omega/2]$, and since $\Delta\omega/\bar{\omega} \ll 1$ then $\boldsymbol{E}_0(t)$ is slowly varying compared to the time interval $2\pi/\bar{\omega}$.

In Cartesian components, $\hat{\boldsymbol{e}}_1$ and $\hat{\boldsymbol{e}}_2$, the field $\boldsymbol{E}_0(t)$ is

$$\boldsymbol{E}_0(t) = \hat{\boldsymbol{e}}_1 E_{01}(t) + \hat{\boldsymbol{e}}_2 E_{02}(t) = \hat{\boldsymbol{e}}_1|E_{01}(t)|\mathrm{e}^{\mathrm{i}\alpha(t)} + \hat{\boldsymbol{e}}_2|E_{02}(t)|\mathrm{e}^{\mathrm{i}\beta(t)}$$

and $\alpha(t)$ and $\beta(t)$ are the phases of the components, which can be functions of time. We assume that the quasi-monochromatic field oscillates in a fixed plane, $\hat{\boldsymbol{e}}_1$-$\hat{\boldsymbol{e}}_2$-plane. This is correct if the field is completely monochromatic ($\boldsymbol{E}_0$ independent of time) as in Section 3. For a quasi-monochromatic field this is an assumption.

[1] The field is analyzed at a field point in space, and the dependence of the spatial variables is suppressed in this section.

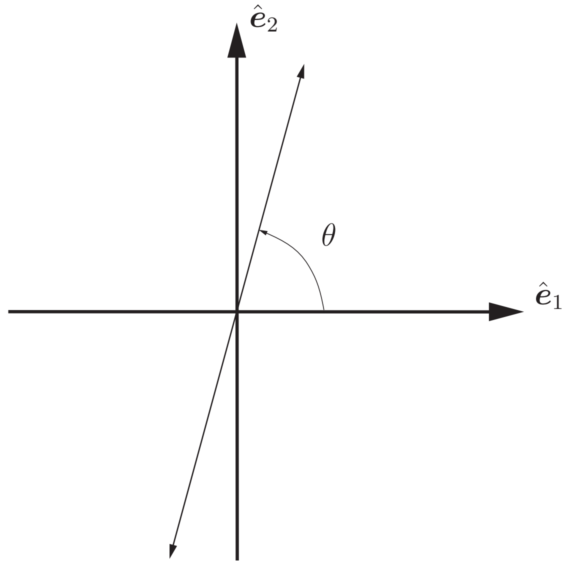


Figure 4.2: A measurement of the intensity along a fixed direction θ.

A quasi-monochromatic field is conveniently examined by investigating its properties along a fixed direction. The projection along a fixed direction $\hat{\boldsymbol{e}}$ is

$$\hat{\boldsymbol{e}} \cdot \boldsymbol{E}_0(t)$$

We parameterize the direction $\hat{\boldsymbol{e}}$ by the angle θ, see Figure 4.2. Explicitly, the representation is

$$\hat{\boldsymbol{e}} = \hat{\boldsymbol{e}}_1 \cos\theta + \hat{\boldsymbol{e}}_2 \mathrm{e}^{\mathrm{i}\delta} \sin\theta \tag{4.1}$$

where we have introduced a phase factor δ, which gives the difference in phase between the $\hat{\boldsymbol{e}}_2$-component and the $\hat{\boldsymbol{e}}_1$-component of the field. Experimentally, the projection along a fixed direction is implemented by, *e.g.*, a polarizer, and the retardation of the phase is implemented by *e.g.*, a plate of retardation. Note that $\hat{\boldsymbol{e}}$ is a complex vector, but that $\hat{\boldsymbol{e}} \cdot \hat{\boldsymbol{e}}^* = 1$. Only if $\delta = 0$, $\hat{\boldsymbol{e}}$ is a real vector.

The intensity of the field with a polarizer, oriented along the direction $\hat{\boldsymbol{e}}$ defined by (4.1), is proportional to the real quantity $I(\theta, \delta)$ defined by[2]

$$I(\theta, \delta) = <(\hat{\boldsymbol{e}} \cdot \boldsymbol{E}_0(t))(\hat{\boldsymbol{e}} \cdot \boldsymbol{E}_0(t))^*> = <|\hat{\boldsymbol{e}} \cdot \boldsymbol{E}_0(t)|^2> \tag{4.2}$$

The temporal average of two complex quantities $f_1(t)$ and $f_2(t)$ over the time interval T' is given by

$$<f_1(t) f_2(t)> = \frac{1}{2T'} \int_{-T'}^{T'} f_1(t) f_2(t) \, \mathrm{d}t$$

The time T' is assumed to be long compared to the time over which the field varies.

[2]In Section 3.3 we showed that in an isotropic material the intensity of the electromagnetic wave is proportional to the square of the electric field.

Define a 2×2-matrix of coherence, $[\mathbf{J}]$, by

$$[\mathbf{J}] = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} = \begin{pmatrix} <E_{01}(t)E_{01}^*(t)> & <E_{01}(t)E_{02}^*(t)> \\ <E_{02}(t)E_{01}^*(t)> & <E_{02}(t)E_{02}^*(t)> \end{pmatrix} \tag{4.3}$$

By the use of row and column vectors, we rewrite the matrix of coherence as (the "dagger" (†) denotes the Hermitian)

$$[\mathbf{J}] = < \begin{pmatrix} E_{01}(t) \\ E_{02}(t) \end{pmatrix} \begin{pmatrix} E_{01}(t)^* & E_{02}(t)^* \end{pmatrix} > = < \begin{pmatrix} E_{01}(t) \\ E_{02}(t) \end{pmatrix} \begin{pmatrix} E_{01}(t) \\ E_{02}(t) \end{pmatrix}^\dagger >$$

where the temporal average of a matrix $[\mathbf{A}]$ has entries that are temporal averages of the entries of the matrix $[\mathbf{A}]$. The matrix of coherence quantifies the time correlation between the Cartesian components of the electric field. The diagonal elements J_{11} and J_{22} are real, positive quantities, but J_{12} and J_{21} are complex numbers. Note that the matrix $[\mathbf{J}]$ is Hermitian, $[\mathbf{J}] = [\mathbf{J}]^*$, since

$$J_{12}^* = <E_{01}(t)E_{02}^*(t)>^* = <E_{02}(t)E_{01}^*(t)> = J_{21}$$

Schwartz' inequality for integrals on the off diagonal elements of the matrix of coherence implies

$$|J_{12}| = |<E_{01}(t)E_{02}^*(t)>| \leq \sqrt{<|E_{01}(t)|^2>}\sqrt{<|E_{02}(t)|^2>} = \sqrt{J_{11}J_{22}}$$

and, consequently, the matrix $[\mathbf{J}]$ has a non-negative determinant.

$$\det[\mathbf{J}] = J_{11}J_{22} - J_{12}J_{21} = J_{11}J_{22} - |J_{12}|^2 \geq 0$$

The intensity of the total field in an isotropic material is proportional to the sum of the diagonal elements of the matrix of coherence, *i.e.*, the trace of the matrix of coherence

$$\mathrm{Tr}[\mathbf{J}] = J_{11} + J_{22} = <|E_{01}(t)|^2> + <|E_{02}(t)|^2> = <|\boldsymbol{E}_0(t)|^2>$$

The intensity along a fixed direction $\hat{\boldsymbol{e}}$, see (4.1), is given by (4.2), and we rewrite this expression by the use of the matrix of coherence $[\mathbf{J}]$. The result is

$$\begin{aligned} I(\theta,\delta) &= J_{11}\cos^2\theta + J_{22}\sin^2\theta + J_{12}\mathrm{e}^{-\mathrm{i}\delta}\cos\theta\sin\theta + J_{21}\mathrm{e}^{\mathrm{i}\delta}\cos\theta\sin\theta \\ &= J_{11}\cos^2\theta + J_{22}\sin^2\theta + 2\,\mathrm{Re}(J_{12}\mathrm{e}^{-\mathrm{i}\delta})\cos\theta\sin\theta \end{aligned}$$

since $J_{12} = J_{21}^*$. To see how this quantity varies as a function of the angle θ, we rewrite the expression. We get, see Problem 4.1

$$I(\theta,\delta) = \frac{1}{2}\underbrace{(J_{11} + J_{22})}_{<|\boldsymbol{E}_0(t)|^2>} + R\cos(2\theta - \alpha) \tag{4.4}$$

where

$$\begin{cases} R = \dfrac{1}{2}\sqrt{(J_{11} - J_{22})^2 + (2\,\mathrm{Re}(J_{12}\mathrm{e}^{-\mathrm{i}\delta}))^2} \\ \tan\alpha = \dfrac{2\,\mathrm{Re}(J_{12}\mathrm{e}^{-\mathrm{i}\delta})}{J_{11} - J_{22}} \end{cases}$$

From these expressions we see that the intensity $I(\theta,\delta)$ varies, as a function of the angles θ and δ, between[3]

$$I_{\min} = \frac{1}{2}(J_{11}+J_{22}) - R_{\max} \leq I(\theta,\delta) \leq \frac{1}{2}(J_{11}+J_{22}) + R_{\max} = I_{\max}$$

where

$$R_{\max} = \frac{1}{2}\sqrt{(J_{11}-J_{22})^2 + 4|J_{12}|^2} = \frac{1}{2}(J_{11}+J_{22})\sqrt{1-\frac{4\det[\mathbf{J}]}{(J_{11}+J_{22})^2}}$$

The quantity $R_{\max}$ is a measure of the variation of the intensity as the angles θ and δ vary. This quantity is most conveniently measured by the degree of polarization P, which is a dimension-less quantity, defined by

$$\begin{aligned} P &= \frac{I_{\max}-I_{\min}}{I_{\max}+I_{\min}} = \frac{2R_{\max}}{J_{11}+J_{22}} \\ &= \sqrt{1-\frac{4\det[\mathbf{J}]}{(J_{11}+J_{22})^2}} = \sqrt{1-\frac{4\det[\mathbf{J}]}{\left(<|\boldsymbol{E}_0(t)|^2>\right)^2}} \end{aligned}$$

This quantity varies between 0 and 1, since it assumes its smallest values when $I_{\max} = I_{\min}$ and is maximal when $I_{\min} = 0$. Therefore, we have

$$P \in [0,1]$$

4.1 Unpolarized field

The electromagnetic field from many natural sources are unpolarized or natural, which implies that the intensity $I(\theta,\delta)$ is the same in all directions $\hat{\boldsymbol{e}}$, *i.e.*, it is independent of the angle θ and the retardation δ. This implies that the quantity $R_{\max}$ is identically zero for all angles δ, which gives

$$\begin{cases} J_{11} = J_{22} \\ J_{12} = J_{21} = 0 \end{cases}$$

The matrix of coherence for an unpolarized field then becomes:

$$[\mathbf{J}] = \begin{pmatrix} J_{11} & 0 \\ 0 & J_{22} \end{pmatrix} = \frac{<|\boldsymbol{E}_0(t)|^2>}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.5}$$

since $<|\boldsymbol{E}_0(t)|^2> = J_{11}+J_{22} = 2J_{11}$.

[3]This result can also be obtained by interpreting the intensity $I(\theta,\delta)$ as a matrix product. To see this, use

$$I(\theta,\delta) = \begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix} \begin{pmatrix} J_{11} & J_{12}e^{-i\delta} \\ J_{21}e^{i\delta} & J_{22} \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

The two eigenvalues of the matrix, $\lambda = \frac{1}{2}(J_{11}+J_{22}) \pm R_{\max}$, give the largest and the smallest values of the intensity.

The determinant of the matrix of coherence for an unpolarized field is

$$\det[\mathbf{J}] = J_{11}J_{22} - J_{12}J_{21} = |J_{11}|^2 = \frac{<|\boldsymbol{E}_0(t)|^2>^2}{4} > 0$$

and the degree of polarization for an unpolarized field is

$$P = \sqrt{1 - \frac{4\det[\mathbf{J}]}{(J_{11}+J_{22})^2}} = \sqrt{1 - \frac{4\det[\mathbf{J}]}{<|\boldsymbol{E}_0(t)|^2>^2}} = 0$$

4.2 Completely polarized field

An unpolarized field is one extreme value of the degree of polarization, $P = 0$. The other extreme value is a completely monochromatic field or a completely polarized field. The complex vector $\boldsymbol{E}_0$ is then constant in time and the matrix of coherence element is

$$[\mathbf{J}] = \begin{pmatrix} E_{01}E_{01}^* & E_{01}E_{02}^* \\ E_{02}E_{01}^* & E_{02}E_{02}^* \end{pmatrix} \tag{4.6}$$

Note that the time average now has disappeared.

The determinant of the matrix of coherence for a monochromatic field is

$$\det[\mathbf{J}] = |E_{01}|^2 |E_{02}|^2 - E_{01}E_{02}^*E_{02}E_{01}^* = 0$$

and the degree of polarization for a monochromatic field is

$$P = \sqrt{1 - \frac{4\det[\mathbf{J}]}{(J_{11}+J_{22})^2}} = 1$$

4.3 General degree of polarization

The two extremes of the degree of polarization — unpolarized and completely polarized field — are characterized by the matrices of coherence of the following form:

$$[\mathbf{J}]_{\text{unpol}} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \qquad [\mathbf{J}]_{\text{pol}} = \begin{pmatrix} B & D \\ D^* & C \end{pmatrix}$$

where A, B and C are non-negative, real numbers, $A \geq 0$, $B \geq 0$, $C \geq 0$, and D is a complex number, satisfying $BC - DD^* = 0$.

We now show that every matrix of coherence, $[\mathbf{J}]$, in a unique way can be written as a sum of an unpolarized and a completely polarized field, *i.e.*,

$$[\mathbf{J}] = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} + \begin{pmatrix} B & D \\ D^* & C \end{pmatrix} = [\mathbf{J}]_{\text{unpol}} + [\mathbf{J}]_{\text{pol}}$$

We prove this by explicitly computing the matrix entries A, B, C, and D. The following connections hold:

$$\begin{cases} J_{11} = A + B \\ J_{12} = D \\ J_{21} = D^* \\ J_{22} = A + C \end{cases}$$

Eliminate B and C in $BC - DD^* = 0$. We get

$$(J_{11} - A)(J_{22} - A) - J_{12}J_{12}^* = 0$$

The two roots, A, of this equation are

$$A = \frac{1}{2}(J_{11} + J_{22}) \pm \frac{1}{2}\sqrt{(J_{11} + J_{22})^2 - 4\det[\mathbf{J}]}$$

where $\det[\mathbf{J}] = J_{11}J_{22} - J_{12}J_{12}^*$. Both these roots are real and positive since

$$\det[\mathbf{J}] = J_{11}J_{22} - |J_{12}|^2 \leq J_{11}J_{22} \leq \frac{1}{4}(J_{11} + J_{22})^2$$

The latter inequality is easily proven by expanding $(J_{11} - J_{22})^2 \geq 0$. Only one of these roots gives positive values of B and C. The unique solution is therefore

$$\begin{cases} A = \frac{1}{2}(J_{11} + J_{22}) - \frac{1}{2}\sqrt{(J_{11} + J_{22})^2 - 4\det[\mathbf{J}]} \\ B = \frac{1}{2}(J_{11} - J_{22}) + \frac{1}{2}\sqrt{(J_{11} + J_{22})^2 - 4\det[\mathbf{J}]} \\ C = \frac{1}{2}(J_{22} - J_{11}) + \frac{1}{2}\sqrt{(J_{11} + J_{22})^2 - 4\det[\mathbf{J}]} \\ D = J_{12} \end{cases}$$

This decomposition provides us with another way of defining the degree of polarization P. The intensity of the matrix $[\mathbf{J}]_{\mathrm{pol}}$ is given by the sum of the diagonal elements.

$$\mathrm{Tr}[\mathbf{J}]_{\mathrm{pol}} = B + C = \sqrt{(J_{11} + J_{22})^2 - 4\det[\mathbf{J}]}$$

The quotient $\mathrm{Tr}[\mathbf{J}]_{\mathrm{pol}}/\mathrm{Tr}[\mathbf{J}]$ is

$$\boxed{P = \frac{\mathrm{Tr}[\mathbf{J}]_{\mathrm{pol}}}{\mathrm{Tr}[\mathbf{J}]} = \sqrt{1 - \frac{4\det[\mathbf{J}]}{(J_{11} + J_{22})^2}}}$$

which coincides with our previous definition of the degree of polarization P. This expression shows that the degree of polarization P for a general quasi-monochromatic field is given by the quotient between the intensity of the completely polarized part, $[\mathbf{J}]_{\mathrm{pol}}$, to the intensity of the total field.

