Detection Theory

Chapter7. Deterministic Signals with Unknown Parameters

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Outline

- Background
- Importance of signal information
- Unknown amplitude
- Unknown arrival time
- Sinusoidal detection
- Classical linear model

Background

- Statistical Decision Theory I
  - PDF for assumed hypothesis is ***completely*** known.
  - Approaches:
    - Neyman-Pearson theorem
    - Bayesian approach based on minimization of Bayes risk
  - Signal model
    - Deterministic signal
    - Random signal

Background (Cont.)

- Statistical Decision Theory II
  - PDF has ***unknown*** parameters (in this chapter ***deterministic*** signals)
  - Approaches:
    - UMP does not ***usually*** exist
    - **G**eneralized **L**ikelihood **R**atio **T**est (GLRT)
      Decide $\mathcal{H}_1$ if

$$L_G(\mathbf{x}) = \frac{p(\mathbf{x};\hat{\theta}_1,\mathcal{H}_1)}{p(\mathbf{x};\hat{\theta}_0,\mathcal{H}_0)} > \gamma$$

    - Bayesian approach
      Decide $\mathcal{H}_1$ if

$$\frac{p(\mathbf{x};\mathcal{H}_1)}{p(\mathbf{x};\mathcal{H}_0)} = \frac{\int p(\mathbf{x}|\hat{\theta}_1,\mathcal{H}_1)p(\hat{\theta}_1)d\hat{\theta}_1}{\int p(\mathbf{x}|\hat{\theta}_0,\mathcal{H}_0)p(\hat{\theta}_0)d\hat{\theta}_0} > \gamma$$

Importance of Signal Information

- Assume there is ***no knowledge*** of signal

$$\mathcal{H}_0 : x[n]=\omega[n] \qquad n=0,1,\ldots,N-1$$
$$\mathcal{H}_1 : x[n]=s[n]+\omega[n] \qquad n=0,1,\ldots,N-1$$

$\omega[n]\sim$ WGN with variance σ^2
$s[n]\sim$ deterministic and completely unknown

- A GLRT decides $\mathcal{H}_1$ if

$$\frac{p(\mathbf{x};\hat{s}[0],\ldots,\hat{s}[N-1],\mathcal{H}_1)}{p(\mathbf{x};\mathcal{H}_0)}>\gamma$$

where $\hat{s}[n]$ is MLE of $s[n]$ for $n=0,1,\ldots,N-1$

Importance of Signal Information (Cont.)

- Under $\mathcal{H}_0$ the PDF of X

$$p(\mathbf{x};\mathcal{H}_0)=\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}\exp\left(-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x[n]^2\right)$$

- Under $\mathcal{H}_1$ the PDF of X

$\hat{s}[n]=x[n]$

$$p(\mathbf{x};\hat{s}[0],\ldots,\hat{s}[N-1],\mathcal{H}_1)=\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}\exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-\hat{s}[n])^2\right]$$

- Thus,

energy detector

$$\frac{\dfrac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}}{\dfrac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}\exp\left(-\dfrac{1}{2\sigma^2}\sum_{n=0}^{N-1}x[n]^2\right)}>\gamma \;\Rightarrow\; T(X)=\sum_{n=0}^{N-1}x[n]^2=\sum_{n=0}^{N-1}x[n]\hat{s}[n]>\gamma'$$

Importance of Signal Information (Cont.)

- Detection performance

$$P_D=Q\left(Q^{-1}(P_{FA})-\sqrt{d^2}\right)$$

$$d_{ED}^2=\frac{\left(\frac{\varepsilon}{\sigma^2}\right)^2}{2N} \qquad d_{MF}^2=\frac{\varepsilon}{\sigma^2}$$

- Loss in performance

$$10\log_{10}\frac{d_{MF}^2}{d_{ED}^2}=10\log_{10}\frac{2N}{\frac{\varepsilon}{\sigma^2}}\ \text{dB}$$

Matched filter *coherently* combines data

Energy detector *incoherently* combines data

Unknown Amplitude

- Detecting a ***deterministic*** signal with ***unknown amplitude*** in WGN

$$\mathcal{H}_0 : x[n]=\omega[n] \qquad n=0,1,\ldots,N-1$$
$$\mathcal{H}_1 : x[n]=A\,s[n]+\omega[n] \qquad n=0,1,\ldots,N-1$$

A: unknown amplitude
$\omega[n]\sim$ WGN with variance σ^2
$s[n]$: deterministic and completely known

- UMP test
- GLRT
- Bayesian approach

Unknown Amplitude (Cont.)

