Detection theory

Random signals with unknown parameters

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Outline

Outline

Introduction

Background

- In chapter 6, detection procedure took a different turn when PDFs of the hypothesis tests were considered unknown.
- The uniformly most powerfully (UMP), generalized likelihood ratio test (GLRT) as well as Bayesian tests are examples of the approaches to such problems.
- Bayesian approach suffered from a number of drawbacks such as its computational complexity as well as its dependence on a prior knowledge of unknown parameters statistics.
- UMP could be only applied to one-sided parameter test.
- GLRT employs the MLE of the unknown parameter and appears to be more attractive due to its lower complexity.
- In chapter 7, the above methods were applied to cases of detecting deterministic signals with unknown parameters.

Example

Detection of $s[n] = A\cos(2\pi f_0 n + \phi)$ where the signal form is known but some of its parameters, such as f_0, n or ϕ may be unknown. If A is, e.g., unknown one could use the following which is basically a correlator-detector.

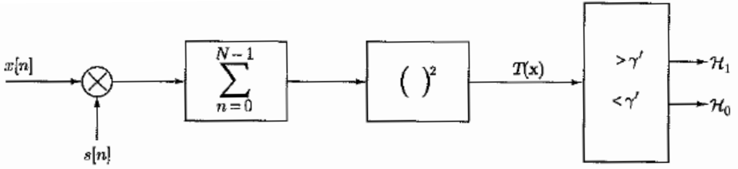


Figure 7.2. GLRT for unknown amplitude signal.

Introduction

- What if the detection problem is that of a Gaussian signal with unknown parameters?!
- Unknown parameter turns out to be the covariance for zero-mean random signals.
- The detection problem may be realized through the previously mentioned approaches.
- GLRT will be adopted as the main detection tool in this case as well.
- Both white noise and colored noise are covered. Colored noise is simply an extension of white noise. Just makes our problem more colorful, though!

Outline

Incompletely known signal covariance

$$\mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I})$$

$$\begin{aligned}\mathcal{H}_0 &: x[n] = w[n] \qquad n = 0, 1, \ldots, N-1 \\ \mathcal{H}_1 &: x[n] = s[n] + w[n] \qquad n = 0, 1, \ldots, N-1\end{aligned} \tag{8.1}$$

Maximizes

$$p(\mathbf{x}; \mathrm{Cs}, \mathcal{H}_1) = \frac{1}{2\pi \det^{\frac{1}{2}}(\mathbf{C}_s + \sigma^2\mathbf{I})} \exp\left[-\frac{1}{2}\mathbf{x}^T(\mathbf{C}_s + \sigma^2\mathbf{I})^{-1}\mathbf{x}\right]$$

Unknown signal power

Incompletely known signal covariance

Maximizes

$$\mathbf{C}_s = P_0\mathbf{C} \qquad \mathbf{C} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}$$

$$p(\mathbf{x}; P_0, \mathcal{H}_1) = \frac{1}{(2\pi)^{\frac{N}{2}} \det^{\frac{1}{2}}(P_0\mathbf{C} + \sigma^2\mathbf{I})} \exp\left[-\frac{1}{2}\mathbf{x}^T(P_0\mathbf{C} + \sigma^2\mathbf{I})^{-1}\mathbf{x}\right]$$

$$\begin{aligned}(P_0\mathbf{C} + \sigma^2\mathbf{I})^{-1} &= (P_0\mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1} + \sigma^2\mathbf{I})^{-1} \\ &= [\mathbf{V}(P_0\boldsymbol{\Lambda} + \sigma^2\mathbf{I})\mathbf{V}^{-1}]^{-1} \\ &= \mathbf{V}(P_0\boldsymbol{\Lambda} + \sigma^2\mathbf{I})^{-1}\mathbf{V}^T.\end{aligned}$$

$$\begin{aligned}\det(P_0\mathbf{C} + \sigma^2\mathbf{I}) &= \det(P_0\mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1} + \sigma^2\mathbf{I}) \\ &= \det[\mathbf{V}(P_0\boldsymbol{\Lambda} + \sigma^2\mathbf{I})\mathbf{V}^{-1}] \\ &= \det(P_0\boldsymbol{\Lambda} + \sigma^2\mathbf{I}) \\ &= \prod_{i=1}^{N}(P_0\lambda_i + \sigma^2).\end{aligned}$$