4.4 Stokes' parameters

Closely related to the state of polarization of a time harmonic or a quasi-harmonic oscillating electromagnetic field is Stokes' parameters[4,5] s_i, $i = 0, 1, 2, 3$. In this section we define these parameters for a monochromatic or a quasi-monochromatic field.

Stoke parameters, which are real numbers, is most easily defined in terms of the entries of the matrix of coherence.

$$\begin{cases} s_0 = J_{11} + J_{22} \\ s_1 = J_{11} - J_{22} \\ s_2 = J_{12} + J_{21} = 2\,\mathrm{Re}\,J_{12} \\ s_3 = \mathrm{i}\,(J_{21} - J_{12}) = 2\,\mathrm{Im}\,J_{12} \end{cases} \tag{4.7}$$

Since the entries of the matrix of coherence can be interpreted as intensity quantities, $I(\theta, \delta)$, along well defined directions θ and retardation δ, Stokes' parameters can be determined experimentally with a polarizer and a plate of retardation. The relationship between these quantities is, see Problem 4.2

$$\begin{cases} s_0 = I(0,0) + I(\pi/2,0) \\ s_1 = I(0,0) - I(\pi/2,0) \\ s_2 = I(\pi/4,0) - I(3\pi/4,0) \\ s_3 = I(\pi/4,\pi/2) - I(3\pi/4,\pi/2) \end{cases}$$

From these expressions of the parameters s_i, $i = 0, 1, 2, 3$, (4.7), we find

$$s_1^2 + s_2^2 + s_3^2 = (J_{11} - J_{22})^2 + (J_{12} + J_{21})^2 - (J_{12} - J_{21})^2 = (J_{11} + J_{22})^2 - 4\det[\mathbf{J}]$$

and we have another way of expression the degree of polarization P, namely

$$P = \frac{\sqrt{s_1^2 + s_2^2 + s_3^2}}{s_0}$$

In Section 4.3 we showed that a general state of polarization can be decomposed uniquely as a sum of an unpolarized state, $P = 0$, and a completely polarized state, $P = 1$. For an unpolarized state we have $s_0 = <|\boldsymbol{E}_0(t)|^2>$ and $s_i = 0$, $i = 1, 2, 3$. The completely polarized state can be interpreted geometrically in Poincaré's sphere,[6] see Section 4.5, but before making this interpretation we rewrite Stokes' parameters of the completely polarized field. Stokes' parameters for a completely polarized field becomes, see (4.6)

$$\begin{cases} s_0 = |E_{01}|^2 + |E_{02}|^2 \\ s_1 = |E_{01}|^2 - |E_{02}|^2 \\ s_2 = 2\,\mathrm{Re}\,E_{01}E_{02}^* \\ s_3 = 2\,\mathrm{Im}\,E_{01}E_{02}^* \end{cases} \tag{4.8}$$

[4]George Gabriel Stokes (1819–1903), Irish mathematician and physicist.

[5]Another notation often occurring in the literature is I, Q, U and V, defined such such that $I = s_0$, $Q = s_1$, $U = s_2$ and $V = s_3$. These parameters were introduced by G. G. Stokes in 1852 to describe light that was not completely polarized.

[6]Henri Poincaré (1854–1912), French mathematician and theoretical physicist.

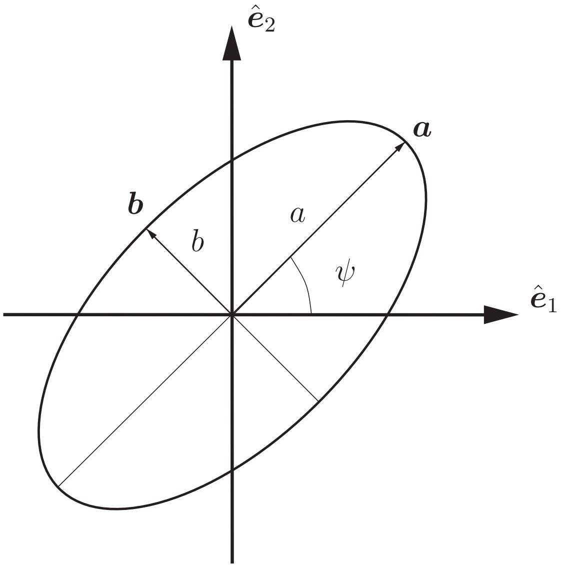


Figure 4.3: The ellipse of polarization and the definition of the tilt angle ψ.

where the time averages are superfluous, since the field $\boldsymbol{E}_0$ is independent of the time.

In Section 3.6 we analyzed the state of polarization of an arbitrary monochromatic electromagnetic field, and we concluded that the complex vector $\boldsymbol{E}_0$ completely characterizes the state of polarization, see (3.38) on page 76. Moreover, the physical field, $\boldsymbol{E}(t)$, traces an ellipse in a fixed plane, which we take as the $\hat{\boldsymbol{e}}_1$-$\hat{\boldsymbol{e}}_2$-plane. We also introduced two real-valued, orthogonal vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. These vectors are the half axes of the ellipse of the polarization. We denote the lengths of these vectors by $a = |\boldsymbol{a}|$ and $b = |\boldsymbol{b}|$, respectively, and the tilt of the ellipse is parameterized by the angle ψ (the angle between the $\hat{\boldsymbol{e}}_1$-axis and $\boldsymbol{a}$), see Figure 4.3. The relation between the complex vector $\boldsymbol{E}_0$ and the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is given by (3.41) on page 77.

$$\boldsymbol{E}_0 = \mathrm{e}^{\mathrm{i}\vartheta}(\boldsymbol{a} + \mathrm{i}\boldsymbol{b})$$

The relation between the Cartesian components E_{0n}, $n = 1, 2$ and the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is

$$\begin{cases} E_{01} = \mathrm{e}^{\mathrm{i}\vartheta}(a\cos\psi \mp \mathrm{i}b\sin\psi) \\ E_{02} = \mathrm{e}^{\mathrm{i}\vartheta}(a\sin\psi \pm \mathrm{i}b\cos\psi) \end{cases}$$

where the plus sign holds if the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are positioned as in the Figure 4.3, and the minus sign if the vector $\boldsymbol{b}$ has the opposite direction.

Stokes' parameters s_i, $i = 0, 1, 2, 3$ for a completely polarized field is now written as, see (4.8)

$$\begin{cases} s_0 = a^2 + b^2 \\ s_1 = \left(a^2 - b^2\right)\cos 2\psi \\ s_2 = \left(a^2 - b^2\right)\sin 2\psi \\ s_3 = \mp 2ab \end{cases} \tag{4.9}$$

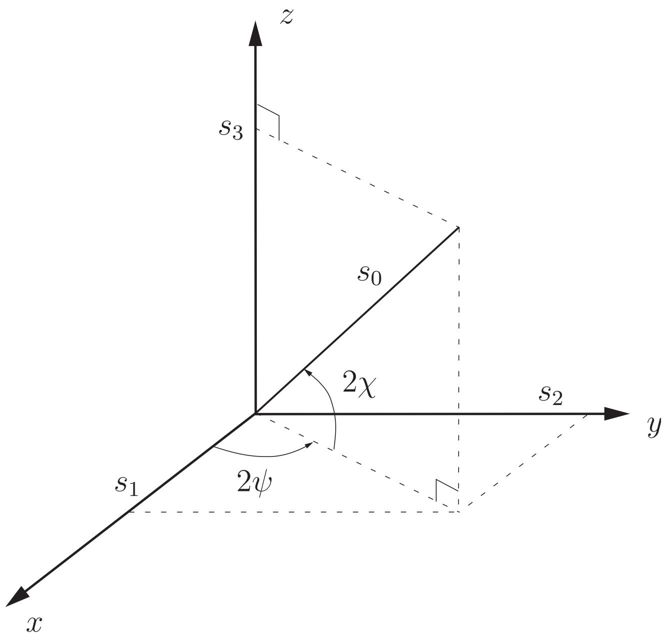


Figure 4.4: Poincaré's sphere and the angles 2ψ and 2χ.

All parameters are not independent. It is easy to prove that

$$s_0^2 = s_1^2 + s_2^2 + s_3^2 \tag{4.10}$$

Two of Stokes' parameters are invariants, *i.e.*, they do not depend on any particular coordinate system, but have the same form in every coordinate system. In Problem 4.3 it is proved

$$\begin{cases} s_0 = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 = |\boldsymbol{E}_0|^2 = a^2 + b^2 \\ s_1^2 + s_2^2 = (|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2)^2 - 4\,|\boldsymbol{a} \times \boldsymbol{b}|^2 = \left(a^2 - b^2\right)^2 \\ s_3 = -2\hat{\boldsymbol{e}} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = -\mathrm{i}\hat{\boldsymbol{e}} \cdot (\boldsymbol{E}_0 \times \boldsymbol{E}_0^*) \end{cases}$$

where $\hat{\boldsymbol{e}} = \hat{\boldsymbol{e}}_1 \times \hat{\boldsymbol{e}}_2$. The sign of s_3 determines whether the field is right- or left-handed polarized, see Table 3.8 on page 79. A negative (positive) value gives a right-(left-)handed polarization.

4.5 Poincaré's sphere

Stokes' parameters, as defined in equation (4.9), can be represented geometrically for a completely polarized field by defining a three-dimensional vector (s_1, s_2, s_3). The relation in equation (4.10) shows that this vector is a sphere at the origin with radius s_0. Two angles characterize this vector — the tilt angle 2ψ and the azimuth angle 2χ in (4.9), see Figure 4.4. The angle 2χ is defined by

$$\sin 2\chi = \frac{s_3}{s_0} = -\frac{2\hat{\boldsymbol{e}} \cdot (\boldsymbol{a} \times \boldsymbol{b})}{a^2 + b^2} = \mp \frac{2ab}{a^2 + b^2}$$

Note that the angle $\pi/2-2\chi$ is the polar angle of the vector (s_1, s_2, s_3), see Figure 4.4. We also see that $\sin 2\chi$ can be written as

$$\sin 2\chi = -\mathrm{i}\hat{\boldsymbol{e}} \cdot (\hat{\boldsymbol{p}}_\mathrm{e} \times \hat{\boldsymbol{p}}_\mathrm{e}^*) \tag{4.11}$$

using the result (3.43) on page 80 in terms of the polarization state vector $\hat{\boldsymbol{p}}_\mathrm{e} = \boldsymbol{E}_0/|\boldsymbol{E}_0|$.

This geometric interpretation of Stokes' parameters shows that the upper (lower) hemi-sphere corresponds to a left-(right-)handed polarized field. The equator $\chi = 0$ corresponds to a the linear polarized field. The poles correspond to LCP or RCP — the north or the south pole, respectively. This geometric interpretation is of great value in the evaluation of the state of polarization of the field.

Problems for Chapter 4

4.1 Do the details in the computations in the expression in (4.4).

4.2 Show that Stokes' parameters s_i, $i = 0, 1, 2, 3$, are related to the intensity measurements $I(\theta, \delta)$ by

$$\begin{cases} s_0 = I(0,0) + I(\pi/2, 0) \\ s_1 = I(0,0) - I(\pi/2, 0) \\ s_2 = I(\pi/4, 0) - I(3\pi/4, 0) \\ s_3 = I(\pi/4, \pi/2) - I(3\pi/4, \pi/2) \end{cases}$$

4.3 Show that Stokes' parameters, s_i, $i = 0, 1, 2, 3$, can be expressed in the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ or $\boldsymbol{E}_0$ in the following way:

$$\begin{cases} s_0 = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 = |\boldsymbol{E}_0|^2 = a^2 + b^2 \\ s_1^2 + s_2^2 = (|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2)^2 - 4\,|\boldsymbol{a} \times \boldsymbol{b}|^2 = \left(a^2 - b^2\right)^2 \\ s_3 = -2\hat{\boldsymbol{e}} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = -\mathrm{i}\hat{\boldsymbol{e}} \cdot (\boldsymbol{E}_0 \times \boldsymbol{E}_0^*) \end{cases}$$

where $\hat{\boldsymbol{e}} = \hat{\boldsymbol{e}}_1 \times \hat{\boldsymbol{e}}_2$.

Appendix A

Vectors and linear transformations

This appendix contains a short overview of the concept of vectors and linear transformations (dyadics) of vectors, and how these are represented in terms of their components. Transformations between different rotated coordinate systems are also reviewed as well as the corresponding transformation of the components of a dyadic.

A.1 Vectors

In this textbook we denote vectors in italic bold face, *e.g.*, $\boldsymbol{u}$. Vectors are in general functions of space and time coordinates, *i.e.*, $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{r}, t)$, but in this appendix these variables are suppressed since they are not essential for the analysis. A vector that depends on the space and time coordinates is called a *vector field*.

In a particular Cartesian coordinate system $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$ the representation of a vector $\boldsymbol{u}$ is [1]

$$\boldsymbol{u} = \hat{\boldsymbol{e}}_1 u_1 + \hat{\boldsymbol{e}}_2 u_2 + \hat{\boldsymbol{e}}_3 u_3$$

A "hat" or caret (ˆ) over a vector, *e.g.*, $\hat{\boldsymbol{e}}_1$, denotes that the vector has unit length (unit vector). The components of the vector, u_i, are obtained by the scalar product

$$u_i = \boldsymbol{u} \cdot \hat{\boldsymbol{e}}_i \qquad i = 1, 2, 3$$

The components of the vector are often written as a column vector.

$$[\boldsymbol{u}] = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

In this textbook we use brackets around the vector, $[\boldsymbol{u}]$, to indicate that we refer to the components of the vector in a particular coordinate system and not the vector $\boldsymbol{u}$ itself. Notice that the vector $\boldsymbol{u}$ is geometric quantity, which is independent of

[1] We assume the unit vectors (basis vectors) $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$ are orthonormal and right-hand oriented.