- **UMP test:** LRT decides $\mathcal{H}_1$ if

$$\frac{p(\mathbf{x};\mathcal{H}_1)}{p(\mathbf{x};\mathcal{H}_0)} > \gamma \quad \Rightarrow \quad \frac{\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}\exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-As[n])^2\right]}{\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}\exp\left(-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x[n]^2\right)} > \gamma$$

UMP does not exist

$$A\sum_{n=0}^{N-1}x[n]s[n] > \gamma' \quad \Rightarrow \quad \begin{cases}\sum_{n=0}^{N-1}x[n]s[n] > \frac{\gamma'}{A} & A>0\\ \sum_{n=0}^{N-1}x[n]s[n] < \frac{\gamma'}{A} & A<0\end{cases}$$

Unknown Amplitude (Cont.)

- **GLRT** decides $\mathcal{H}_1$ if

$$\frac{p(X;\hat{A},\mathcal{H}_1)}{p(X;\mathcal{H}_0)} > \gamma \quad \Rightarrow \quad \frac{\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}\exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-\hat{A}s[n])^2\right]}{\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}\exp\left(-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x[n]^2\right)} > \gamma$$

$\hat{A}$ is MLE of A under $\mathcal{H}_1$

$$\hat{A} = \frac{\sum_{n=0}^{N-1}x[n]s[n]}{\sum_{n=0}^{N-1}s^2[n]}$$

$$T(X) = \left(\sum_{n=0}^{N-1}x[n]s[n]\right)^2 > \gamma' \quad \text{or} \quad \left|\sum_{n=0}^{N-1}x[n]s[n]\right| > \sqrt{\gamma'}$$

Unknown Amplitude (Cont.)

- **GLRT** detector performance

$$\left|\sum_{n=0}^{N-1}x[n]s[n]\right| > \sqrt{\gamma'}$$

$$P_{FA} = \Pr\left\{|u(x)| > \sqrt{\gamma'};\mathcal{H}_0\right\}$$

$$P_D = \Pr\left\{|u(x)| > \sqrt{\gamma'};\mathcal{H}_1\right\}$$

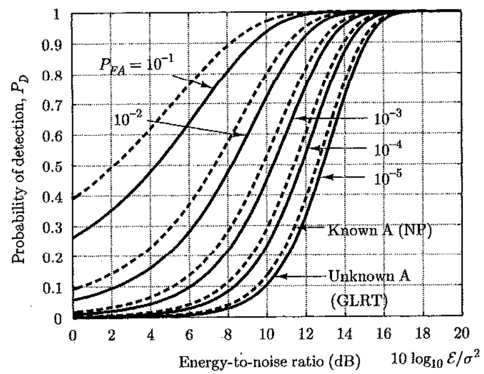
$$u(x) = \sum_{n=0}^{N-1}x[n]s[n] \sim \begin{cases} N\left(0,\sigma^2\sum_{n=0}^{N-1}s^2[n]\right) & \text{under } \mathcal{H}_0\\ N\left(A\sum_{n=0}^{N-1}s^2[n],\sigma^2\sum_{n=0}^{N-1}s^2[n]\right) & \text{under } \mathcal{H}_1\end{cases}$$

Unknown Amplitude (Cont.)

- **GLRT** detector performance

$$P_D = Q\left(Q^{-1}(P_{FA}/2)-\sqrt{d^2}\right)+Q\left(Q^{-1}(P_{FA}/2)+\sqrt{d^2}\right)$$

where $d^2 = \frac{\mathcal{E}}{\sigma^2}$



Unknown Amplitude (Cont.)

- **Bayesian** approach, under $\mathcal{H}_1$

$$\mathbf{x} = \mathbf{s}A + \mathbf{w}$$

$$\mathbf{x} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w}$$

A random variable with PDF $N\left(\mu_A, \sigma_A^2\right)$

$\mathbf{s} = \left[s[0], s[1], \ldots, s[N-1]\right]^T$

$\mathbf{w}$ WGN with variance $(0, \sigma^2\mathbf{I})$

- NP test decides $\mathcal{H}_1$ if (results in Section 5.6)

$$\mu_\theta = \mu_A$$
$$\mathbf{C}_\theta = \sigma_A^2$$
$$\mathbf{C}_\omega = \sigma^2\mathbf{I}$$

$$T'(\mathbf{x}) = \mathbf{x}^T\left(\mathbf{H}\mathbf{C}_\theta\mathbf{H}^T + \mathbf{C}_\omega\right)^{-1}\mathbf{H}\mu_\theta + \frac{1}{2}\mathbf{x}^T\mathbf{C}_\omega^{-1}\mathbf{H}\mathbf{C}_\theta\mathbf{H}^T\left(\mathbf{H}\mathbf{C}_\theta\mathbf{H}^T + \mathbf{C}_\omega\right)^{-1}\mathbf{x} > \gamma'$$

Unknown Amplitude (Cont.)

