Detection Theory

Chapter 5. Random Signals

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Outline

- Background
- General Gaussion detection
- Estimator-correlator
- Linear model
- Estimator-correlator for large data records
- Example: tapped delay line channel model

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So far, detection under:

- Neyman-Pearson criteria (max P_D s.t. PFA = constant) :likelihood ratio test, threshold set by PFA.

- Minimize Bayesian risk (assign costs to decisions, assign priors to the different hypotheses): likelihood ratio test, threshold set by priors + costs.

- Known deterministic signals in Gaussian noise: correlators

Now we look at detecting random signals.

We cant use the MF, since we do not know it!

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General Gaussion detection

- Some processes are better represented as random (e.g. speech)
- Rather than assume them completely random, assume signal comes from a random process with known covariance structure

Consider a binary hypothesis testing model of the following form:

$$\mathrm{H}_0 : \mathrm{x} = \mathrm{w}$$

$$\mathrm{H}_1 : \mathrm{x} = \mathrm{s} + \mathrm{w}$$

where w $\sim N(0, C_w)$ and s $\sim N(\mu_s, C_s)$ and s, w are independent.

General Gaussion detection

NP detector decides H_1 if

$$\frac{p(x\,;H_1)}{p(x\,;H_0)} > \gamma,$$

$H_0 : x = w$

$H_1 : x = s + w$

under H_0, $x \sim N(\mu_s, C_s + C_w)$, then the PDF of x is

$$\frac{1}{(2\pi)^{\frac{N}{2}} \det^{\frac{1}{2}}(C_s + C_w)} \exp[-\frac{1}{2}(x - \mu_s)^T (C_s + C_w)^{-1}(x - \mu_s)]$$

under H_0, $x \sim N(0, C_w)$, then the PDF of x is

$$\frac{1}{(2\pi)^{\frac{N}{2}} \det^{\frac{1}{2}}(C_w)} \exp[-\frac{1}{2} x^T C_w^{-1} x]$$

General Gaussion detection

Hence,

$$\frac{\frac{1}{(2\pi)^{\frac{N}{2}}\det^{\frac{1}{2}}(C_s + C_w)}\exp[-\frac{1}{2}(x - \mu_s)^T(C_s + C_w)^{-1}(x - \mu_s)]}{\frac{1}{(2\pi)^{\frac{N}{2}}\det^{\frac{1}{2}}(C_w)}\exp[-\frac{1}{2}x^T C_w^{-1} x]} > \gamma$$

Logarithm

$$\frac{\det^{\frac{1}{2}}(C_w)}{\det^{\frac{1}{2}}(C_s + C_w)}\exp[\frac{1}{2}(x^T C_w^{-1} x - (x - \mu_s)^T(C_s + C_w)^{-1}(x - \mu_s)) > \gamma$$

Scaling factor

Independe
nt of x

$$T(x) = x^T C_w^{-1} x - (x - \mu_s)^T (C_s + C_w)^{-1}(x - \mu_s) > \gamma'$$

General Gaussion detection

$$T(x) = x^T C_w^{-1} x - (x - \mu_s)^T (C_s + C_w)^{-1} (x - \mu_s) > \gamma'$$

$$T(x) = x^T C_w^{-1} x - x^T (C_s + C_w)^{-1} x + x^T (C_s + C_w)^{-1} \mu_s + \mu_s^T (C_s + C_w)^{-1} x - \mu_s^T (C_s + C_w)^{-1} \mu_s$$

Equal

$$T(x) = x^T C_w^{-1} x - x^T (C_s + C_w)^{-1} x + 2 x^T (C_s + C_w)^{-1} \mu_s - \mu_s^T (C_s + C_w)^{-1} \mu_s$$

Scaling factor

Independent of x

$$T'(x) = \frac{1}{2} x^T (C_w^{-1} - (C_s + C_w)^{-1}) x + x^T (C_s + C_w)^{-1} \mu_s$$

How?