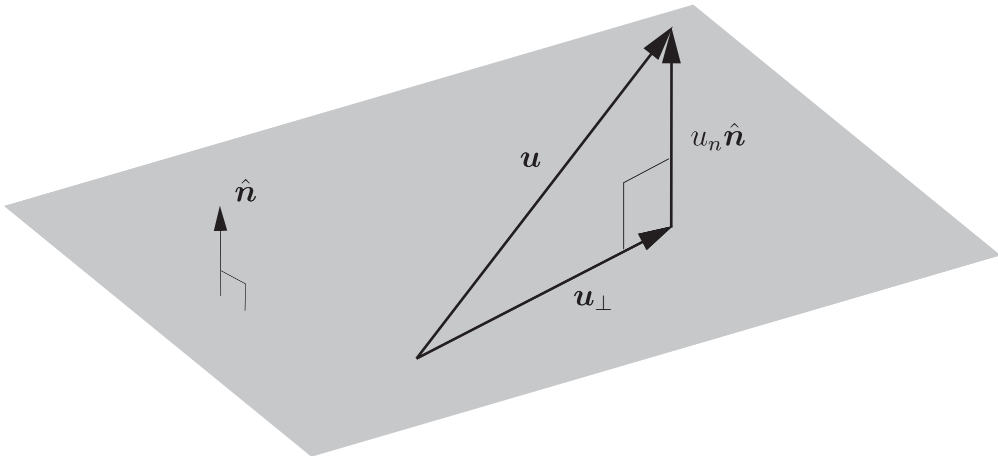


Figure A.1: Projection of a vector $\boldsymbol{u}$ in a component, $u_n\hat{\boldsymbol{n}}$, along the direction $\hat{\boldsymbol{n}}$ and a component, $\boldsymbol{u}_\perp$, perpendicular to this direction.

coordinate system. The coordinate representation, $[\boldsymbol{u}]$, on the other hand, depends on the coordinate system we use to represent the vector in, *i.e.*, the components are different in different coordinate systems, but the vector itself remains the same.

In wave propagation problems, the vectors are decomposed in its components along a given direction (usually the direction of propagation) $\hat{\boldsymbol{n}}$, and a component lying in the plane orthogonal to the direction $\hat{\boldsymbol{n}}$. We denote this decomposition or projection of a vector $\boldsymbol{u}$ as a sum of two parts $\boldsymbol{u}_\perp$ and u_n, see Figure A.1

$$\boldsymbol{u} = \boldsymbol{u}_\perp + u_n\hat{\boldsymbol{n}}$$

where

$$\begin{cases} u_n = \boldsymbol{u}\cdot\hat{\boldsymbol{n}} \\ \boldsymbol{u}_\perp = \boldsymbol{u} - \hat{\boldsymbol{n}}(\boldsymbol{u}\cdot\hat{\boldsymbol{n}}) = -\hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\hat{\boldsymbol{u}}) \end{cases} \tag{A.1}$$

by the use of the BAC-CAB rule, $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})$.

A.2 Linear transformations, matrices and dyadics

Often, we have to deal with linear mappings from one vector field to other vector field, *i.e.*, mapping of a vector $\boldsymbol{u}$ to another vector $\boldsymbol{v}$ — both in general functions of the space and time coordinates $\boldsymbol{r}$ and t. The most simple type of linear transformation is

$$\boldsymbol{v} = \boldsymbol{a}\underbrace{(\boldsymbol{b}\cdot\boldsymbol{u})}_{\text{scalar}}$$

The vector $\boldsymbol{u}$ is here mapped to a new vector $\boldsymbol{v}$ in a new direction along the vector $\boldsymbol{a}$. The scaling of the vector is made with the vector $\boldsymbol{b}$ by the scalar product $\boldsymbol{b}\cdot\boldsymbol{u}$. This mapping is a called a *(simple) dyadic*, and we use the symbol $\boldsymbol{ab}$ for this transformation, together with the dyadic product (either with or without parenthesis around the transformation $\boldsymbol{ab}$) defined as

$$\boldsymbol{v} = (\boldsymbol{ab}) \cdot \boldsymbol{u} = \boldsymbol{ab} \cdot \boldsymbol{u} \overset{\text{def}}{=} \boldsymbol{a} (\boldsymbol{b} \cdot \boldsymbol{u})$$

Note that the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in the transformation $\boldsymbol{ab}$ are forming a new quantity without any signs between the vectors. The components of the vector $\boldsymbol{v}$ are

$$v_i = a_i \sum_{j=1}^{3} b_j u_j \qquad i = 1, 2, 3$$

Simple dyadics can be used as building blocks for general linear transformations of vector fields. The sum of two dyadics, *e.g.*, $\mathbf{A} = \boldsymbol{a}_1 \boldsymbol{b}_1 + \boldsymbol{a}_2 \boldsymbol{b}_2$, is a new dyadic defined in the following natural way:

$$\boldsymbol{v} = \mathbf{A} \cdot \boldsymbol{u} = (\boldsymbol{a}_1 \boldsymbol{b}_1 + \boldsymbol{a}_2 \boldsymbol{b}_2) \cdot \boldsymbol{u} = \boldsymbol{a}_1 \boldsymbol{b}_1 \cdot \boldsymbol{u} + \boldsymbol{a}_2 \boldsymbol{b}_2 \cdot \boldsymbol{u}$$

Note that the linear transformation (dyadic) $\mathbf{A}$ is written in roman bold face to distinguish the quantity from the vector $\boldsymbol{A}$ (italic bold face).

In a specific coordinate system $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$ we represent the general linear transformation $\mathbf{A}$ from a vector field $\boldsymbol{u}$ to another vector field $\boldsymbol{v}$ by simple dyadics $\hat{\boldsymbol{e}}_i \hat{\boldsymbol{e}}_j$, $i, j = 1, 2, 3$.

$$\mathbf{A} = \sum_{i,j=1}^{3} A_{ij} \hat{\boldsymbol{e}}_i \hat{\boldsymbol{e}}_j \tag{A.2}$$

The action on a vector field $\boldsymbol{u}$ becomes

$$\boldsymbol{v} = \mathbf{A} \cdot \boldsymbol{u} \overset{\text{def}}{=} \sum_{i,j=1}^{3} \hat{\boldsymbol{e}}_i A_{ij} (\hat{\boldsymbol{e}}_j \cdot \boldsymbol{u}) = \sum_{i,j=1}^{3} \hat{\boldsymbol{e}}_i A_{ij} u_j$$

or in its components

$$v_i = \sum_{j=1}^{3} A_{ij} u_j \qquad i = 1, 2, 3$$

The component representation of the linear transformation or dyadic $\mathbf{A}$ is represented by a matrix

$$[\mathbf{A}] = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

Just as with vectors, we denote the matrix of the dyadic $\mathbf{A}$ by enclosing the dyadic by brackets, *i.e.*, $[\mathbf{A}]$, to distinguish the coordinate representation of the dyadic from the dyadic $\mathbf{A}$ itself, which is independent of any coordinate representation. This convention is similar to the one we use for a vector $\boldsymbol{u}$ and its coordinate representation $[\boldsymbol{u}]$.

By the notion $\boldsymbol{u}_i = \sum_{j=1}^{3} A_{ij}\hat{\boldsymbol{e}}_j$, $i = 1, 2, 3$, we see that the general linear transformation $\mathbf{A}$ always can be written as

$$\mathbf{A} = \hat{\boldsymbol{e}}_1\boldsymbol{u}_1 + \hat{\boldsymbol{e}}_2\boldsymbol{u}_2 + \hat{\boldsymbol{e}}_3\boldsymbol{u}_3$$

which shows that three dyadics is enough to represent a general linear transformation or dyadic.

So far, the action of vector, $\boldsymbol{u}$, on a dyadic, $\mathbf{A}$, has been from the right by a scalar multiplication. However, some of the advantages with the dyadic notation are seen if we let the action take place from the left. This action leads to the transpose operation. To this end, make a scalar multiplication from the left, that gives a new vector field $\boldsymbol{v}$ defined by, see (A.2)

$$\boldsymbol{v} = \boldsymbol{u} \cdot \mathbf{A} \stackrel{\text{def}}{=} \sum_{i,j=1}^{3} (\hat{\boldsymbol{e}}_i \cdot \boldsymbol{u})\, A_{ij}\hat{\boldsymbol{e}}_j = \sum_{i,j=1}^{3} \hat{\boldsymbol{e}}_j A_{ij} u_i = \sum_{i=1}^{3} u_i \boldsymbol{u}_i$$

or in component form

$$v_i = \sum_{j=1}^{3} A_{ji} u_j \qquad i = 1, 2, 3$$

We observe that a scalar multiplication from the left gives the transpose of the matrix $[\mathbf{A}]$. This operation defines the transposed dyadic, $\mathbf{A}^t$.

$$\boldsymbol{v} = \mathbf{A}^t \cdot \boldsymbol{u} \stackrel{\text{def}}{=} \boldsymbol{u} \cdot \mathbf{A}$$

and

$$[\mathbf{A}^t] = [\mathbf{A}]^t$$

In particular,

$$\boldsymbol{u}_1 \cdot (\mathbf{A} \cdot \boldsymbol{u}_2) = \big(\boldsymbol{u}_2 \cdot \mathbf{A}^t\big) \cdot \boldsymbol{u}_1 = \boldsymbol{u}_2 \cdot \big(\mathbf{A}^t \cdot \boldsymbol{u}_1\big)$$

for all complex-valued vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

The inverse dyadic, $\mathbf{A}^{-1}$, of a dyadic $\mathbf{A}$ can be defined as the matrix inverse of its matrix representation in a specific coordinate system, *i.e.*,

$$[\mathbf{A}^{-1}] = [\mathbf{A}]^{-1} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^{-1}$$

if the coordinate representation of $\mathbf{A}$ is

$$[\mathbf{A}] = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

Similarly, we define the (complex) conjugate of a dyadic, $\mathbf{A}^*$, and its conjugate transpose or Hermitian transpose, $\mathbf{A}^\dagger$, of a dyadic $\mathbf{A}$ by the coordinate representations

$$[\mathbf{A}^*] = [\mathbf{A}]^* = \begin{pmatrix} A_{11}^* & A_{12}^* & A_{13}^* \\ A_{21}^* & A_{22}^* & A_{23}^* \\ A_{31}^* & A_{32}^* & A_{33}^* \end{pmatrix} \qquad [\mathbf{A}^\dagger] = [\mathbf{A}]^\dagger = \begin{pmatrix} A_{11}^* & A_{21}^* & A_{31}^* \\ A_{12}^* & A_{22}^* & A_{32}^* \\ A_{13}^* & A_{23}^* & A_{33}^* \end{pmatrix}$$

We find that $\mathbf{A}^\dagger = \mathbf{A}^{*t} = \mathbf{A}^{t*}$. A dyadic is called symmetric (Hermitian) if $\mathbf{A}^t = \mathbf{A}$ ($\mathbf{A}^\dagger = \mathbf{A}$). If the dyadic satisfies $\mathbf{A}^t = -\mathbf{A}$ ($\mathbf{A}^\dagger = -\mathbf{A}$), the dyadic is called anti-symmetric (anti-Hermitian). A general dyadic can be decomposed in a symmetric (Hermitian) and an anti-symmetric (anti-Hermitian) part as

$$\mathbf{A} = \mathbf{A}_\mathrm{s} + \mathbf{A}_\mathrm{a} \qquad (\mathbf{A} = \mathbf{A}_\mathrm{H} + \mathbf{A}_\text{a-H})$$

where

$$\begin{cases} \mathbf{A}_\mathrm{s} = \dfrac{1}{2}\left(\mathbf{A} + \mathbf{A}^t\right) \\ \mathbf{A}_\mathrm{a} = \dfrac{1}{2}\left(\mathbf{A} - \mathbf{A}^t\right) \end{cases} \qquad \left(\begin{cases} \mathbf{A}_\mathrm{H} = \dfrac{1}{2}\left(\mathbf{A} + \mathbf{A}^\dagger\right) \\ \mathbf{A}_\text{a-H} = \dfrac{1}{2}\left(\mathbf{A} - \mathbf{A}^\dagger\right) \end{cases}\right)$$

Similarly, a complex-valued dyadic can be decomposed in its real and imaginary parts as

$$\mathbf{A} = \mathbf{A}_\mathrm{r} + \mathrm{i}\mathbf{A}_\mathrm{i}$$

where

$$\begin{cases} \mathbf{A}_\mathrm{r} = \dfrac{1}{2}\left(\mathbf{A} + \mathbf{A}^\dagger\right) \\ \mathbf{A}_\mathrm{i} = \dfrac{1}{2\mathrm{i}}\left(\mathbf{A} - \mathbf{A}^\dagger\right) \end{cases}$$

Notice that both $\mathbf{A}_\mathrm{r}$ and $\mathbf{A}_\mathrm{i}$ are Hermitian dyadics, and that the off-diagonal elements of $\mathbf{A}_\mathrm{r}$ and $\mathbf{A}_\mathrm{i}$ in general are not real numbers (the diagonal elements are). For any $\boldsymbol{u} \in \mathbb{C}^3$

$$\begin{cases} \mathrm{Re}\left\{\boldsymbol{u}^* \cdot \mathbf{A} \cdot \boldsymbol{u}\right\} = \boldsymbol{u}^* \cdot \mathbf{A}_\mathrm{r} \cdot \boldsymbol{u} \\ \mathrm{Im}\left\{\boldsymbol{u}^* \cdot \mathbf{A} \cdot \boldsymbol{u}\right\} = \boldsymbol{u}^* \cdot \mathbf{A}_\mathrm{i} \cdot \boldsymbol{u} \end{cases}$$

Analogously, the vector product between a vector field $\boldsymbol{u}$ and a dyadic $\mathbf{A}$ is defined.

$$\mathbf{B} = \mathbf{A} \times \boldsymbol{u} \stackrel{\text{def}}{=} \sum_{i,j=1}^{3} \hat{\boldsymbol{e}}_i A_{ij} \left(\hat{\boldsymbol{e}}_j \times \boldsymbol{u}\right)$$

The vector product can also be applied from the left. We get

$$\mathbf{B} = \boldsymbol{u} \times \mathbf{A} \stackrel{\text{def}}{=} \sum_{i,j=1}^{3} \left(\boldsymbol{u} \times \hat{\boldsymbol{e}}_i\right) A_{ij} \hat{\boldsymbol{e}}_j$$

A.2.1 Projections

Projections of a vector $\boldsymbol{u}$ on a plane with unit normal vector $\hat{\boldsymbol{n}}$ is a linear mapping with a dyadic, see (A.1)

$$\boldsymbol{u}_\perp = \boldsymbol{u} - \hat{\boldsymbol{n}}(\hat{\boldsymbol{n}} \cdot \boldsymbol{u}) = \mathbf{I}_\perp \cdot \boldsymbol{u}$$

where the projection dyadic $\mathbf{I}_\perp$ is

$$\mathbf{I}_\perp = \mathbf{I}_3 - \hat{\boldsymbol{n}}\hat{\boldsymbol{n}}$$

In the same manner, an arbitrary dyadic $\mathbf{A}$ can be decomposed into components parallel and perpendicular to a given fixed direction $\hat{\boldsymbol{n}}$. We obtain this decomposition by employing $\mathbf{I}_3 = \mathbf{I}_\perp + \hat{\boldsymbol{n}}\hat{\boldsymbol{n}}$

$$\boldsymbol{v} = \mathbf{A} \cdot \boldsymbol{u} = (\mathbf{I}_\perp + \hat{\boldsymbol{n}}\hat{\boldsymbol{n}}) \cdot \mathbf{A} \cdot (\mathbf{I}_\perp + \hat{\boldsymbol{n}}\hat{\boldsymbol{n}}) \cdot \boldsymbol{u}$$

This decomposition implies that every dyadic $\mathbf{A}$ can be written as:

$$\mathbf{A} = \mathbf{A}_{\perp\perp} + \hat{\boldsymbol{n}}\boldsymbol{A}_n + \boldsymbol{A}_\perp\hat{\boldsymbol{n}} + \hat{\boldsymbol{n}}A_{nn}\hat{\boldsymbol{n}} \tag{A.3}$$

where we introduced the notation

$$\begin{cases} \mathbf{A}_{\perp\perp} = \mathbf{I}_\perp \cdot \mathbf{A} \cdot \mathbf{I}_\perp \\ \boldsymbol{A}_n = \hat{\boldsymbol{n}} \cdot \mathbf{A} \cdot \mathbf{I}_\perp \\ \boldsymbol{A}_\perp = \mathbf{I}_\perp \cdot \mathbf{A} \cdot \hat{\boldsymbol{n}} \\ A_{nn} = \hat{\boldsymbol{n}} \cdot \mathbf{A} \cdot \hat{\boldsymbol{n}} \end{cases}$$

Note that the dyadic $\mathbf{A}_{\perp\perp}$ is a two-dimensional dyadic (mapping vectors in the plane with normal $\hat{\boldsymbol{n}}$ into the same plane). Similarly, the vectors $\boldsymbol{A}_n$ and $\boldsymbol{A}_\perp$ are two-dimensional vectors, and A_{nn} is a scalar. If we let the direction $\hat{\boldsymbol{n}}$ be the unit vector $\hat{\boldsymbol{e}}_3$, we obtain the coordinate representations of these quantities (we use the index 3 instead of n)

$$[\mathbf{A}_{\perp\perp}] = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad [\boldsymbol{A}_\perp] = \begin{pmatrix} A_{13} \\ A_{23} \\ 0 \end{pmatrix} \qquad [\boldsymbol{A}_3] = \begin{pmatrix} A_{31} \\ A_{32} \\ 0 \end{pmatrix}$$

Such a decomposition is proved useful in the analysis of wave propagation in planar structures.