- NP test statistic

$$T'(\mathbf{x}) = \frac{\mu_A}{\sigma^2 + \sigma_A^2\mathbf{s}^T\mathbf{s}}\mathbf{x}^T\mathbf{s} + \frac{\sigma_A^2}{2\sigma^2(\sigma^2 + \sigma_A^2\mathbf{s}^T\mathbf{s})}(\mathbf{x}^T\mathbf{s})^2$$

correlator

Squared-correlator

  - if $\sigma_A^2 = 0$ ⇒ $T'(\mathbf{x}) = \frac{1}{\sigma^2}\mathbf{x}^T\mu_A\mathbf{s}$ (known amplitude case)
  - If $\sigma_A^2 \to \infty$ ⇒ $T'(\mathbf{x}) = \frac{1}{2\sigma^2\mathbf{s}^T\mathbf{s}}(\mathbf{x}^T\mathbf{s})^2$ (unknown amplitude)
  - Using Bayesian detector we require knowledge of μ_A and σ_A^2.

Unknown Arrival Time

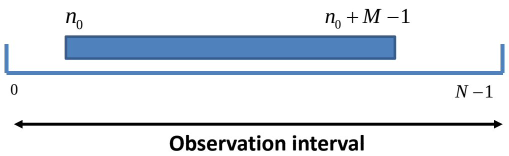
- Detecting a ***deterministic*** signal with ***unknown arrival time*** in WGN

$$\mathcal{H}_0 : x[n] = \omega[n] \qquad n = 0, 1, \ldots, N-1$$
$$\mathcal{H}_1 : x[n] = s[n-n_0] + \omega[n] \qquad n = 0, 1, \ldots, N-1$$

n_0 unknown delay

$\omega[n]$ WGN with variance σ^2

$s[n]$ deteministic and known $[0, M-1]$



Unknown Arrival Time (Cont.)

- **GLRT** decides $\mathcal{H}_1$ if

$$\frac{p(\mathbf{x}; \hat{n}_0, \mathcal{H}_1)}{p(\mathbf{x}; \mathcal{H}_0)} > \gamma$$

$\hat{n}_0$ is MLE of n_0 under $\mathcal{H}_1$

$$\max\left(\sum_{n=n_0}^{n_0+M-1} x[n]s[n-n_0]\right)$$

- Under $\mathcal{H}_1$

$$p(\mathbf{x}; n_0, \mathcal{H}_1) = \prod_{n=0}^{n_0-1}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2\sigma^2}x^2[n]\right) \cdot \prod_{n=n_0}^{M-1}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{1}{2\sigma^2}\left(x[n] - s[n-n_0]\right)^2\right] \cdot \prod_{n=n_0+M}^{N-1}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2\sigma^2}x^2[n]\right)$$

Unknown Arrival Time (Cont.)

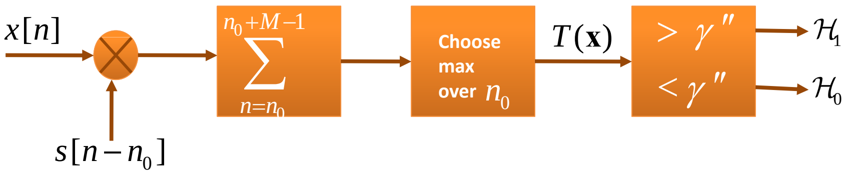
- Test statistic

$$\sum_{n=\hat{n}_0}^{\hat{n}_0+M-1} s^2[n-\hat{n}_0]=\varepsilon$$

$$-\frac{1}{2\sigma^2}\sum_{n=\hat{n}_0}^{\hat{n}_0+M-1}\left(-2x[n]s[n-\hat{n}_0]-s^2[n-\hat{n}_0]\right)>\gamma'$$

$$T(\mathbf{x})=\sum_{n=\hat{n}_0}^{\hat{n}_0+M-1} x[n]s[n-\hat{n}_0]>\gamma''$$

$$T(\mathbf{x})=\max_{n_0\in[0,N-M]}\sum_{n=n_0}^{n_0+M-1} x[n]s[n-n_0]>\gamma''$$



Unknown Arrival Time (Cont.)