General Gaussion detection

Matrix inversion lemma :

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(DA^{-1}B + C^{-1})^{-1}DA^{-1}$$

Asumme $A = C_w$, $B = D = I$, $C = C_s$ then,

$$(C_w + C_s)^{-1} = C_w^{-1} - C_w^{-1}(C_w^{-1} + C_s^{-1})^{-1}C_w^{-1}$$

$$C_w^{-1} - (C_s + C_w)^{-1} = C_w^{-1}(C_w^{-1} + C_s^{-1})^{-1}C_w^{-1}$$

Now let we take a look at the right side of the equation,

$$C_w^{-1}(C_w^{-1} + C_s^{-1})^{-1}C_w^{-1}$$

multiply $C_sC_s^{-1} = I$ after first C_w^{-1}, we can get

$$C_w^{-1}(C_w^{-1} + C_s^{-1})^{-1}C_w^{-1}$$
$$= C_w^{-1}C_sC_s^{-1}(C_w^{-1} + C_s^{-1})^{-1}C_w^{-1}$$

$$\Rightarrow \quad C_w^{-1}C_s(C_w^{-1}C_s + I)^{-1}C_w^{-1}$$

General Gaussion detection

multiply $C_s C_s^{-1} = I$ after first C_w^{-1}, we can get

$$C_w^{-1}(C_w^{-1}+C_s^{-1})^{-1}C_w^{-1}$$
$$= C_w^{-1}C_s C_s^{-1}(C_w^{-1}+C_s^{-1})^{-1}C_w^{-1}$$

$$C_w^{-1}C_s(C_w^{-1}C_s+I)^{-1}C_w^{-1}$$

$$C_w^{-1}C_s(C_s+C_w)^{-1}$$

Finally, we can find out,

$$C_w^{-1} - (C_s + C_w)^{-1} = C_w^{-1}C_s(C_s + C_w)^{-1}$$

Hence,

$$T'(x) = \frac{1}{2}x^T(C_w^{-1} - (C_s + C_w)^{-1})x + x^T(C_s + C_w)^{-1}\mu_s$$

$$T'(x) = \frac{1}{2}x^T C_w^{-1}C_s(C_s + C_w)^{-1}x + x^T(C_s + C_w)^{-1}\mu_s$$

Key points of the whole chapter

General Gaussion detection

$$T'(x) = x^T (C_s + C_w)^{-1} \mu_s + \frac{1}{2} x^T C_w^{-1} C_s (C_s + C_w)^{-1} x$$

$$H_0 : x = w$$
$$H_1 : x = s + w$$

Linear form

Quadratic form

Linear form:

$\text{if } C_s = 0,\ s \sim N(\mu_s, 0)$

↓

Deteminstic signal

↓

$$T'(x) = x^T C_w^{-1} \mu_s$$

↓

Prewhitener and matched fileter

Quadratic form:

$\text{if } \mu_s = 0,\ s \sim N(0, C_s)$

↓

Random signal with zero mean

↓

$$T'(x) = \frac{1}{2} x^T C_w^{-1} C_s (C_s + C_w)^{-1} x = \frac{1}{2} x^T C_w^{-1} \hat{s}$$

$\hat{s} = C_s (C_s + C_w)^{-1} x$ is the MMSE estimator of s.

↓

Prewhitener followed by an estimator-correlator

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Estimator-correlator

An random process with zero mean and covariance matrix C_s

$$x \sim \begin{cases} N(0, \sigma^2 I) & \text{under } H_0 \\ N(0, C_s + \sigma^2 I) & \text{under } H_{1.} \end{cases}$$