A.3 Rotation of coordinate system

Several textbooks deal with rotations in space, *e.g.*, the excellent book by Kuipers [15], which also contains the concept of quaternions and its connections to rotations in $\mathbb{R}^3$. For details we refer to these textbooks.

Two coordinate systems $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$ and $(\hat{\boldsymbol{e}}_1', \hat{\boldsymbol{e}}_2', \hat{\boldsymbol{e}}_3')$, both orthonormal and right-hand oriented, share the common origin. An example of two such systems is depicted in Figure A.2.

Since the unprimed unit vectors form a basis, each of the primed unit vectors $(\hat{\boldsymbol{e}}_1', \hat{\boldsymbol{e}}_2', \hat{\boldsymbol{e}}_3')$ can be expressed as a linear combination of the unprimed unit vectors $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$. We have

$$\begin{cases} \hat{\boldsymbol{e}}_1' = \hat{\boldsymbol{e}}_1 a_{11} + \hat{\boldsymbol{e}}_2 a_{12} + \hat{\boldsymbol{e}}_3 a_{13} \\ \hat{\boldsymbol{e}}_2' = \hat{\boldsymbol{e}}_1 a_{21} + \hat{\boldsymbol{e}}_2 a_{22} + \hat{\boldsymbol{e}}_3 a_{23} \\ \hat{\boldsymbol{e}}_3' = \hat{\boldsymbol{e}}_1 a_{31} + \hat{\boldsymbol{e}}_2 a_{32} + \hat{\boldsymbol{e}}_3 a_{33} \end{cases}$$

or

$$\hat{\boldsymbol{e}}_i' = \sum_{j=1}^{3} \hat{\boldsymbol{e}}_j a_{ij} \qquad i = 1, 2, 3$$

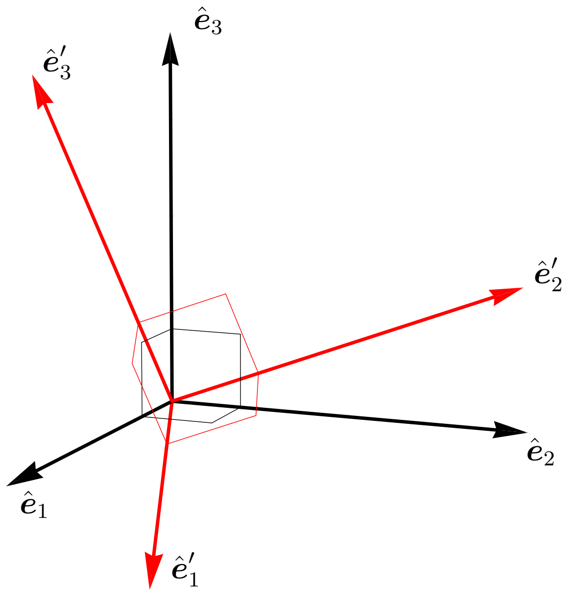


Figure A.2: Two rotated coordinate systems $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$ and $(\hat{\boldsymbol{e}}'_1, \hat{\boldsymbol{e}}'_2, \hat{\boldsymbol{e}}'_3)$ with a common origin. The original is shown in black and the rotated in red.

Since we have assumed that the unit vectors $(\hat{\boldsymbol{e}}'_1, \hat{\boldsymbol{e}}'_2, \hat{\boldsymbol{e}}'_3)$ and $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$ both are right-hand oriented, the determinant of the matrix $[\mathbf{A}]$, with entries a_{ij}, is 1, *i.e.*, $\det[\mathbf{A}] = 1$.

The components a_{ij}, $i, j = 1, 2, 3$, are the direction cosines[2] between the axis i and j.

$$a_{ij} = \hat{\boldsymbol{e}}'_i \cdot \hat{\boldsymbol{e}}_j \qquad i, j = 1, 2, 3$$

In general, we have

$$\hat{\boldsymbol{e}}'_i \cdot \hat{\boldsymbol{e}}_j = a_{ij} \neq a_{ji} = \hat{\boldsymbol{e}}'_j \cdot \hat{\boldsymbol{e}}_i \qquad i, j = 1, 2, 3$$

Similarly, the unprimed unit vectors $\hat{\boldsymbol{e}}_i$ can be written as a linear combination of the primed ones $(\hat{\boldsymbol{e}}'_1, \hat{\boldsymbol{e}}'_2, \hat{\boldsymbol{e}}'_3)$. We get

$$\hat{\boldsymbol{e}}_i = \hat{\boldsymbol{e}}'_1 \left(\hat{\boldsymbol{e}}'_1 \cdot \hat{\boldsymbol{e}}_i\right) + \hat{\boldsymbol{e}}'_2 \left(\hat{\boldsymbol{e}}'_2 \cdot \hat{\boldsymbol{e}}_i\right) + \hat{\boldsymbol{e}}'_3 \left(\hat{\boldsymbol{e}}'_3 \cdot \hat{\boldsymbol{e}}_i\right) \quad i = 1, 2, 3$$

or expressed in the direction cosines a_{ij}

$$\begin{cases} \hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{e}}'_1 a_{11} + \hat{\boldsymbol{e}}'_2 a_{21} + \hat{\boldsymbol{e}}'_3 a_{31} \\ \hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{e}}'_1 a_{12} + \hat{\boldsymbol{e}}'_2 a_{22} + \hat{\boldsymbol{e}}'_3 a_{32} \\ \hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{e}}'_1 a_{13} + \hat{\boldsymbol{e}}'_2 a_{23} + \hat{\boldsymbol{e}}'_3 a_{33} \end{cases}$$

In short

$$\hat{\boldsymbol{e}}_i = \sum_{j=1}^{3} \hat{\boldsymbol{e}}'_j a_{ji} \qquad i = 1, 2, 3$$

[2]Also given as $a_{ij} = \cos(x_i, x_j)$ where (x_i, x_j) is the angle between $\hat{\boldsymbol{e}}'_i$ and $\hat{\boldsymbol{e}}_j$ axes.

From these expressions we see that if the transformation $\hat{\boldsymbol{e}}_i \to \hat{\boldsymbol{e}}'_i$ is made with a_{ij} then the transformation $\hat{\boldsymbol{e}}'_i \to \hat{\boldsymbol{e}}_i$ is made with a_{ji}. The matrix $[\mathbf{A}]$ therefore is an orthogonal matrix, *i.e.*, $[\mathbf{A}]^{-1} = [\mathbf{A}]^t$.

We are now ready to give the formal definition of a vector. A vector $\boldsymbol{u}$ is a geometric quantity with components (u_1, u_2, u_3) in the system $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$, which are related to the components (u'_1, u'_2, u'_3) in the system $(\hat{\boldsymbol{e}}'_1, \hat{\boldsymbol{e}}'_2, \hat{\boldsymbol{e}}'_3)$ by the direction cosines, a_{ij}, $i, j = 1, 2, 3$, in the following way:

$$u'_i = \sum_{j=1}^{3} a_{ij} u_j \qquad i = 1, 2, 3 \tag{A.4}$$

or expressed as column vectors and standard matrix multiplication

$$\begin{pmatrix} u'_1 \\ u'_2 \\ u'_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

Definition A.1. *A physical quantity* $\boldsymbol{u}$ *which components* u_i *and* u'_i *in two rotated coordinate systems, respectively, are related to each other by* (A.4) *is called a* **vector**[3].

This definition implies that for every vector $\boldsymbol{u}$ we have

$$\begin{aligned} \boldsymbol{u} =& \hat{\boldsymbol{e}}'_1 u'_1 + \hat{\boldsymbol{e}}'_2 u'_2 + \hat{\boldsymbol{e}}'_3 u'_3 = \sum_{i=1}^{3} u'_i \hat{\boldsymbol{e}}'_i \\ =& \sum_{i,j=1}^{3} a_{ij} u_j \hat{\boldsymbol{e}}'_i = \sum_{j=1}^{3} u_j \hat{\boldsymbol{e}}_j = \hat{\boldsymbol{e}}_1 u_1 + \hat{\boldsymbol{e}}_2 u_2 + \hat{\boldsymbol{e}}_3 u_3 \end{aligned}$$

By this definition every vector becomes a quantity which is independent of the coordinate representation, just as we required in the beginning of this appendix.

In a similar way, we define a dyadic (or a tensor of the second kind).

Definition A.2. *A physical quantity* $\mathbf{D}$ *which components* D_{ij} *and* D'_{ij} *in two rotated coordinate systems, respectively, are related by*

$$D'_{ij} = \sum_{k,l=1}^{3} a_{ik} a_{jl} D_{kl} \qquad i, j = 1, 2, 3$$

is called a **dyadic**. *This relation can also be expressed as similarity transformation*

$$[\mathbf{D}]' = [\mathbf{A}]\,[\mathbf{D}]\,[\mathbf{A}^t] \tag{A.5}$$

[3] A vector is also called a polar vector which distinguishes it from an axial vector which transforms by (A.4) where $\det[\mathbf{A}] = -1$. If the vector depends on the space coordinates, *i.e.*, $\boldsymbol{u}$ is a vector field, then also the space coordinates are transformed as

$$u'_i(x'_1, x'_2, x'_3) = \sum_{j=1}^{3} a_{ij} u_j(x_1, x_2, x_3) \qquad i = 1, 2, 3$$

where x'_1, x'_2, x'_3 and x_1, x_2, x_3 are the components of the position vector in the primed and the unprimed systems, respectively.

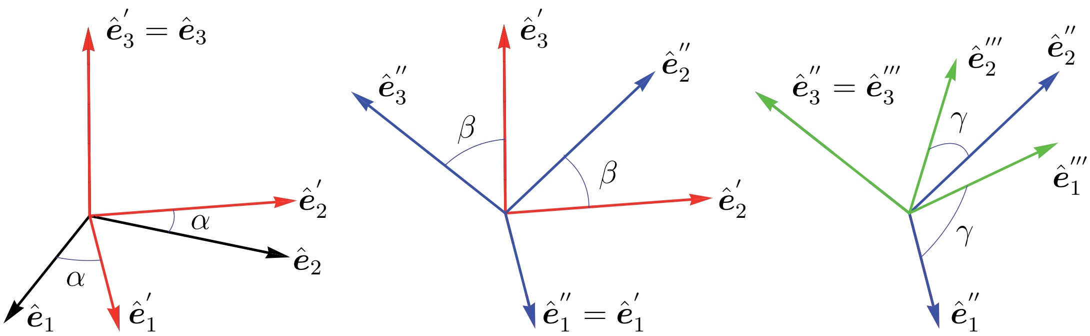


Figure A.3: Definition of the three Euler angles α, β and γ. The first rotation has red coordinate axes, the second rotation blue ones, and the final rotation has green coordinate axes.

A.3.1 Euler angles

The two rotated coordinate systems, $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$ and $(\hat{\boldsymbol{e}}_1', \hat{\boldsymbol{e}}_2', \hat{\boldsymbol{e}}_3')$, are related to each other by the direction cosines. These relations can also be expressed in an alternative way by the three Euler angles α, β, γ. These angles are defines by three consecutive rotations, see Figure A.3. The three rotations are explicitly given by:

1. A rotation with the angle α around the $\hat{\boldsymbol{e}}_3$ axis
2. A rotation with the angle β around the $\hat{\boldsymbol{e}}_1'$ axis
3. A rotation with the angle γ around the $\hat{\boldsymbol{e}}_3''$ axis

The three different rotations are represented by the following matrices:

1. The first rotation is represented by

$$[\mathbf{R}_1] = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2. The second rotation is represented by

$$[\mathbf{R}_2] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{pmatrix}$$

3. The third rotation is represented by

$$[\mathbf{R}_3] = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

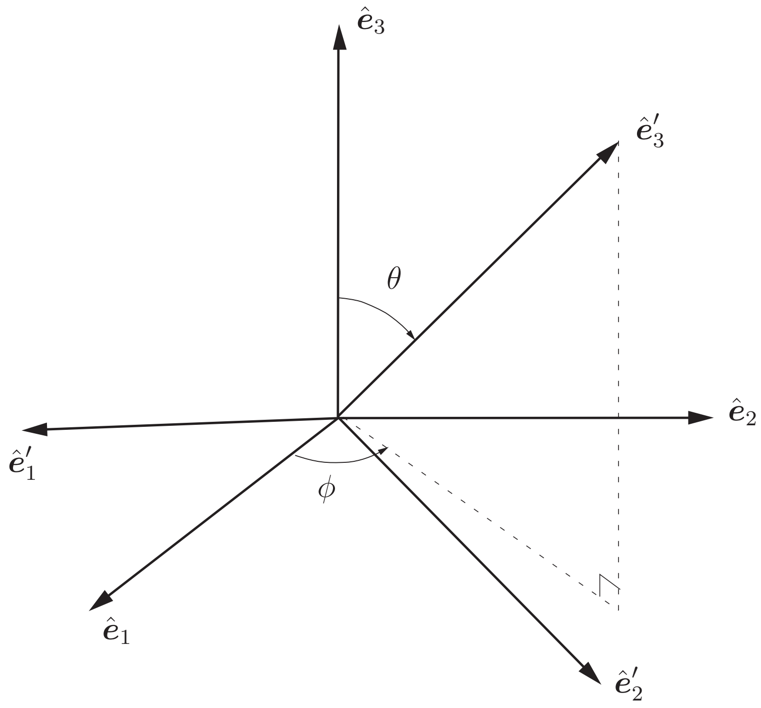


Figure A.4: The angles of rotation θ and ϕ. The unit vector $\hat{\boldsymbol{e}}'_1$ lies in the $\hat{\boldsymbol{e}}_1$-$\hat{\boldsymbol{e}}_2$-plane in this figure.