Unknown signal power

Incompletely known signal covariance

$$\ln p(\mathbf{x}; P_0, \mathcal{H}_1)$$

$$= -\frac{N}{2}\ln 2\pi - \frac{1}{2}\sum_{i=1}^{N}\ln(P_0\lambda_i + \sigma^2) - \frac{1}{2}\mathbf{x}^T\mathbf{V}(P_0\mathbf{\Lambda} + \sigma^2\mathbf{I})^{-1}\mathbf{V}^T\mathbf{x} \quad \& \quad \mathbf{V}^T\mathbf{x} = \begin{bmatrix} \mathbf{v}_1^T\mathbf{x} \\ \mathbf{v}_2^T\mathbf{x} \\ \vdots \\ \mathbf{v}_N^T\mathbf{x} \end{bmatrix}$$

$$\ln p(\mathbf{x}; P_0, \mathcal{H}_1)$$

$$= -\frac{N}{2}\ln 2\pi - \frac{1}{2}\sum_{i=1}^{N}\ln(P_0\lambda_i + \sigma^2) - \frac{1}{2}\sum_{i=1}^{N}\frac{(\mathbf{v}_i^T\mathbf{x})^2}{P_0\lambda_i + \sigma^2} \qquad (8.4)$$

Minimizes

$$J(P_0) = \sum_{i=1}^{N}\left[\ln(P_0\lambda_i + \sigma^2) + \frac{(\mathbf{v}_i^T\mathbf{x})^2}{P_0\lambda_i + \sigma^2}\right]. \qquad (8.5)$$

MLE

General solution unknown

Unknown signal power (White signal)

Incompletely known signal covariance

$$J(P_0) = \sum_{i=1}^{N} \left[\ln(P_0\lambda_i + \sigma^2) + \frac{(\mathbf{v}_i^T\mathbf{x})^2}{P_0\lambda_i + \sigma^2} \right] \xrightarrow[\lambda_i = \lambda]{\mathbf{C} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1} = \lambda\mathbf{I}}$$

$$J(P_0) = N\ln(P_0\lambda + \sigma^2) + \frac{1}{P_0\lambda + \sigma^2}\sum_{i=1}^{N}(\mathbf{v}_i^T\mathbf{x})^2.$$

But

$$\begin{aligned}
\sum_{i=1}^{N}(\mathbf{v}_i^T\mathbf{x})^2 &= \sum_{i=1}^{N}\mathbf{x}^T\mathbf{v}_i\mathbf{v}_i^T\mathbf{x} \\
&= \mathbf{x}^T\sum_{i=1}^{N}\mathbf{v}_i\mathbf{v}_i^T\mathbf{x} \\
&= \mathbf{x}^T\mathbf{V}\mathbf{V}^T\mathbf{x} \\
&= \mathbf{x}^T\mathbf{x}
\end{aligned}$$

so that

$$J(P_0) = N\ln(P_0\lambda + \sigma^2) + \frac{\mathbf{x}^T\mathbf{x}}{P_0\lambda + \sigma^2}. \tag{8.6}$$

Unknown signal power (White signal)

Incompletely known signal covariance

1. If $P_0 \geq 0$

$$\hat{P}_0^+ = \hat{P}_0 > 0 \longrightarrow \hat{P}_0^+ = \frac{\frac{1}{N}\sum_{n=0}^{N-1} x^2[n] - \sigma^2}{\lambda}$$

2. If $P_0 \leq 0 \Rightarrow P_0 = 0$

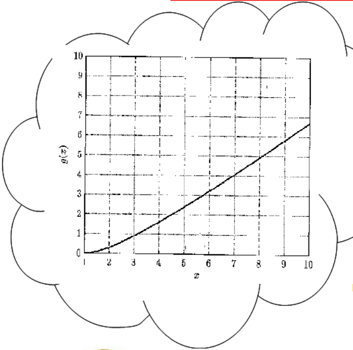
$$\hat{P}_0 = \max\left(0, \left(\frac{1}{N}\sum_{n=0}^{N-1} x^2[n] - \sigma^2\right) \Big/ \lambda\right)$$