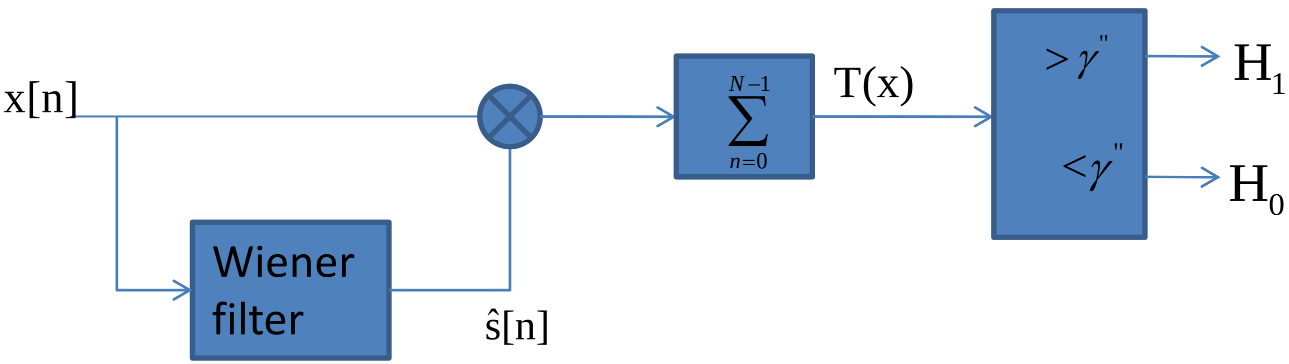
Detect random signal under WGN noise

$$T(x) = \frac{1}{2} x^T C_w^{-1} \hat{s} > \gamma'$$

$$T(x) = x^T \hat{s} > \gamma''$$

$$\hat{s} = C_s (C_s + C_w)^{-1} x$$

$$= C_s (C_s + \sigma^2 I)^{-1} x$$



$$\hat{s} = C_s (C_s + \sigma^2 I)^{-1} x$$

Estimator-correlator-eigendecomposition

Assume C_s is a symmetric and semidefinite matrix with eigendecomopsition

$$V^T C_s V = \Lambda_s$$

where V is an orthogonal matrix: $V^T = V^{-1}$

Then

$$T(x) = x^T C_s (C_s + \sigma^2 I)^{-1} x$$

replace $C_s = V\Lambda_s V^T$

$$x^T V\Lambda_s V^T (V\Lambda_s V^T + \sigma^2 I)^{-1} x$$

Group and multiply with $VV^T = I$

$$(V^T x)^T \Lambda_s V^{-1} (V\Lambda_s V^T + \sigma^2 I)^{-1} V V^T x$$

$$(V^T x)^T \Lambda_s (V^{-1} V\Lambda_s V^T V + V^{-1}\sigma^2 I V)^{-1} V^T x$$