In total, the rotation is made by

$$[\mathbf{A}] = [\mathbf{R}_3]\,[\mathbf{R}_2]\,[\mathbf{R}_1]$$

In this textbook, we are often using similarity transformations of linear transformations (dyadics). These are made by a *rotation matrix* $[\mathbf{R}]$ consisting of a combination of two rotations. The spherical angles θ and ϕ are defined in Figure A.4. The relation between these angles and the Euler angles α, β and γ is:

$$\alpha = \phi - \pi/2, \qquad \beta = -\theta, \qquad \gamma = 0$$

We get

$$\begin{aligned}
[\mathbf{R}] = [\mathbf{R}_2]\,[\mathbf{R}_1] &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha\cos\beta & \cos\alpha\cos\beta & \sin\beta \\ \sin\alpha\sin\beta & -\cos\alpha\sin\beta & \cos\beta \end{pmatrix} \\
&= \begin{pmatrix} \sin\phi & -\cos\phi & 0 \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix}
\end{aligned} \tag{A.6}$$

Appendix B

The Fourier transform

A series of useful results related to Fourier[1] transforms in one or several dimensions is collected in this appendix. This overview also contains the Hilbert transform, Section B.2, and Meĭman's theorem, Section B.3, that often are used in conjunction with Fourier transforms. The class of functions of positive type and Herglotz functions are also reviewed in Sections B.4 and B.5, respectively.

B.1 The Fourier transform

In this section, we summarize the most import properties of the Fourier transform in $\mathbb{R}^n$. For mathematical details and proofs we refer to the literature, see *e.g.*, [24]. The definition of the Fourier transform of an integrable complex-valued function $f(\boldsymbol{x})$ is

$$\hat{f}(\boldsymbol{\xi}) = \int\limits_{\mathbb{R}^n} f(\boldsymbol{x}) \mathrm{e}^{-\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{x}} \,\mathrm{d}x^n$$

The Fourier transform satisfy

$$\left|\hat{f}(\boldsymbol{\xi})\right| \leq \int\limits_{\mathbb{R}^n} |f(\boldsymbol{x})| \,\mathrm{d}x^n$$

and it has an inverse

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \hat{f}(\boldsymbol{\xi}) \mathrm{e}^{\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{x}} \,\mathrm{d}\xi^n$$

The Fourier transform of a derivative is

$$\widehat{\partial_i f}(\boldsymbol{\xi}) = \mathrm{i}\,(\boldsymbol{\xi}\cdot\hat{\boldsymbol{e}}_i)\,\hat{f}(\boldsymbol{\xi}), \qquad i = 1, 2, \ldots, n$$

where $\hat{\boldsymbol{e}}_i$ is a unit vector in the x_i-direction. The Parseval's formula[2] reads

$$\int\limits_{\mathbb{R}^n} \hat{f}(\boldsymbol{\xi})\hat{g}^*(\boldsymbol{\xi}) \,\mathrm{d}\xi^n = (2\pi)^n \int\limits_{\mathbb{R}^n} f(\boldsymbol{x}) g^*(\boldsymbol{x}) \,\mathrm{d}x^n$$

[1] Jean Baptiste Joseph Fourier (1768–1830), French mathematician and physicist.
[2] Marc-Antoine Parseval (1755–1836), French mathematician.

Convolution transform into products, *i.e.*,

$$\widehat{f * g}(\boldsymbol{\xi}) = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} f(\boldsymbol{x}-\boldsymbol{y})g(\boldsymbol{y})\ \mathrm{d}y^n \mathrm{e}^{-\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{x}}\ \mathrm{d}x^n = \hat{f}(\boldsymbol{\xi})\hat{g}(\boldsymbol{\xi})$$

and inverse

$$\widehat{fg}(\boldsymbol{\xi}) = \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} \hat{f}(\boldsymbol{\xi}-\boldsymbol{\eta})\hat{g}(\boldsymbol{\eta})\ \mathrm{d}\eta^n = \frac{1}{(2\pi)^n}\hat{f} * \hat{g}(\boldsymbol{\xi})$$

B.2 Hilbert transform and Plemelj's formulas

Related to the Fourier transform is the Hilbert[3] transform. Let $f(z)$ be a holomorphic (analytic) function in the upper complex z-plane ($\operatorname{Im} z > 0$, denoted $\mathbb{C}_+$), such that[4] $|f(z)| \to 0$ as $|z| \to \infty$ in $\mathbb{C}_+$. Then, by Cauchy's theorem,[5] we get the Hilbert transform [1]

$$f(x) = \frac{1}{\mathrm{i}\pi}\lim_{R\to\infty} P\!\!\int_{-R}^{R} \frac{f(x')}{x'-x}\,\mathrm{d}x', \quad x \in \mathbb{R}$$

where $P\!\int \mathrm{d}x'$ denotes Cauchy's principle value defined by

$$P\!\!\int_{-\infty}^{\infty} \frac{f(x')}{x'-x}\,\mathrm{d}x' = \lim_{\epsilon\to 0}\left(\int_{-\infty}^{x-\epsilon} \frac{f(x')}{x'-x}\,\mathrm{d}x' + \int_{x+\epsilon}^{\infty} f(x)\ \mathrm{d}x'\right)$$

Plemelj's formulas[6] are obtained by taking the real and imaginary parts of this equation, *i.e.*, with $f(z) = f_{\mathrm{r}}(z) + \mathrm{i}f_{\mathrm{i}}(z)$, where the real and imaginary parts of the complex-function $f(z)$ are denoted $f_{\mathrm{r}}(z)$ and $f_{\mathrm{i}}(z)$, respectively.

$$\begin{cases} f_{\mathrm{r}}(x) = \dfrac{1}{\pi}P\!\displaystyle\int_{-\infty}^{\infty} \frac{f_{\mathrm{i}}(x')}{x'-x}\,\mathrm{d}x' \\ f_{\mathrm{i}}(x) = -\dfrac{1}{\pi}P\!\displaystyle\int_{-\infty}^{\infty} \frac{f_{\mathrm{r}}(x')}{x'-x}\,\mathrm{d}x' \end{cases} \quad x \in \mathbb{R} \tag{B.1}$$

We proceed and decompose $f_{\mathrm{r}}(x)$ and $f_{\mathrm{i}}(x)$ in their even and odd parts as

$$f_{\mathrm{r,i}}(x) = f^{\mathrm{e}}_{\mathrm{r,i}}(x) + f^{\mathrm{o}}_{\mathrm{r,i}}(x), \quad x \in \mathbb{R}$$

where

$$\begin{cases} f^{\mathrm{e}}_{\mathrm{r,i}}(x) = \dfrac{1}{2}\left(f_{\mathrm{r,i}}(x) + f_{\mathrm{r,i}}(-x)\right) \\ f^{\mathrm{o}}_{\mathrm{r,i}}(x) = \dfrac{1}{2}\left(f_{\mathrm{r,i}}(x) - f_{\mathrm{r,i}}(-x)\right) \end{cases}$$

[3] David Hilbert (186—1943), German mathematician.

[4] Note that the only function that can satisfy these conditions in the entire complex plane is $f(z) = 0$.

[5] Augustin-Louis Cauchy (1789–1857), French mathematician.

[6] Josip Plemelj (1873–1967), Slovene mathematician.

Then

$$\begin{cases} f^{\mathrm{e}}_{\mathrm{r,i}}(-x) = f^{\mathrm{e}}_{\mathrm{r,i}}(x) \\ f^{\mathrm{o}}_{\mathrm{r,i}}(-x) = -f^{\mathrm{o}}_{\mathrm{r,i}}(x) \end{cases} \quad x \in \mathbb{R}$$

Using the symmetries of the even and the odd parts of $f_{\mathrm{r,i}}(x)$, Plemelj's formulas then become

$$\begin{cases} f^{\mathrm{e}}_{\mathrm{r}}(x) = \dfrac{2}{\pi} P\displaystyle\int_0^\infty \frac{x' f^{\mathrm{o}}_{\mathrm{i}}(x')}{x'^2 - x^2}\,\mathrm{d}x' \\ f^{\mathrm{o}}_{\mathrm{r}}(x) = \dfrac{2x}{\pi} P\displaystyle\int_0^\infty \frac{f^{\mathrm{e}}_{\mathrm{i}}(x')}{x'^2 - x^2}\,\mathrm{d}x' \end{cases} \qquad \begin{cases} f^{\mathrm{e}}_{\mathrm{i}}(x) = -\dfrac{2}{\pi} P\displaystyle\int_0^\infty \frac{x' f^{\mathrm{o}}_{\mathrm{r}}(x')}{x'^2 - x^2}\,\mathrm{d}x' \\ f^{\mathrm{o}}_{\mathrm{i}}(x) = -\dfrac{2x}{\pi} P\displaystyle\int_0^\infty \frac{f^{\mathrm{e}}_{\mathrm{r}}(x')}{x'^2 - x^2}\,\mathrm{d}x' \end{cases} \tag{B.2}$$

since

$$\frac{1}{x' - x} + \frac{1}{x' + x} = \frac{2x'}{x'^2 - x^2}, \qquad \frac{1}{x' - x} - \frac{1}{x' + x} = \frac{2x}{x'^2 - x^2}$$

The Fourier transform of a real-valued, temporal quantity satisfies $f(z) = (f(-z^*))^*$, *i.e.*, the real part is even, and the imaginary part is odd, under the transformation $z \to -z^*$, for all $z \in \mathbb{C}_+$. Under this assumption we get for real x, $f^{\mathrm{o}}_{\mathrm{r}}(x) = f^{\mathrm{e}}_{\mathrm{i}}(x) = 0$, and

$$\begin{cases} f_{\mathrm{r}}(x) = f^{\mathrm{e}}_{\mathrm{r}}(x) = \dfrac{2}{\pi} P\displaystyle\int_0^\infty \frac{x' f^{\mathrm{o}}_{\mathrm{i}}(x')}{x'^2 - x^2}\,\mathrm{d}x' = \dfrac{2}{\pi} P\displaystyle\int_0^\infty \frac{x' f_{\mathrm{i}}(x')}{x'^2 - x^2}\,\mathrm{d}x' \\ f_{\mathrm{i}}(x) = f^{\mathrm{o}}_{\mathrm{i}}(x) = -\dfrac{2x}{\pi} P\displaystyle\int_0^\infty \frac{f^{\mathrm{e}}_{\mathrm{r}}(x')}{x'^2 - x^2}\,\mathrm{d}x' = -\dfrac{2x}{\pi} P\displaystyle\int_0^\infty \frac{f_{\mathrm{r}}(x')}{x'^2 - x^2}\,\mathrm{d}x' \end{cases} \tag{B.3}$$

The requirements of a function in order to apply the Hilbert transform is conveniently summarized in the theorem by Titchmarsh [24].[7]

Theorem B.1 (Titchmarsh)**.** *If $f(x)$ is square integrable on the real axis, i.e., $f \in L^2(\mathbb{R})$, the following three conditions are equivalent:*

1. *the inverse Fourier transform of $f(x)$ vanishes for $x < 0$, i.e.,*

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) \mathrm{e}^{-\mathrm{i}x\xi}\,\mathrm{d}\xi = 0, \quad x < 0$$

2. *$f(x)$ is, for almost all x, the limit as $y \to 0^+$ of the function $f(z) = f(x + \mathrm{i}y)$, which is holomorphic in the upper half of the z-plane and satisfies*

$$\int_{-\infty}^{\infty} |f(x + \mathrm{i}y)|^2\,\mathrm{d}x < \infty, \quad y > 0$$

3. *the real and imaginary parts of $f(x) = f_{\mathrm{r}}(x) + \mathrm{i} f_{\mathrm{i}}(x)$ satisfy Plemelj's formulas*

$$\begin{cases} f_{\mathrm{r}}(x) = \dfrac{1}{\pi} P\displaystyle\int_{-\infty}^\infty \frac{f_{\mathrm{i}}(x')}{x' - x}\,\mathrm{d}x' \\ f_{\mathrm{i}}(x) = -\dfrac{1}{\pi} P\displaystyle\int_{-\infty}^\infty \frac{f_{\mathrm{r}}(x')}{x' - x}\,\mathrm{d}x' \end{cases}$$

[7] Edward Charles Titchmarsh (1899–1963), British mathematician.

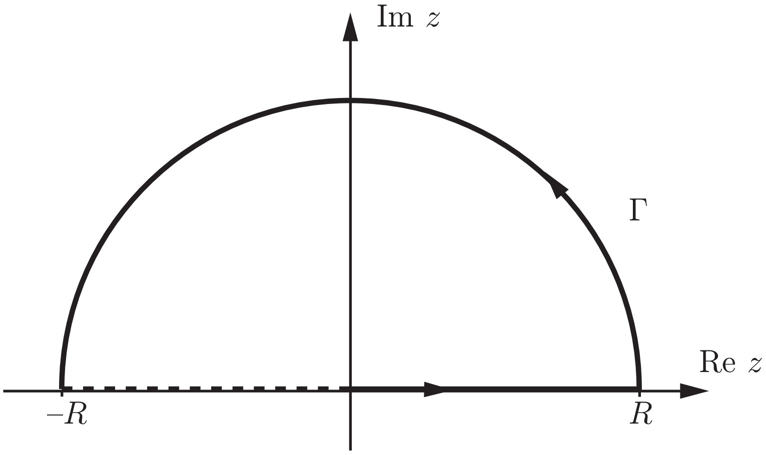


Figure B.1: The contour Γ in the complex z-plane. The dashed part of the contour denotes the negative real axis.

B.3 Meĭman's theorem

The location of roots of an analytic function $f(z)$ in the upper half plane is of great importance in many applications. In particular, functions $f(z)$ that are Fourier transforms of real-valued quantities, and thus satisfying the cross symmetry

$$f(z) = (f(-z^*))^*, \quad \operatorname{Im} z \geq 0$$

are of interest. This cross symmetry implies that the functions $f(z)$ are real-valued along the positive imaginary axis. Specifically, $f(0)$ is a real number. Meĭman's theorem implies that the imaginary axis is the only place where the functions are real-valued in the upper half plane, provided the imaginary part of $f(z)$ is positive on the real axis.

Theorem B.2 (Meĭman)**.** *Let the function $f(z)$ be analytic in the upper half plane $\mathbb{C}_+$, $\operatorname{Im} z > 0$, and continuous on the real axis, satisfying*

$$|f(z)| \to 0 \textit{ as } |z| \to \infty \textit{ in } \mathbb{C}_+ \cup \mathbb{R}$$

and the cross symmetry

$$f(z) = (f(-z^*))^* \textit{ in } \mathbb{C}_+ \cup \mathbb{R}$$

Then, if $\operatorname{Im} f(x) > 0$ for all $x > 0$ (and by the cross symmetry $\operatorname{Im} f(x) < 0$ for all $x < 0$), $f(z)$ does not take any real values at any finite point in $\mathbb{C}_+$, except on the imaginary axis, where it decreases monotonically to zero as $z \to \mathrm{i}\infty$ along the imaginary axis.

For convenience we give the proof.