- Detection performance of **GLRT** is ***difficult***!

  PDF of ***correlated*** Gaussian random variable has to be determined.

$$P_{FA}=\Pr\left\{\max_{n_0\in[0,N-M]}\sum_{n=n_0}^{n_0+M-1} x[n]s[n-n_0]>\gamma';\mathcal{H}_0\right\}$$

$$P_D=\Pr\left\{\max_{n_0\in[0,N-M]}\sum_{n=n_0}^{n_0+M-1} x[n]s[n-n_0]>\gamma';\mathcal{H}_1\right\}$$

- Delay is less than sampling interval

$$T(\mathbf{x})=\max_{n_0\in[0,N-M]}\int_{-\frac{1}{2}}^{\frac{1}{2}} X(f)\left|S(f)\exp(-j2\pi f n_0)\right|^* df$$

Sinusoidal Detection

- Detection of a sinusoid in WGN

$$\mathcal{H}_0: x[n]=\omega[n] \qquad n=0,1,\dots,N-1$$

$$\mathcal{H}_1: x[n]=\begin{cases}\omega[n] & n=0,1,\dots,n_0-1,n_0+M,\dots,N-1\\ A\cos(2\pi f_0+\varphi)+\omega[n] & n=n_0,n_0+1,\dots,n_0+M-1\end{cases}$$

- GLRT approach
  - A unknown
  - A,φ unknown
  - A,φ,f_0 unknown
  - A,φ,f_0,n_0 unknown

Sinusoidal Detection (Cont.)

- Detection of a sinusoid in WGN with ***unknown* amplitude** and **phase, GLRT** decides $\mathcal{H}_1$ if

$$\frac{p(\mathbf{x};\hat{A},\hat{\varphi},\mathcal{H}_1)}{p(\mathbf{x};\mathcal{H}_0)}>\gamma$$

$\hat{A}$ and $\hat{\varphi}$ are MLE of A and φ

$$\hat{A}=\sqrt{\hat{\alpha}_1^2+\hat{\alpha}_2^2} \qquad \hat{\alpha}_1=\frac{2}{N}\sum_{n=0}^{N-1}x[n]\cos 2\pi f_0 n$$

$$\hat{\varphi}=\arctan\left(\frac{-\hat{\alpha}_2}{\hat{\alpha}_1}\right) \qquad \hat{\alpha}_2=\frac{2}{N}\sum_{n=0}^{N-1}x[n]\sin 2\pi f_0 n$$

Sinusoidal Detection (Cont.)

- We decide $\mathcal{H}_1$ if

$$\frac{\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}\exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-\hat{A}\cos(2\pi f_0 n+\hat{\varphi}))^2\right]}{\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}\exp\left(-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2[n]\right)}>\gamma$$

$$\left|\sum_{n=0}^{N-1}x[n]\exp(-j2\pi f_0 n)\right|^2=I(f_0)>\gamma'$$

periodogram detector

incoherent or quadrature MF

Sinusoidal Detection (Cont.)

- **GLRT** detection performance

$$P_{FA}=\Pr\{I(f_0)>\gamma';\mathcal{H}_0\}$$

$$P_D=\Pr\{I(f_0)>\gamma';\mathcal{H}_1\}$$

$$I(f_0)=\xi_1^2+\xi_2^2$$

jointly Gaussian

$$\boldsymbol{\xi}=[\xi_1\ \xi_2]^T$$

$$\xi_1=\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}x[n]\cos 2\pi f_0 n$$

$$\xi_2=\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}x[n]\sin 2\pi f_0 n$$

$$\boldsymbol{\xi}\sim\begin{cases}N\left(0,\frac{\sigma^2}{2}I\right) & \text{under } \mathcal{H}_0\\ N\left(\begin{bmatrix}\frac{\sqrt{N}}{2}A\cos\varphi\\ -\frac{\sqrt{N}}{2}A\sin\varphi\end{bmatrix},\frac{\sigma^2}{2}I\right) & \text{under } \mathcal{H}_1\end{cases}$$

Sinusoidal Detection (Cont.)