Unknown signal power (White signal)

Incompletely known signal covariance

$$\ln L_G(\mathbf{x}) = \frac{N}{2}\boxed{\left[\left(\frac{\hat{P}_0\lambda}{\sigma^2}+1\right) - \ln\left(\frac{\hat{P}_0\lambda}{\sigma^2}+1\right) - 1\right]} > \ln\gamma$$

$$g\left(\underbrace{\frac{\hat{P}_0\lambda}{\sigma^2}+1}_{x}\right)$$



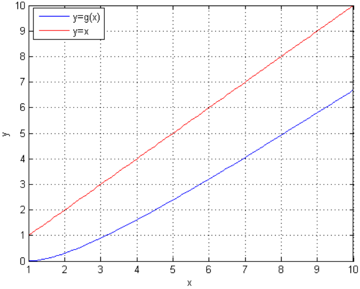
Does inverse of g(x) exist?

$$\hat{P}_0 > \frac{\sigma^2}{\lambda}\left[g^{-1}\left(\frac{2}{N}\ln\gamma\right) - 1\right] = \gamma'.$$

Unknown signal power (White signal)

Special case

What if $g(x)$ is replaced with its approximation, i.e., $y = x \Longrightarrow \widehat{P_0} > \frac{\ln(\gamma)2\sigma^2}{N\lambda}$. Further simplifications give, $(1/N)\sum_{i=0}^{N-1} x^2 \geq \frac{(2\ln(\gamma)+N)\sigma^2}{N}$.



Unknown signal power (Low-rank covariance matrix)

Incompletely known signal covariance

$$\mathbf{C}_s = P_0 \lambda_1 \mathbf{v}_1 \mathbf{v}_1^T$$

Rank one covariance matrix

e.g.

$$s[n] = A \text{ where } A \sim \mathcal{N}(0, \sigma_A^2)$$

MLE:

$$J(P_0) = \sum_{i=1}^{N} \left[\ln(P_0\lambda_i + \sigma^2) + \frac{(\mathbf{v}_i^T \mathbf{x})^2}{P_0 \lambda_i + \sigma^2} \right]$$

$$J(P_0) = \ln(P_0\lambda_1 + \sigma^2) + \frac{(\mathbf{v}_1^T \mathbf{x})^2}{P_0\lambda_1 + \sigma^2} + (N-1)\ln\sigma^2 + \frac{1}{\sigma^2}\sum_{i=2}^{N}(\mathbf{v}_i^T\mathbf{x})^2. \qquad (8.10)$$

$$\hat{P}_0 = \max\left(0, \frac{(\mathbf{v}_1^T\mathbf{x})^2 - \sigma^2}{\lambda_1}\right)$$

GLRT:

$$L_G(\mathbf{x}) = \frac{\dfrac{1}{(2\pi)^{\frac{N}{2}} \det^{\frac{1}{2}}(\hat{P}_0\mathbf{C} + \sigma^2\mathbf{I})} \exp\left[-\dfrac{1}{2}\mathbf{x}^T(\hat{P}_0\mathbf{C} + \sigma^2\mathbf{I})^{-1}\mathbf{x}\right]}{\dfrac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left(-\dfrac{1}{2\sigma^2}\mathbf{x}^T\mathbf{x}\right)} \rightarrow \bar{x}^2 > \frac{\gamma''}{N}$$

GLRT Criteria

Incompletely known signal covariance

The GLRT decision criteria

In general the detection problem for an arbitrary $C_s(\theta)$ (where θ is a vector of unknown variables) can be summarized as follows. The GLRT decides H_1 if,

MLE of $\boldsymbol{\theta}$

$$2 \ln L_G(\mathbf{x}) = \frac{1}{\sigma^2} \mathbf{x}^T \underbrace{\mathbf{C}_s(\hat{\boldsymbol{\theta}})(\mathbf{C}_s(\hat{\boldsymbol{\theta}}) + \sigma^2 \mathbf{I})^{-1}}_{\text{MMSE estimator}} \mathbf{x} - \ln \det \left(\frac{\mathbf{C}_s(\hat{\boldsymbol{\theta}})}{\sigma^2} + \mathbf{I} \right) \qquad (8.14)$$

Outline

Large data record approximations

Provided that the following criteria hold, the GLRT depends on the PSD of the signal as shown below.