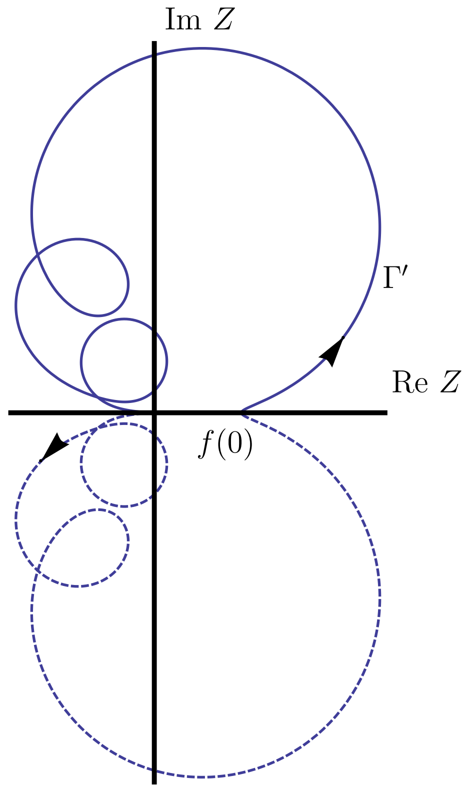


Figure B.2: The contour Γ' in the complex $Z = f(z)$-plane, under the assumption that $f(0) > 0$. If $f(0) < 0$, the contour lies in the left half of the complex Z-plane. The solid part of the contour is the map of the positive real axis and the dashed part the map of the negative real axis.

Proof: For any real constant α, the function $f(z) - \alpha$ is analytic in $\mathbb{C}_+$ and the argument principle [8] implies that[8]

$$\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} \frac{\mathrm{d}f(z)}{\mathrm{d}z} \frac{\mathrm{d}z}{f(z) - \alpha} = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma'} \frac{\mathrm{d}Z}{Z - \alpha}$$

is equal to the number of roots of the function $f(z) - \alpha$ in $\mathbb{C}_+$ inside the contour Γ, *i.e.*, the number of points at which $f(z) = \alpha$ inside Γ. The appropriate contour Γ is shown in Figure B.1. The contour Γ' is the map of the contour Γ under $f(z)$, *i.e.*, $\Gamma' = f(\Gamma)$ and shown in Figure B.2, where we explicitly assume that $f(0) > 0$.

[8]The argument principle states that if $f(z)$ is a meromorphic function inside and on some closed contour γ, with f having no zeros or poles on γ, then the following formula holds

$$N - P = \frac{1}{2\pi \mathrm{i}} \oint_{\gamma} \frac{f'(z)}{f(z)}\, \mathrm{d}z$$

where N and P denote the number of zeros and poles of $f(z)$ inside the contour γ, respectively, with each zero and pole counted as many times as its multiplicity and order, respectively. This statement of the theorem assumes that the contour γ is simple, *i.e.*, without self-intersections, and that it is oriented counter-clockwise. If the curve is self-intersecting, the number of turns around the root or pole has to included.

The semi-circle in Figure B.1 is mapped to the origin in Figure B.2. The positive real axis, solid curve in Figure B.1, is mapped to a curve in the upper part of the complex Z-plane in Figure B.2, due to the assumption that $\operatorname{Im} f(z) > 0$ on the positive real axis (solid curve). The negative real axis is mapped to the dashed part of the curve in Figure B.2.

As a consequence, the contour Γ' does not intersect the real z-axis for any finite real value except at $f(0)$. From the argument principle we then conclude that if $0 < \alpha < f(0)$, so that α lies inside the contour Γ' (or $f(0) < \alpha < 0$ if $f(0) < 0$), there is only one root to $f(z) = \alpha$ in the upper half plane, and if $0 < f(0) < \alpha$ ($\alpha < f(0) < 0$) there are no roots to $f(z) = \alpha$. By continuity, there must be one root on the imaginary axis, since $f(z)$ varies from $f(0)$ to zero as $z \in \mathrm{i}[0, \infty)$. The argument principle showed that this is the only root in the upper half plane. Moreover, the variation along the imaginary axis must be monotonic since a maximum or a minimum implies that there must be more than one root of $f(z) = \alpha$, which cannot happen by the arguments above. $\square$

A consequence of Meĭman's theorem is the function $f(z)$ does not have any roots in the upper half plane, including the real axis.

B.3.1 Zeros in the upper complex half plane

Related to Meĭman's theorem and the absence of zeros in the upper complex half plane is the following theorem:

Theorem B.3. *Let the function $f(z)$ be a non-constant analytic function in the upper half plane $\mathbb{C}_+$, $\operatorname{Im} z > 0$, and continuous in the closed domain $\overline{\mathbb{C}}_+$. Moreover, assume*

$$|f(z)| \to 0 \text{ uniformly as } |z| \to \infty \text{ in } \overline{\mathbb{C}}_+$$

and

$$\operatorname{Re} f(x) \geq 0, \quad x \in \mathbb{R}$$

Then, $\operatorname{Re} f(z) > 0$, for all $z \in \mathbb{C}_+$. In particular, there are no zeros of $f(z)$ in $\mathbb{C}_+$.

Proof: Denote the real part of f by u, *i.e.*, $u(z) = \operatorname{Re} f(z)$. The real-valued function u is a harmonic function in the upper half plane. Let z_0 be an arbitrary point in $\mathbb{C}_+$, and let $\epsilon > 0$ be an arbitrary real number. Since the function $|u(z)| \leq |f(z)| \to 0$ as $|z| \to \infty$ in $\overline{\mathbb{C}}_+$, there is an $R_0 > |z_0|$ such that the values on the semi-circle γ of radius R satisfy

$$u(z) \geq -\epsilon, \quad \text{for all } |z| = R \geq R_0$$

The minimum principle for harmonic functions implies that $u(z_0)$ is always larger (unless u is a constant) than its values on the boundary. This shows that

$$u(z_0) > -\epsilon$$

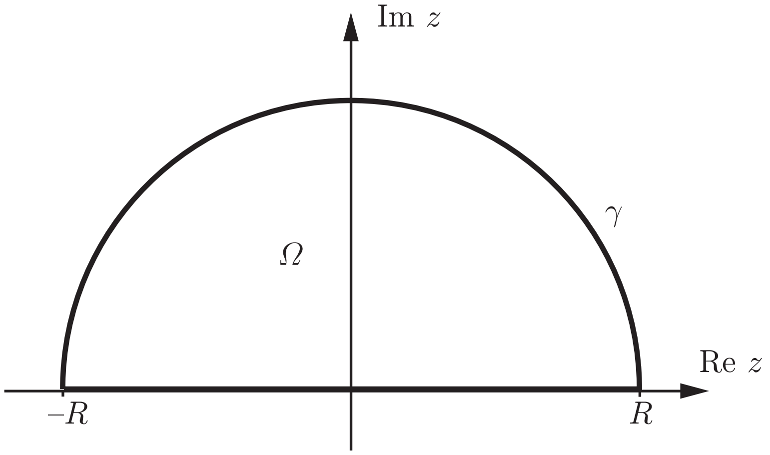


Figure B.3: The semi-circle γ in the complex upper half-plane used in the proof of Theorem B.3.

Note that the condition on the real line $u(x) \geq 0$ is always satisfied. Since ϵ and z_0 are arbitrary, we conclude, together with the assumption on the real axis, that

$$u(z) \geq 0, \quad \text{for all } z \in \overline{\mathbb{C}}_+$$

We now repeat the arguments, but using the sharper bound, $u(z) \geq 0$, for all $z \in \overline{\mathbb{C}}_+$. This holds in particular on the semi-circle γ, *i.e.*,

$$u(z) \geq 0, \quad \text{for all } z \in \gamma$$

The minimum principle then shows that

$$u(z_0) > 0$$

which then proves the theorem, since z_0 is an arbitrary point in $\mathbb{C}_+$. □

The same conclusion can be made by focusing on the imaginary part, and we have the following corollary:

Corollary B.1. *Let the function $f(z)$ be a non-constant analytic function in the upper half plane $\mathbb{C}_+$, $\operatorname{Im} z > 0$, and continuous in the closed domain $\overline{\mathbb{C}}_+$. Moreover, assume*

$$|f(z)| \to 0 \text{ uniformly as } |z| \to \infty \text{ in } \overline{\mathbb{C}}_+$$

and

$$\operatorname{Im} f(x) \geq 0, \quad x \in \mathbb{R}$$

Then, $\operatorname{Im} f(z) > 0$, for all $z \in \mathbb{C}_+$. In particular, there are no zeros of $f(z)$ in $\mathbb{C}_+$.

B.4 Functions of positive type

The definition and some properties of functions positive type are reviewed in this section. We start with the concept of a positive matrix (or more exactly, a non-negative matrix).

An $n \times n$ complex-valued matrix A is called a positive matrix provided

$$\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} z_i z_j^* \geq 0$$

for all $z_i \in \mathbb{C}$, $i = 1, 2, \ldots, n$. If we let all but one of the numbers z_i be zero, say $z_i = 1$, we see that all diagonal elements have to be real and non-negative, *i.e.*, $A_{ii} \geq 0$. If we let two numbers, say z_i and z_j, $i \neq j$, be non-zero and the rest equal to zero, we obtain

$$A_{ii}|z_i|^2 + A_{jj}|z_j|^2 + A_{ij} z_i z_j^* + A_{ji} z_j z_i^* \geq 0 \tag{B.4}$$

The imaginary part of this expression has to be zero, which implies

$$(A_{ij} - A_{ji}^*) z_i z_j^* + (A_{ji} - A_{ij}^*) z_j z_i^* = 0$$

implying[9] that $A_{ij} = A_{ji}^*$, *i.e.*, A is Hermitian symmetric. We are now in a position to rewrite (B.4) with $z_j = 1$ as[10]

$$\left| \sqrt{A_{ii}} z_i + \frac{A_{ij}}{\sqrt{A_{ii}}} \right|^2 \geq \frac{|A_{ij}|^2}{A_{ii}} - A_{jj}$$

and with $z_i = -A_{ij}/A_{ii}$, we get

$$|A_{ij}|^2 \leq A_{ii} A_{jj}$$

which also holds if one of the diagonal elements is zero.

A function of positive type is defined. We restrict ourselves to continuous functions, but the definition can be generalized to hold for distributions as well.

Definition B.1. *A complex-valued function $f(x) \in C^0(\mathbb{R})$ is of positive type if for every positive integer n and every set of real numbers x_i, $i = 1, 2, \ldots, n$, the matrix A defined as*

$$A_{ij} = f(x_i - x_j)$$

is a positive matrix.

From above, we get

$$\begin{cases} f(0) \geq 0 \\ f(-x) \geq (f(x))^* \\ |f(x)| \leq f(0) \end{cases}$$

[9] Take *e.g.*, $z_i = 1, \mathrm{i}$ and $z_j = 1$.

[10] Use $|a + b|^2 = |a|^2 + |b|^2 + 2\,\mathrm{Re}(ab^*)$. If $A_{ii} = 0$, change the role of i and j. If both $A_{ii} = A_{jj} = 0$, then $A_{ij} = 0$.

The definition of a function of positive type can equally well be formulated as the following condition on the convolution integral:

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x-y)\,(\phi(y))^* \,\mathrm{d}y \right) \phi(x)\,\mathrm{d}x \geq 0$$

for all continuous functions $\phi(x) \in C_0^0(\mathbb{R})$ with compact support, since the integral is a limit of Riemann sums

$$\sum_{i,j} f(x_i - x_j)\phi(x_i)\,(\phi(x_j))^*\,\Delta x_i \Delta x_j$$

which is always positive if f is a function of positive type.

A simple example of a function of positive type is the exponential $\exp\{\mathrm{i}xk\}$, since

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{e}^{\mathrm{i}(x_i - x_j)k} z_i z_j^* = \sum_{i=1}^{n} \left|\mathrm{e}^{\mathrm{i}x_i k} z_i\right|^2 \geq 0$$

which is generalized to the Fourier transform of a positive function $F(x) \geq 0$, since

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \left(\int_{\mathbb{R}} F(k)\mathrm{e}^{\mathrm{i}k(x-y)}\,\mathrm{d}k \right) \phi(x)\,(\phi(y))^*\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}} F(k) \left| \int_{\mathbb{R}} \phi(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x \right|^2 \mathrm{d}k \geq 0$$

The opposite is also true, *i.e.*, a function with positive Fourier transform is a function of positive type. This result is stated more precisely in a theorem by Bochner.[11] We have

Theorem B.4 (Bochner)**.** *The functions of positive type are exactly the functions of the form*

$$f(x) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}\mu(k)$$

where μ is a finite positive Borel[12] *measure, i.e., $\mu(\mathbb{R}) < \infty$.*

In the main text, we often encounters causal functions $f(t)$, *i.e.*, $f(t) = 0$ for $t < 0$. If we denote the Fourier transform of this function by

$$\hat{f}(k) = \int_0^\infty f(t)\mathrm{e}^{\mathrm{i}kt}\,\mathrm{d}t$$

If $f(t)$ is real-valued, $\hat{f}(-k) = (\hat{f}(k))^*$. Moreover, the Fourier transform of f can be uniquely be extended into the upper complex plane of k, *i.e.*, $\operatorname{Im} k \geq 0$.

We extend the function $f(t)$ to an even function of t by defining $F(t) = f(|t|)$. The Fourier transform of $F(t)$ is related to $\hat{f}(k)$ as

$$\hat{F}(k) = \int_{-\infty}^{\infty} F(t)\mathrm{e}^{\mathrm{i}kt}\,\mathrm{d}t = \int_0^\infty F(t)\mathrm{e}^{\mathrm{i}kt}\,\mathrm{d}t + \int_0^\infty F(-t)\mathrm{e}^{-\mathrm{i}kt}\,\mathrm{d}t = 2\operatorname{Re}\hat{f}(k)$$

[11] Salomon Bochner (1899–1982), American mathematician.
[12] Félix Édouard Justin Émile Borel (1871–1956), French mathematician.

Suppose the causal function $f(t)$ satisfies

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{t} f(t-t')\phi(t')\,dt'\right)\phi(t)\,dt \geq 0, \text{ for all } \phi(t)\in C^0(\mathbb{R})$$

then $F(t) = f(|t|)$ is a function of positive type, since

$$\begin{aligned}&\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} F(t-t')\phi(t')\,dt'\right)\phi(t)\,dt\\ &=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{t} f(t-t')\phi(t')\,dt'\right)\phi(t)\,dt+\int_{-\infty}^{\infty}\left(\int_{t}^{\infty} f(t'-t)\phi(t')\,dt'\right)\phi(t)\,dt\end{aligned}$$

which is identical to

$$\begin{aligned}&\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} F(t-t')\phi(t')\,dt'\right)\phi(t)\,dt\\ &=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{t} f(t-t')\phi(t')\,dt'\right)\phi(t)\,dt+\int_{-\infty}^{\infty}\left(\int_{-\infty}^{t'} f(t'-t)\phi(t)\,dt\right)\phi(t')\,dt' \geq 0\end{aligned}$$

As a consequence of Bochner's theorem, Theorem B.4, a causal function $f(t)$ that satisfies

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{t} f(t-t')\phi(t')\,dt'\right)\phi(t)\,dt \geq 0, \text{ for all } \phi(t)\in C_0^0(\mathbb{R})$$

has a Fourier transform that has a positive real part, *i.e.*,

$$\operatorname{Re}\hat{f}(k) \geq 0$$

B.5 Herglotz functions

Analytic functions play a central role in the theory of electromagnetics, and several functions encountered in this book have specific analytic properties. A special class of analytic functions are the Herglotz[13] functions, which are defined as

Definition B.2. *A function $f(z)$ is called a Herglotz function if*

1. *$f(z)$ is defined everywhere in the upper half plane, $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, and $f(z)$ is an analytic function in $\mathbb{C}_+$*

2. *$\operatorname{Im} f(z) \geq 0$ when $z \in \mathbb{C}_+$*

i.e., the function maps the upper half space into itself.