- **GLRT** detection performance

$$P_{FA}=Q_{\chi_2^2}\left(\frac{2\gamma'}{\sigma^2}\right)=\exp\left(-\frac{\gamma'}{\sigma^2}\right)$$

$$P_D=Q_{\chi_2'^2(\lambda)}\left(\frac{2\gamma'}{\sigma^2}\right)$$

$$P_D=Q_{\chi_2'^2(\lambda)}\left(2\ln\frac{1}{P_{FA}}\right)$$

$$\lambda=\frac{NA^2}{2\sigma^2}$$

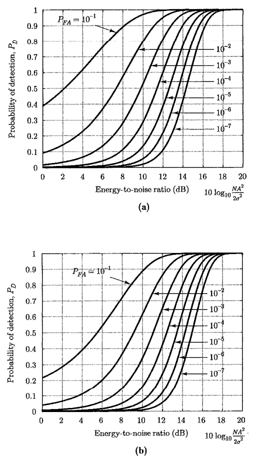


Figure 7.6. GLRT detection performance for sinusoid in WGN (a) Unknown amplitude (b) Unknown amplitude, phase.

Sinusoidal Detection (Cont.)

- Detection of a sinusoid in WGN with **unknown** A, φ, f_0, n_0
  GLRT decides $\mathcal{H}_1$ if

$$\frac{p(\mathbf{x};\hat{A},\hat{\varphi},\hat{f}_0,\hat{n}_0,\mathcal{H}_1)}{p(\mathbf{x};\mathcal{H}_0)}>\gamma$$

$\hat{A},\ \hat{\varphi},\ \hat{f}_0$ and $\hat{n}_0$ are MLE of A, φ, f_0 and n_0

$$\hat{A}=\sqrt{\hat{\alpha}_1^2+\hat{\alpha}_2^2}\qquad \hat{\alpha}_1=\frac{2}{M}\sum_{n=n_0}^{n_0+M-1}x[n]\cos 2\pi\hat{f}_0(n-n_0)$$

$$\hat{\varphi}=\arctan\left(\frac{-\hat{\alpha}_2}{\hat{\alpha}_1}\right)\qquad \hat{\alpha}_2=\frac{2}{M}\sum_{n=n_0}^{n_0+M-1}x[n]\sin 2\pi\hat{f}_0(n-n_0)$$

Sinusoidal Detection (Cont.)

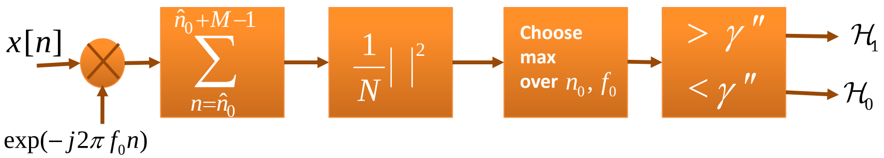
- After simplification,

$$\ln \frac{p(\mathbf{x}; \hat{A}, \hat{\varphi}, \hat{f}_0, n_0, \mathcal{H}_1)}{p(\mathbf{x}; \mathcal{H}_0)} = \frac{I_{n_0}(\hat{f}_0)}{\sigma^2}$$

$$\text{where } I_{n_0}(\hat{f}_0) = \frac{1}{M}\left|\sum_{n=n_0}^{n_0+M-1} x[n]\exp(-j2\pi \hat{f}_0 n)\right|^2$$

- We decide $\mathcal{H}_1$ if

$$\max_{n_0, f_0} \frac{I_{n_0}(f_0)}{\sigma^2} > \gamma'$$



Classical Linear Model

- Linear Bayesian model
  - Detection problem with ***unknown*** signal parameters converts to ***general Gaussian problem***
  - ***NP*** detector
- Classical linear model
  - Parameters assumed ***deterministic***
  - GLRT

GLRT Classical Linear Model

- Assume that

$$\mathbf{x} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w}$$

$\mathbf{H}$ is known $N \times p$ observation matrix of rank p
$\boldsymbol{\theta}$ is $p \times 1$ vector of parameters
$\mathbf{w}$ is $N \times 1$ noise vector with PDF $N(0, \sigma^2 I)$

- GLRT for hypothesis testing problem

$$\mathcal{H}_0 : A\theta = b$$
$$\mathcal{H}_1 : A\theta \neq b$$

$\mathbf{A}$ is $r \times p$ matrix of rank r
$\mathbf{b}$ is $r \times 1$ vector
$\mathbf{A}\boldsymbol{\theta} = \mathbf{b}$ is consistent set of linear equations

GLRT Classical Linear Model (Cont.)