1. $s(n)$ is WSS
2. a large data record of $s(n)$ is available

$$\begin{aligned} l(\mathbf{x}) &= \ln \frac{p(\mathbf{x};\mathcal{H}_1)}{p(\mathbf{x};\mathcal{H}_0)} \\ &= -\frac{N}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\ln\left(\frac{P_{ss}(f)}{\sigma^2}+1\right) - \frac{P_{ss}(f)}{P_{ss}(f)+\sigma^2}\frac{I(f)}{\sigma^2}\right] df \qquad (8.15) \end{aligned}$$

Periodogram

Wiener filter

MLE of $\boldsymbol{\theta}$ is found through minimization of

$$J(\boldsymbol{\theta}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\ln\left(\frac{P_{ss}(f;\boldsymbol{\theta})}{\sigma^2}+1\right) - \frac{P_{ss}(f;\boldsymbol{\theta})}{P_{ss}(f;\boldsymbol{\theta})+\sigma^2}\frac{I(f)}{\sigma^2}\right] df \qquad (8.16)$$

Outline

Weak signal detection

- A strong/weak signal can be differentiated through its power, i.e., P_0.
- $P_0 \geq 0$. Thus, it is a one-sided parameter.
- A LMP test can be applied
- From chapter 6, the test can be constructed as

$$T(x) = \frac{\partial \ln p(x; P_0, H_1)}{\partial P_0} \geq \gamma$$

where $P_0 = 0$

Outline

Signal processing example

Periodic random signals Periodic random signals are encountered frequently in practice. Speech signals, rotary parts in an aircraft engine are examples of periodic random signals.

Example

Probably, a good communication example is the response of a wireless channel to a passband signal of $s(t) = \cos(2\pi f_c t)$ given as, $h(t) = h_I(t)\cos(2\pi f_c t) - h_Q(t)\sin(2\pi f_c t)$ where

$$h_I(t) = \sum_{i=1}^{L(t)} \alpha_n(t)\cos(\phi_n(t)),\ h_I(t) = \sum_{i=1}^{L(t)} \alpha_n(t)\cos(\phi_n(t))$$ and

$\phi_n(t) = 2\pi f_c \tau_n(t) - \phi_{D\,n}$. **Question** What are the PDFs for α_n and ϕ_n?

Signal processing example

The signal model

- A WSS Gaussian random process
- Dc value of zero
- No aliasing, i.e., no component equal or beyond the Nyquist frequency
- The period is known and equals M

The PSD and autocorrelation of the signal:

$$P_{ss}(f) = \sum_{i=1}^{\frac{M}{2}-1} \frac{P_i}{2}\delta\left(f - \frac{i}{M}\right) \qquad 0 \le f \le \frac{1}{2}$$

$$r_{ss}[k] = \sum_{i=1}^{L} P_i \cos\left(2\pi \frac{i}{M} k\right)$$

Signal processing example

Example

The above model arises, e.g., if the signal is the sum of sinosoids with a Rayliegh distributed amplitude and uniformly distributed phase such as,

$$s[n] = \sum_{i=1}^{L} A_i \cos(2\pi f_i n + \phi_i)$$

The log-likelihood ratio for the above model when the signal length is an exact multiple of signal period, i.e., $N = KM$, is,

$$\begin{aligned} l(\mathbf{x}) &= \ln \frac{p(\mathbf{x}; \mathbf{P}, \mathcal{H}_1)}{p(\mathbf{x}; \mathcal{H}_0)} \\ &= -\sum_{i=1}^{L} \left[\ln \left(\frac{NP_i/2}{\sigma^2} + 1 \right) - \underbrace{\frac{NP_i/2}{NP_i/2 + \sigma^2}}_{\text{Wiener filter}} \frac{\overbrace{I(f_i)}^{\text{Periodogram}}}{\sigma^2} \right]. \end{aligned} \qquad (8.28)$$