Notice that if $f(z)$ is a Herglotz function it has no zeroes or poles in $\mathbb{C}_+$, since $\operatorname{Im} f(z) > 0$ in the upper half plane. Moreover, if $f(z)$ is a Herglotz function, then $-1/f(z)$ is a Herglotz function.

The properties of a Herglotz function are summarized in the following theorem [21]:

[13] Gustav Herglotz (1881–1953), German mathematician.

Theorem B.5. *Let $f(z)$ be a Herglotz function. Then*

1. *$f(z)$ has finite normal limits $f(x+\mathrm{i}0) = \lim_{\varepsilon\downarrow 0} f(x+\mathrm{i}\varepsilon)$ for a.e. $x \in \mathbb{R}$.*
2. *If $f(z)$ has a zero normal limit on a subset of $\mathbb{R}$ having positive Lebesque[14] measure, then $f \equiv 0$.*
3. *There exists a Borel measure μ on $\mathbb{R}$ satisfying*
$$\int_{\mathbb{R}} \frac{\mathrm{d}\mu(x)}{1+x^2} < \infty$$
such that the Nevanlinna[15] representation
$$f(z) = A + Bz + \int_{-\infty}^{\infty} \frac{1+zx}{(1+x^2)(x-z)}\,\mathrm{d}\mu(x), \quad \operatorname{Im} z > 0$$
where
$$A = \operatorname{Re} f(\mathrm{i}), \quad \text{and} \quad B = \lim_{y\to\infty} \frac{f(\mathrm{i}y)}{\mathrm{i}y} \geq 0$$
holds.
4. *Let $(x_1, x_2) \subset \mathbb{R}$, then the Stieltjes[16] inversion formula for μ reads*
$$\frac{1}{2}\mu(\{x_1\}) + \frac{1}{2}\mu(\{x_2\}) + \mu((x_1,x_2)) = \frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\int_{x_1}^{x_2} \operatorname{Im}(f(x+\mathrm{i}\varepsilon))\,\mathrm{d}x$$
5. *The absolutely continuous part μ_{ac} of μ w.r.t. Lebesque measure $\mathrm{d}x$ on $\mathbb{R}$ is given by*
$$\mathrm{d}\mu_{ac}(x) = \frac{1}{\pi}\operatorname{Im}(f(x+\mathrm{i}0))\,\mathrm{d}x$$

With enough regularity on the real axis the Herglotz functions admit the integral representation
$$f(z) = A + Bz + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(1+zx)\operatorname{Im} f(x)}{(1+x^2)(x-z)}\,\mathrm{d}x, \quad \operatorname{Im} z > 0$$
where
$$A = \operatorname{Re} f(\mathrm{i}), \quad \text{and} \quad B = \lim_{y\to\infty} \frac{f(\mathrm{i}y)}{\mathrm{i}y} \geq 0$$

Example B.1
To illustrate the power of this representation, assume the function $f(z)$ is the complex square root, *i.e.*,
$$f(z) = z^{1/2}$$

[14] Henri Léon Lebesgue (1875–1941), French mathematician.
[15] Rolf Herman Nevanlinna (1895–1980), Finnish mathematician.
[16] Thomas Joannes Stieltjes (1856–1894), Dutch mathematician.

We fix the branch of the square root by assigning $f(x) = \sqrt{x}$ if $x > 0$ (limit value from above the real axis). This function is then a Herglotz function and the Nevanlinna representation is

$$f(z) = A + Bz + \int_{-\infty}^{\infty} \frac{1 + zx}{(1 + x^2)(x - z)} \, \mathrm{d}\mu(x), \quad \operatorname{Im} z > 0$$

where

$$A = \operatorname{Re} f(\mathrm{i}) = \operatorname{Re} \frac{1 + \mathrm{i}}{\sqrt{2}} = \frac{1}{\sqrt{2}}, \quad \text{and} \quad B = \lim_{y \to \infty} \frac{f(\mathrm{i}y)}{\mathrm{i}y} = \frac{1 + \mathrm{i}}{\mathrm{i}\sqrt{2}} \lim_{y \to \infty} \frac{\sqrt{y}}{y} = 0$$

and the measure

$$\frac{\mathrm{d}\mu(x)}{\mathrm{d}x} = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \operatorname{Im}(f(x + \mathrm{i}\varepsilon)) = \frac{1}{\pi} \begin{cases} 0, & x > 0 \\ \sqrt{-x}, & x < 0 \end{cases}$$

and the Nevanlinna representation becomes

$$z^{1/2} = \frac{1}{\sqrt{2}} + \frac{1}{\pi} \int_{-\infty}^{0} \frac{1 + zx}{(1 + x^2)(x - z)} \sqrt{-x} \, \mathrm{d}x, \quad \operatorname{Im} z > 0$$

A simple change of variables, $x \to -t^2$ gives

$$z^{1/2} = \frac{1}{\sqrt{2}} + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(zt^2 - 1)t^2}{(1 + t^4)(t^2 + z)} \, \mathrm{d}t, \quad \operatorname{Im} z > 0$$

That this is an identity for $\operatorname{Im} z > 0$ is readily proved by the use of residue calculus (close the contour in the upper half plane and use the residues at the poles at $t = (\pm 1 + \mathrm{i})/\sqrt{2}$ and $t = \mathrm{i}z^{1/2}$). ∎

The square root is an example of a Herglotz function. Other examples are:

$$\alpha z, \quad \beta, \quad -\frac{\alpha}{z}, \quad \ln z, \quad \mathrm{i} \ln(1 - \mathrm{i}z)$$

where $\alpha > 0$ and $\operatorname{Im} \beta \geq 0$. A more advanced example of a Herglotz function is

$$h_\Delta(z) = \frac{1}{\pi} \int_{-\Delta}^{\Delta} \frac{1}{z - t} \, \mathrm{d}t = \frac{1}{\pi} \ln \frac{z - \Delta}{z + \Delta}$$

The real and the imaginary parts of this function as well as the imaginary part in the upper complex plane are depicted in Figure B.4.

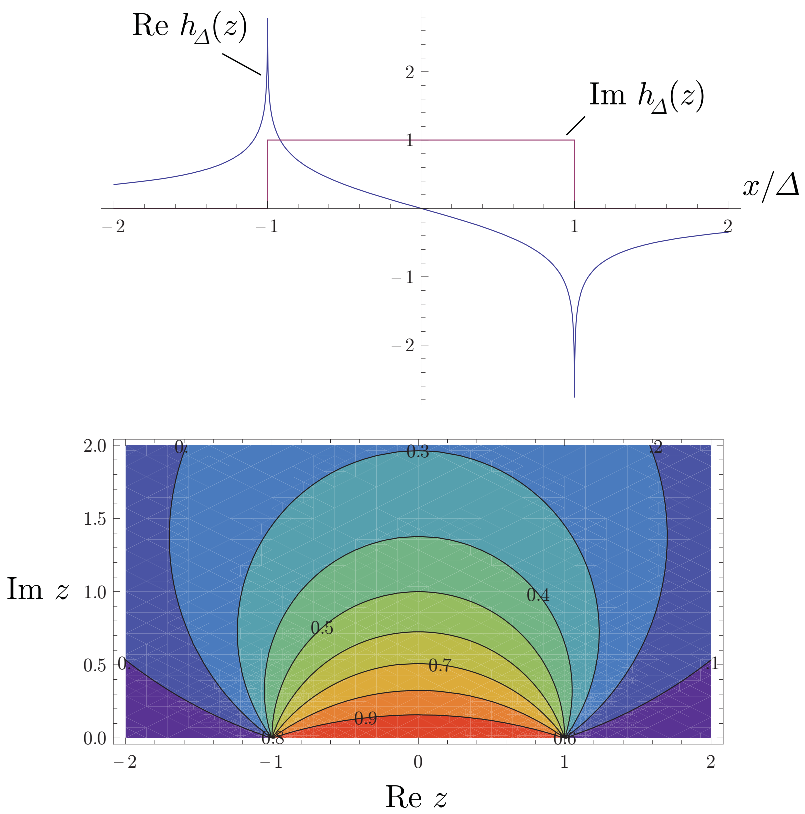


Figure B.4: The real and the imaginary parts of the function $h_\Delta(z)$ and the imaginary part in the upper complex plane.

Appendix C

Notation

Appropriate notion leads to a more easy understood, systematic, and structured text, and, in the same token, implies a tendency of making less errors and slips. Most of the notation is explained at the place in the text were they are introduced, but some more general notion that is often used is collected in this appendix.

- Vector-valued quantities (mostly in $\mathbb{R}^3$) is denoted in slanted bold face, *e.g.*, $\boldsymbol{a}$ and $\boldsymbol{b}$, and vectors of unit length have a "hat" or caret (ˆ) over a symbol, *e.g.*, $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{\rho}}$.

- The (Euclidean) scalar product between two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, is denoted in the usual standard way by a dot $(\cdot)$, *i.e.*, $\boldsymbol{a} \cdot \boldsymbol{b}$. If the vectors are complex-valued the appropriate scalar product is $\boldsymbol{a}^* \cdot \boldsymbol{b}$, where the star $*$ denotes the complex conjugate of the vector.

- We make a distinction between a vector $\boldsymbol{a}$ and its representation in components in a specific coordinate system, and denote the components as a colonn vector or with brackets around the vector, *i.e.*,

$$[\boldsymbol{a}] = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

  where

$$\boldsymbol{a} = \hat{\boldsymbol{x}} a_x + \hat{\boldsymbol{y}} a_y + \hat{\boldsymbol{z}} a_z$$

- Linear vector-valued transformations are denoted in bold roman fonts, *e.g.*, $\mathbf{A}$. A linear transformation $\mathbf{A}$ acting on a vector field $\boldsymbol{a}$ gives a new vector field $\boldsymbol{b}$ and we use the notation

$$\boldsymbol{b} = \mathbf{A} \cdot \boldsymbol{a}$$

  In a specific coordinate system the linear transformation $\mathbf{A}$ is represented by a 3×3 matrix $[\mathbf{A}]$, where we again use brackets around $\mathbf{A}$ to emphasize that we refer to its components. The components of the vector $\boldsymbol{b}$ is then

$$[\boldsymbol{b}] = [\mathbf{A}] \cdot [\boldsymbol{a}]$$

or

$$\begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{yx} & A_{yy} & A_{yz} \\ A_{zx} & A_{zy} & A_{zz} \end{pmatrix} \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

- The unity and the null dyadics in n dimensions are denoted $\mathbf{I}_n$ and $\mathbf{0}_n$ respectively, and the corresponding matrix representations are denoted $[\mathbf{I}]_n$ and $[\mathbf{0}]_n$, respectively. In three dimensions we have

$$[\mathbf{I}]_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad [\mathbf{0}]_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

or in two dimensions

$$[\mathbf{I}]_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad [\mathbf{0}]_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

- The dyadic $\mathbf{J}$ (matrix $[\mathbf{J}]$) performs a rotation of a projection on the x-y-plane followed by a rotation of $\pi/2$ in the x-y-plane,

$$[\mathbf{J}] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

- The transpose of a matrix is denoted by a superscript (^{t}) and the Hermitian of a matrix with the superscript dagger (†), *i.e.*,

$$A^t_{ij} = A_{ji}$$
$$A^\dagger_{ij} = A^*_{ji}$$

- The symbol ■ denotes the end of an example.

- The real and the imaginary part of a complex number $z = x + \mathrm{i}y$ are denoted $\operatorname{Re} z$ and $\operatorname{Im} z$, respectively, dvs.

$$\operatorname{Re} z = x$$
$$\operatorname{Im} z = y$$

A star (*) is used to denote the complex conjugate of a complex number, *i.e.*, $z^* = x - \mathrm{i}y$.

- The Heaviside's step function, $H(t)$, is defined in the usual way as

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

- The Kronecker's delta (function) symbol, δ_{ij}, is defined as

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

- The cylindrical coordinate system (ρ, ϕ, z) is defined by

$$\begin{cases} \rho = \sqrt{x^2 + y^2} \\ \phi = \begin{cases} \arccos \frac{x}{\sqrt{x^2+y^2}} & y \geq 0 \\ 2\pi - \arccos \frac{x}{\sqrt{x^2+y^2}} & y < 0 \end{cases} \\ z = z \end{cases}$$

  The domain of the coordinates are $\rho \in [0, \infty)$, $\phi \in [0, 2\pi)$, and $z \in (-\infty, \infty)$.

- The spherical coordinate system (r, θ, ϕ) is defined as

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \arccos \dfrac{z}{\sqrt{x^2 + y^2 + z^2}} \\ \phi = \begin{cases} \arccos \frac{x}{\sqrt{x^2+y^2}} & y \geq 0 \\ 2\pi - \arccos \frac{x}{\sqrt{x^2+y^2}} & y < 0 \end{cases} \end{cases}$$

  The domain of the coordinates are $r \in [0, \infty)$, $\theta \in [0, \pi]$, and $\phi \in [0, 2\pi)$.

Appendix D

Units and constants

The explicit form of the equations in electromagnetism varies depending on the system of units that we use. The SI-system is the one that is used in most literature nowadays, and this textbook is no exception. The relevant constant in the SI-system that is used in the text is collected in this appendix.

The speed of light in vacuum c_0 has the value (exact value)

$$c_0 = 299\,792\,458 \text{ m/s}$$

μ_0 and ϵ_0 denote the permeability and the permittivity of vacuum, respectively. Their exact values are

$$\begin{aligned} \mu_0 &= 4\pi \cdot 10^{-7} \text{ N/A}^2 \\ \epsilon_0 &= \frac{1}{c_0^2 \mu_0} \text{ F/m} \end{aligned}$$

Approximative values of these constants are

$$\begin{aligned} \mu_0 &\approx 12.566\,370\,614 \cdot 10^{-7} \text{ N/A}^2 \\ \epsilon_0 &\approx 8.854\,187\,817 \cdot 10^{-12} \text{ F/m} \end{aligned}$$

The wave impedance of vacuum is denoted

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = c_0\mu_0 = 299\,792\,458 \cdot 4\pi \cdot 10^{-7}\ \Omega \approx 376.730\,314\ \Omega$$

The charge of the electron, $-e$, and its mass, m, have the values

$$\begin{aligned} e &\approx 1.602\,177\,33 \cdot 10^{-19} \text{ C} \\ m &\approx 9.109\,389\,8 \cdot 10^{-31} \text{ kg} \\ e/m &\approx 1.758\,819\,63 \cdot 10^{11} \text{ C/kg} \end{aligned}$$

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Answers to problems

1.1 Apply the theorem of divergence (Gauss' theorem) to the vector field $\boldsymbol{B} = \boldsymbol{A} \times \boldsymbol{a}$, where $\boldsymbol{a}$ is an arbitrary constant vector.

1.2 Ampère's law $\nabla \times \boldsymbol{H} = \boldsymbol{0}$ implies that there exists a potential Φ such that

$$\boldsymbol{H} = -\nabla\Phi$$

Use the divergence theorem to prove the problem.