- We decide $\mathcal{H}_1$ if

$$T(\mathbf{x}) = 2\ln L_G(\mathbf{x}) = \frac{(\mathbf{A}\hat{\boldsymbol{\theta}}_1 - \mathbf{b})^T \left[\mathbf{A}(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{A}^T\right]^{-1}(\mathbf{A}\hat{\boldsymbol{\theta}}_1 - \mathbf{b})}{\sigma^2} > \gamma'$$

$$\hat{\boldsymbol{\theta}}_1 = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{x}$$

$\hat{\boldsymbol{\theta}}_1$ is the MLE of $\boldsymbol{\theta}$ under $\mathcal{H}_1$

GLRT Classical Linear Model (Cont.)

- Detection performance

$$P_{FA} = Q_{\chi_r^2}(\gamma')$$

$$P_D = Q_{\chi_r^2(\lambda)}(\gamma')$$

$$\lambda = \frac{(\mathbf{A}\boldsymbol{\theta}_1 - \mathbf{b})^T \left[\mathbf{A}(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{A}^T\right]^{-1}(\mathbf{A}\boldsymbol{\theta}_1 - \mathbf{b})}{\sigma^2}$$

GLRT Classical Linear Model (Cont.)

- Unknown amplitude signal in WGN

$$\mathcal{H}_0 : x[n] = \omega[n] \qquad n = 0,1,\ldots,N-1$$
$$\mathcal{H}_1 : x[n] = As[n] + \omega[n] \quad n = 0,1,\ldots,N-1$$

⇓

$$\mathcal{H}_0 : x[n] = As[n] + \omega[n], A = 0 \quad n = 0,1,\ldots,N-1$$
$$\mathcal{H}_1 : x[n] = As[n] + \omega[n], A \neq 0 \quad n = 0,1,\ldots,N-1$$

$$\mathbf{x} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w}$$

$$\mathbf{H} = [s[0], s[1], \ldots, s[N-1]]^T$$
$$\boldsymbol{\theta} = A$$

⇒

$$\mathcal{H}_0 : \theta = 0$$
$$\mathcal{H}_1 : \theta \neq 0$$

GLRT Classical Linear Model (Cont.)

- We decide $\mathcal{H}_1$ if

$$T(\mathbf{x}) = 2\ln L_G(\mathbf{x}) = \frac{(\mathbf{A}\hat{\boldsymbol{\theta}}_1 - \mathbf{b})^T \left[\mathbf{A}(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{A}^T\right]^{-1}(\mathbf{A}\hat{\boldsymbol{\theta}}_1 - \mathbf{b})}{\sigma^2} > \gamma'$$

$$A = 1, b = 0$$

⇓

$$T(\mathbf{x}) = \frac{\hat{\boldsymbol{\theta}}_1 \mathbf{H}^T\mathbf{H}\hat{\boldsymbol{\theta}}_1}{\sigma^2} = \frac{\mathbf{H}^T\mathbf{H}\hat{\theta}_1^2}{\sigma^2}$$

$$\hat{\boldsymbol{\theta}}_1 = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{x} = \frac{\sum_{n=0}^{N-1} x[n]s[n]}{\sum_{n=0}^{N-1} s^2[n]}$$

⇒

$$T(\mathbf{x}) = \frac{\left(\sum_{n=0}^{N-1} x[n]s[n]\right)^2}{\sigma^2 \sum_{n=0}^{N-1} s^2[n]} > \gamma'$$

GLRT Classical Linear Model (Cont.)

- Detection performance

$$P_{FA} = Q_{\chi_r^2}(\gamma')$$

$$P_D = Q_{\chi_r^2(\lambda)}(\gamma')$$

$$\text{where } \lambda = \frac{\mathbf{H}^T\mathbf{H}\theta_1^2}{\sigma^2} = \frac{A^2 \sum_{n=0}^{N-1} s^2[n]}{\sigma^2} = \frac{\varepsilon}{\sigma^2}$$

⇓

$$P_D = Q\left(Q^{-1}(P_{FA}/2) - \sqrt{d^2}\right) + Q\left(Q^{-1}(P_{FA}/2) + \sqrt{d^2}\right)$$

$$\text{where } d^2 = \frac{\varepsilon}{\sigma^2}$$