Signal processing example

Detection criteria Finding the MLE of P_k we have that

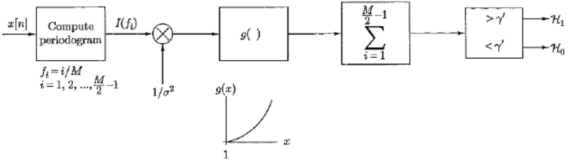
$$\hat{P}_k = \max\left(0, \frac{2}{N}(I(f_k) - \sigma^2)\right)$$

By mathematical manipulations of (8.28) it is easy to show that,

$$\sum_{i=1}^{L} g\left(\frac{I(f_i)}{\sigma^2}\right) > \gamma'$$

Furthermore, by linearizing $g(x)$, i.e., $g(x) \approx x$, we have that,

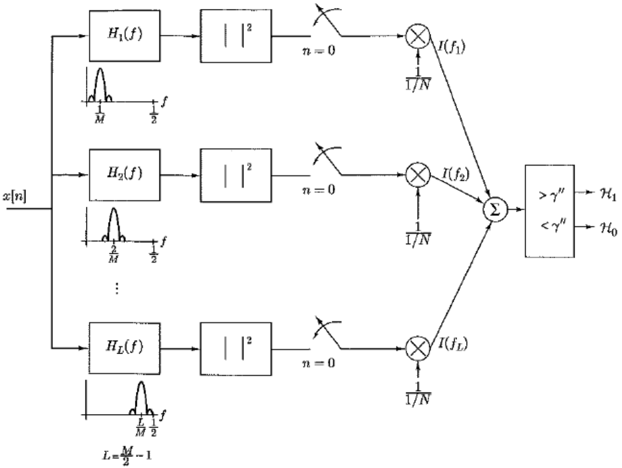
$$\sum_{i=1}^{L} I(f_i) > \sigma^2\gamma.$$



Comb-filter detector

Signal processing example

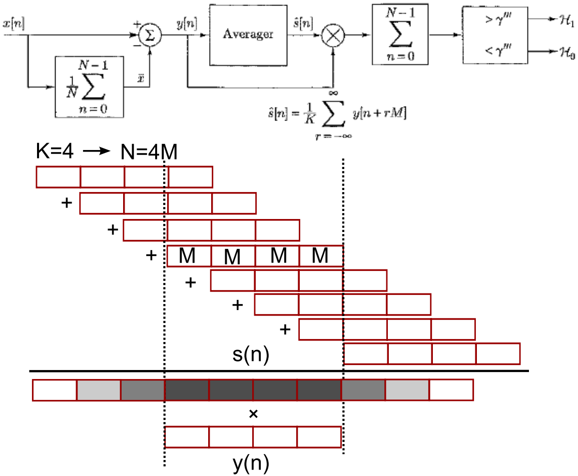
Comb filter (bank) Through mathematical reformulation the GLRT can be expresses as,



Estimation-correlation detector

Signal processing example

Another way of implementing the same system,



Average-energy detector

Signal processing example

Another realization (probably the easiest!),

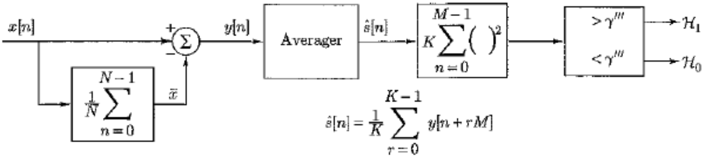
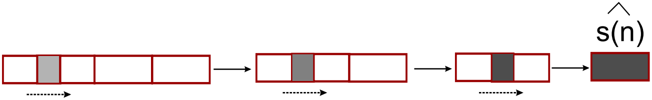


Figure 8.5. Averager-energy detector for periodic Gaussian random signal in WGN.



Outline

Summary I

1. Weak signal detection can be done involving no MLE evaluations. Besides, an LMP can be applied to such detection problems.
2. Aside from large data record approximations, which may simplify the NP detection in a number of scenarios, we learned that the GLRT for a general problem is,

MLE of $\boldsymbol{\theta}$

$$2\ln L_G(\mathbf{x}) = \frac{1}{\sigma^2}\mathbf{x}^T \underbrace{\mathbf{C}_s(\hat{\boldsymbol{\theta}})(\mathbf{C}_s(\hat{\boldsymbol{\theta}}) + \sigma^2\mathbf{I})^{-1}}_{\text{MMSE estimator}}\mathbf{x} - \ln\det\left(\frac{\mathbf{C}_s(\hat{\boldsymbol{\theta}})}{\sigma^2} + \mathbf{I}\right) \qquad (8.14)$$

The above problem is generally hard to solve due to matrix inversions as well as possibility of multiple unknown parameters. The following was introduced as cases where a closed form solution may be sought.