1.3 On the surface of the conductor we have $\boldsymbol{S} = -\hat{\boldsymbol{\rho}}\frac{1}{2}a\sigma E^2$ where the electric field on the surface of the conductor is related to the current by $I = \pi a^2\sigma E$. The terms in Poynting's theorem are

$$\iint_S \boldsymbol{S} \cdot \hat{\boldsymbol{\nu}} \, \mathrm{d}S = -\pi a^2 l\sigma E^2$$
$$\iiint_V \boldsymbol{E} \cdot \boldsymbol{J} \, \mathrm{d}v = \pi a^2 l\sigma E^2$$

1.4 The electric and the magnetic fields between the plates are

$$\boldsymbol{E}(\boldsymbol{r}, t) = \hat{\boldsymbol{z}} E_0 J_0\left(\frac{\omega\rho}{c_0}\right)\cos(\omega t + \alpha)$$
$$\boldsymbol{H}(\boldsymbol{r}, t) = -\hat{\boldsymbol{\phi}}\frac{E_0}{\eta_0} J_1\left(\frac{\omega\rho}{c_0}\right)\sin(\omega t + \alpha)$$

where the wave impedance of vacuum is denoted by

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

The phase α and the amplitude E_0 are arbitrary. The terms in Poynting's theorem are

$$\iint_S \boldsymbol{S} \cdot \hat{\boldsymbol{\nu}} \, \mathrm{d}S = \pi a d\frac{E_0^2}{\eta_0} J_0\left(\frac{\omega a}{c_0}\right) J_1\left(\frac{\omega a}{c_0}\right)\sin(2\omega t + 2\alpha)$$
$$\iiint_V \left[\boldsymbol{H} \cdot \frac{\partial}{\partial t}\boldsymbol{B} + \boldsymbol{E} \cdot \frac{\partial}{\partial t}\boldsymbol{D}\right] \mathrm{d}v = -\pi a d\frac{E_0^2}{\eta_0} J_0\left(\frac{\omega a}{c_0}\right) J_1\left(\frac{\omega a}{c_0}\right)\sin(2\omega t + 2\alpha)$$

1.5 $\omega_r 2\sqrt{2}c_0/a \qquad f_r = 13.5$ GHz

2.1

$$\boldsymbol{P}(t) = \begin{cases} \boldsymbol{0} & t \le 0 \\ \epsilon_0 \alpha \tau \boldsymbol{E}_0 \left[1 - \mathrm{e}^{-t/\tau}\right] & 0 < t < T \\ \epsilon_0 \alpha \tau \boldsymbol{E}_0 \mathrm{e}^{-t/\tau} \left[\mathrm{e}^{T/\tau} - 1\right] & t \ge T \end{cases}$$

2.2

$$\boldsymbol{M}(z,t) = \hat{\boldsymbol{x}} H(\tau) \frac{H_0 \alpha}{\beta} \left(1 - \mathrm{e}^{-\beta\tau}\right) \qquad \text{where } \tau = t - \frac{z}{c_0}$$

2.3

$$\boldsymbol{E}_1(t) = \begin{cases} \boldsymbol{0} & t \le 0 \\ \hat{\boldsymbol{\nu}} E \left[1 + \frac{\alpha}{\beta}\left(1 - \cos\beta t\right)\right] & 0 < t < T \\ \hat{\boldsymbol{\nu}} E \frac{\alpha}{\beta} \left[\cos\beta(t-T) - \cos\beta t\right] & t \ge T \end{cases}$$

2.4

$$\boldsymbol{P}(y,t) = \hat{\boldsymbol{x}} H(\tau) \frac{E\epsilon_0 \alpha \beta}{\omega_0^2 - \beta^2} \left(\cos\beta\tau - \cos\omega_0\tau\right) \qquad \text{where } \tau = t - \frac{y}{c_0}$$

2.5

$$\left[\boldsymbol{\Sigma}\right](t) = \omega_{\mathrm{p}}^2 \mathrm{e}^{-\nu t} \begin{pmatrix} \cos\omega_g t & \sin\omega_g t & 0 \\ -\sin\omega_g t & \cos\omega_g t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2.6

$$\begin{cases} \chi_{xx}(t) = \chi_{yy}(t) = \dfrac{\omega_{\mathrm{p}}^2}{\nu^2 + \omega_g^2} \left(\omega_g \mathrm{e}^{-\nu t} \sin\omega_g t + \nu\left(1 - \mathrm{e}^{-\nu t}\cos\omega_g t\right)\right) \\ \chi_{xy}(t) = -\chi_{yx}(t) = \dfrac{\omega_{\mathrm{p}}^2}{\nu^2 + \omega_g^2} \left(\omega_g \left(1 - \mathrm{e}^{-\nu t}\cos\omega_g t\right) - \nu \mathrm{e}^{-\nu t} \sin\omega_g t\right) \\ \chi_{xz}(t) = \chi_{yz}(t) = \chi_{zx}(t) = \chi_{zy}(t) = 0 \\ \chi_{zz}(t) = \dfrac{\omega_{\mathrm{p}}^2}{\nu} \left(1 - \mathrm{e}^{-\nu t}\right) \end{cases}$$

3.1

$$\begin{cases} \boldsymbol{A} = \hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}} \\ \boldsymbol{B} = (\hat{\boldsymbol{x}} + \xi\hat{\boldsymbol{y}}) + \mathrm{i}(-\xi\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}) \end{cases}$$

where ξ is an arbitrary real number.

3.3

$$2\alpha\beta + \alpha^2 + \omega^2 \ge 0 \text{ which implies } \quad \beta \ge -\frac{\alpha}{2}$$

3.4

$$\alpha^2 - \beta^2 + \omega^2 \ge 0 \text{ which implies } \quad |\beta| \le \alpha$$

3.5

$$[\boldsymbol{\epsilon}] = \begin{pmatrix} \epsilon & \mathrm{i}\epsilon_g & 0 \\ -\mathrm{i}\epsilon_g & \epsilon & 0 \\ 0 & 0 & \epsilon_z \end{pmatrix}$$

where

$$\begin{cases} \epsilon = 1 - \dfrac{\omega_p^2}{\omega^2 - \omega_g^2} \\ \epsilon_g = -\dfrac{\omega_p^2 \omega_g}{\omega(\omega^2 - \omega_g^2)} \\ \epsilon_z = 1 - \dfrac{\omega_p^2}{\omega^2} \end{cases}$$

3.6

$$\begin{cases} \boldsymbol{\mu} = \begin{pmatrix} \mu & \mathrm{i}\mu_g & 0 \\ -\mathrm{i}\mu_g & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ [\boldsymbol{\epsilon}] = \epsilon\, [\mathbf{I}]_3 \end{cases}$$

where

$$\begin{cases} \mu = 1 - \dfrac{\omega_0 \omega_m}{\omega^2 - \omega_0^2} \\ \mu_g = \dfrac{\omega \omega_m}{\omega^2 - \omega_0^2} \end{cases}$$

The two frequencies ω_0 (the gyromagnetic frequency) and ω_m (the saturation frequency) are explicitly given by

$$\begin{cases} \omega_0 = -g\mu_0 H_0 \\ \omega_m = -g\mu_0 M_0 \end{cases}$$

3.7

$$\epsilon(\omega) = 1 - \frac{\omega_s^2}{\omega^2} + \frac{\mathrm{i}\omega_n^2}{\omega(\nu - \mathrm{i}\omega)}$$

where the plasma frequencies for the superconducting and "normal" state, respectively, are

$$\omega_s^2 = \frac{N_s e^2}{m\epsilon_0} \qquad \omega_n^2 = \frac{N_n e^2}{m\epsilon_0}$$

3.8 a) Linearly polarized field.

b) Right circular polarized field.

c) Left circular polarized field.

d) An ellipse with half axes a and b along the $\hat{\boldsymbol{e}}_1$- and the $\hat{\boldsymbol{e}}_2$-axis, respectively. The field is right handed elliptic polarized and $\boldsymbol{E}(t=0) = \hat{\boldsymbol{e}}_1 a$.

e) An ellipse with half axes $\sqrt{2}a\cos\pi/8$ and $\sqrt{2}a\sin\pi/8$, respectively. The latter axis is tilted 45° against the positive $\hat{\boldsymbol{e}}_1$-axis. The field is right handed elliptic polarized and $\boldsymbol{E}(t=0) = a\left(\hat{\boldsymbol{e}}_1 + \hat{\boldsymbol{e}}_2/\sqrt{2}\right)$.

3.9 a)

$$\boldsymbol{E}_0 = \hat{\boldsymbol{e}}_1 E_1 + \hat{\boldsymbol{e}}_2 E_2 \quad E_1, E_2 \text{ complex number}$$

van be written as

$$\boldsymbol{E}_0 = \underbrace{\frac{E_1 - \mathrm{i}E_2}{2}(\hat{\boldsymbol{e}}_1 + \mathrm{i}\hat{\boldsymbol{e}}_2)}_{\text{RCP}} + \underbrace{\frac{E_1 + \mathrm{i}E_2}{2}(\hat{\boldsymbol{e}}_1 - \mathrm{i}\hat{\boldsymbol{e}}_2)}_{\text{LCP}}$$

b)

$$|a| = |b|$$

i.e.,

$$\boldsymbol{E}_0 = a(\boldsymbol{E}_+ + \mathrm{e}^{\mathrm{i}\alpha}\boldsymbol{E}_-)$$

3.11

$$\omega = \omega_0 + \omega_m$$

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Transformation of unit vectors

Cylindrical coordinates (ρ, ϕ, z)

$$\begin{cases} \rho = \sqrt{x^2+y^2} \\ \phi = \begin{cases} \arccos \frac{x}{\sqrt{x^2+y^2}} & y > 0 \\ 2\pi - \arccos \frac{x}{\sqrt{x^2+y^2}} & y < 0 \end{cases} \\ z = z \end{cases}$$

Spherical coordinates (r, θ, ϕ)

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \arccos \frac{z}{\sqrt{x^2+y^2+z^2}} \\ \phi = \begin{cases} \arccos \frac{x}{\sqrt{x^2+y^2}} & y > 0 \\ 2\pi - \arccos \frac{x}{\sqrt{x^2+y^2}} & y < 0 \end{cases} \end{cases}$$

$(r, \theta, \phi) \longrightarrow (x, y, z)$

$$\begin{cases} \hat{\boldsymbol{r}} = \hat{\boldsymbol{x}} \sin\theta\cos\phi + \hat{\boldsymbol{y}} \sin\theta\sin\phi + \hat{\boldsymbol{z}}\cos\theta \\ \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{x}} \cos\theta\cos\phi + \hat{\boldsymbol{y}} \cos\theta\sin\phi - \hat{\boldsymbol{z}}\sin\theta \\ \hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{x}} \sin\phi + \hat{\boldsymbol{y}}\cos\phi \end{cases}$$

$(x, y, z) \longrightarrow (r, \theta, \phi)$

$$\begin{cases} \hat{\boldsymbol{x}} = \hat{\boldsymbol{r}} \sin\theta\cos\phi + \hat{\boldsymbol{\theta}} \cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi \\ \hat{\boldsymbol{y}} = \hat{\boldsymbol{r}} \sin\theta\sin\phi + \hat{\boldsymbol{\theta}} \cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi \\ \hat{\boldsymbol{z}} = \hat{\boldsymbol{r}} \cos\theta - \hat{\boldsymbol{\theta}}\sin\theta \end{cases}$$

$(\rho, \phi, z) \longrightarrow (x, y, z)$

$$\begin{cases} \hat{\boldsymbol{\rho}} = \hat{\boldsymbol{x}}\cos\phi + \hat{\boldsymbol{y}}\sin\phi = (\hat{\boldsymbol{x}}x + \hat{\boldsymbol{y}}y)/\sqrt{x^2+y^2} \\ \hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{x}}\sin\phi + \hat{\boldsymbol{y}}\cos\phi = (-\hat{\boldsymbol{x}}y + \hat{\boldsymbol{y}}x)/\sqrt{x^2+y^2} \\ \hat{\boldsymbol{z}} = \hat{\boldsymbol{z}} \end{cases}$$

$(x, y, z) \longrightarrow (\rho, \phi, z)$

$$\begin{cases} \hat{\boldsymbol{x}} = \hat{\boldsymbol{\rho}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi \\ \hat{\boldsymbol{y}} = \hat{\boldsymbol{\rho}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi \\ \hat{\boldsymbol{z}} = \hat{\boldsymbol{z}} \end{cases}$$

$(r, \theta, \phi) \longrightarrow (\rho, \phi, z)$

$$\begin{cases} \hat{\boldsymbol{r}} = \hat{\boldsymbol{\rho}}\sin\theta + \hat{\boldsymbol{z}}\cos\theta \\ \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\rho}}\cos\theta - \hat{\boldsymbol{z}}\sin\theta \\ \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}} \end{cases}$$

$(\rho, \phi, z) \longrightarrow (r, \theta, \phi)$

$$\begin{cases} \hat{\boldsymbol{\rho}} = \hat{\boldsymbol{r}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta \\ \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{z}} = \hat{\boldsymbol{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta \end{cases}$$

Important vector identities

(1) $(\boldsymbol{a} \times \boldsymbol{c}) \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{c}\,((\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c})$

(2) $(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c})$

(3) $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b})$

(4) $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b})$

Integration formulas

Stokes' theorem and analogous theorems

(1) $$\iint\limits_S (\nabla \times \boldsymbol{A}) \cdot \hat{\boldsymbol{\nu}}\ \mathrm{d}S = \int\limits_C \boldsymbol{A} \cdot\, d\boldsymbol{r}$$

(2) $$\iint\limits_S \hat{\boldsymbol{\nu}} \times \nabla\varphi\ \mathrm{d}S = \int\limits_C \varphi\, d\boldsymbol{r}$$

(3) $$\iint\limits_S (\hat{\boldsymbol{\nu}} \times \nabla) \times \boldsymbol{A}\ \mathrm{d}S = \int\limits_C d\boldsymbol{r} \times \boldsymbol{A}$$

Gauss' theorem and analogous theorems

(1) $$\iiint\limits_V \nabla \cdot \boldsymbol{A}\ \mathrm{d}v = \iint\limits_S \boldsymbol{A} \cdot \hat{\boldsymbol{\nu}}\ \mathrm{d}S$$

(2) $$\iiint\limits_V \nabla\varphi\ \mathrm{d}v = \iint\limits_S \varphi\hat{\boldsymbol{\nu}}\ \mathrm{d}S$$

(3) $$\iiint\limits_V \nabla \times \boldsymbol{A}\ \mathrm{d}v = \iint\limits_S \hat{\boldsymbol{\nu}} \times \boldsymbol{A}\ \mathrm{d}S$$

Green's theorems

(1) $$\iiint\limits_V (\psi\nabla^2\varphi - \varphi\nabla^2\psi)\ \mathrm{d}v = \iint\limits_S (\psi\nabla\varphi - \varphi\nabla\psi) \cdot \hat{\boldsymbol{\nu}}\ \mathrm{d}S$$

(2) $$\iiint\limits_V (\psi\nabla^2\boldsymbol{A} - \boldsymbol{A}\nabla^2\psi)\ \mathrm{d}v$$
$$= \iint\limits_S (\nabla\psi \times (\hat{\boldsymbol{\nu}} \times \boldsymbol{A}) - \nabla\psi(\hat{\boldsymbol{\nu}} \cdot \boldsymbol{A}) - \psi(\hat{\boldsymbol{\nu}} \times (\nabla \times \boldsymbol{A})) + \hat{\boldsymbol{\nu}}\psi(\nabla \cdot \boldsymbol{A}))\ \mathrm{d}S$$