Summary II

- Unknown signal power but known signal covariance.

OOPS!

It can lead to nonlinear equations in signal power. It can be done in closed form for,

  - white signal, i.e., $\lambda_i = \lambda$.
  - Low-rank covariance matrix, i.e., only one nonzero λ.

- Detection of a WSS Gaussian periodic signal of unknown PSD which is the extension of the low-rank covariance matrix case.

Outline

Problems I

1. Problem (8.13).
2. Problem (8.14).
3. A commercial aircraft crashed in a mountainous area. The airplanes emergency beacon transmits a signal for the rescue team to locate the plane. Suppose that the beacon is a simple signal equal to $s(n) = P_0 \cos(2\pi f_c n + \phi_0)$, where f_c is the normalized carrier frequency and ϕ_0 is a random offset phase. The wireless channel in the vicinity of the crash site, where the signal is detectable, is assumed to be a WSSUS channel where its power delay profile (PDP) is $PDP(k) = P_h \sum_{i=1}^{L} \delta(k-i)$ in which P_h is a constant value. Assuming the receivers noise is $w(n) \sim N(0, \sigma^2)$, derive the appropriate NP test statistics for the detection of the beacon signal for the following scenarios.

Problems II

- Due to severe weather conditions no helicopter can be dispatched to the area. So, the rescue is done through the ground operation team carrying a detection device for the beacon signal. (the effect of Doppler may be ignored!)

- A helicopter is dispatched to the area. It moves at a speed of v km/h (Assume no LOS exists between the transmitter and the helicopter) which introduces a certain normalized Doppler (f_d). Compare the performance to the above scenario. Is there any difference? Please elaborate.

Problems III

Hint

Suppose the received signal is modeled as,
$r(n) = r_I(n)\cos(2\pi f_c n) - r_Q(n)\sin(2\pi f_c n)$ where
$r_I(n) = \sum_{i=1}^{L} \alpha_n(n)\cos(\phi_n(n))$ and $r_Q(n) = -\sum_{i=1}^{L} \alpha_n(n)\sin(\phi_n(n))$. Also the autocorrelation of in-phase and quadrature components may be expressed as, $A_r(k) = P_t J_0(2\pi f_d k)$ where f_d is the normalized Doppler shift and P_t is the total received power. Also assume the cross-correlation between in-phase quadrature components is zero. The observation interval (N) is long and is exactly a multiple of the main period in the signal. You may need to use the equations on page 317 while modifying the derivations given in Appendix 8A to solve this problem.

Problems IV

4. Suppose the aircraft in the above problem crashes in an area where there is only one resolvable non-line of sight component. Due to poor quality of the RF front-end of the emergency beacon device (which introduces severe nonlinearity in the mixer), there is leakage of the main modulated signal into 4 harmonics of the carrier frequency. In other words the transmitted signal is transmitted in $f \in (f_c, 2f_c, 3f_c, 4f_c, 5f_c)$. Assume the channel response related to different carrier frequencies are different and uncorrelated. Plot the probability of detection for the following scenarios.
   - Linearize any nonlinear function when deriving the NP test.
   - Leave any nonlinear function intact when deriving the NP test.
   - Find the fundamental carrier frequency using (8.35) in the book when the exact value is not known. Is it possible to find it when nonlinear functions are left intact?

Problems V

- Suppose the beacon device is of high quality. So there is no inter modulation (IM) distortion. In other words, there is no leakage into harmonics of the carrier frequency. Compare the performance of the detector to the previous case.

For the problem, assume the channel is time invariant, i.e., ignore the Doppler effect. Also, the system parameters of interest are, $f_c = 1$ GHz, $f_s = 10$ GHz, $P_0 = 1$ mW, $P_{FA} = 0.01, 0.001$ and $SNR = [-3, 25]$ dB.

OBS!

Use Monte Carlo simulations if necessary.