$$(V^T x)^T \Lambda_s (\Lambda_s + \sigma^2 I)^{-1} V^T x$$

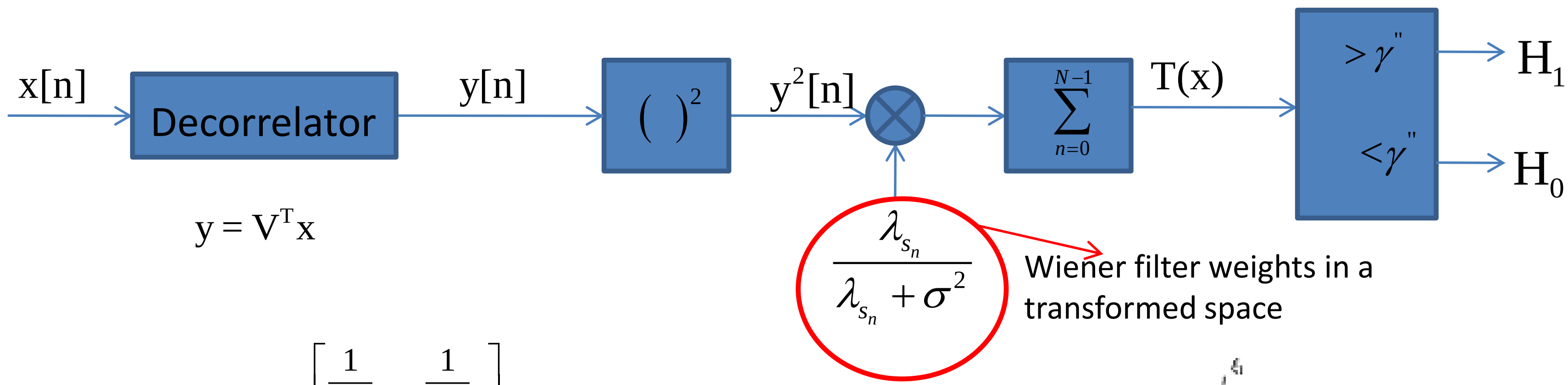
if $y = V^T x$, then

$$T(x) = y^T \Lambda_s (\Lambda_s + \sigma^2 I)^{-1} y$$

$$= \sum_{0}^{N-1} \frac{\lambda_{s_n}}{\lambda_{s_n} + \sigma^2} y^2[n]$$

where, λ_{s_n} is the eigenvalue of C_s

Estimator-correlator-eigendecomposition



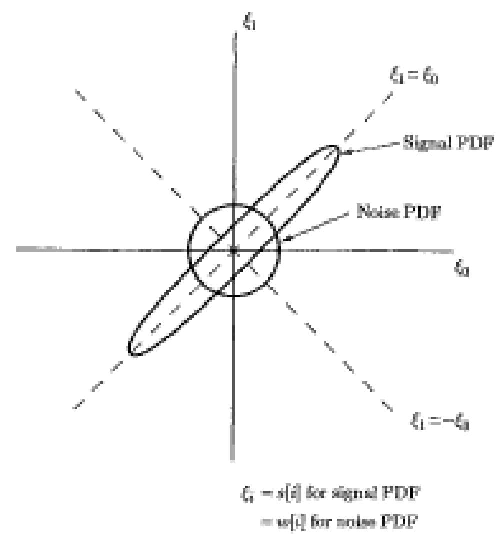
$$y = V^T x$$

$$C_s = \sigma_s^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \quad V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

if $\rho \approx 1$ and $\sigma_s^2 \gg \sigma^2$, then

$$\frac{\lambda_{s_0}}{\lambda_{s_0} + \sigma^2} = \frac{\sigma_s^2(1+\rho)}{\sigma_s^2(1+\rho) + \sigma^2} \approx 1$$

$$\frac{\lambda_{s_1}}{\lambda_{s_1} + \sigma^2} = \frac{\sigma_s^2(1-\rho)}{\sigma_s^2(1-\rho) + \sigma^2} \approx 0$$



Thus, the contribution of y[0] to T(x) is weighted more heavily, y[1] is essentially discarded

Estimator-correlator: detection performance

- Difficult to detemine analytically
- T(x) is a weighted sum of independent χ_1^2 random varables
- Numerical evaluation is necessary in general

$$P_{FA} = \int_{\gamma''}^{\infty} \int_{-\infty}^{\infty} \prod_{n=0}^{N-1} \frac{1}{\sqrt{1-2ja_n\omega}} \exp(-j\omega t) \frac{d\omega}{2\pi} dt$$

$$P_D = \int_{\gamma''}^{\infty} \int_{-\infty}^{\infty} \prod_{n=0}^{N-1} \frac{1}{\sqrt{1-2j\lambda_{s_n}\omega}} \exp(-j\omega t) \frac{d\omega}{2\pi} dt$$

where,

$$a_n = \frac{\lambda_{s_n}\sigma^2}{\lambda_{s_n} + \sigma^2}$$

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Linear Model

$$x = \begin{cases} \mathrm{w} & \text{under } \mathrm{H}_0 \\ \mathrm{A}\mathbf{1} + \mathrm{w} & \text{under } \mathrm{H}_1 \end{cases}$$

A is a random vector, we have some pre-knowlegde about the possible values of A

Apply the Bayesian linear model, the detection problem is changed:

$$\mathrm{H}_0 : \mathrm{x} = \mathrm{w}$$
$$\mathrm{H}_1 : \mathrm{x} = \mathrm{H}\theta + \mathrm{w}$$

where H is a known observation matrix, θ is a random vector with $\theta \sim N(0, \mathrm{C}_\theta)$

then $\mathrm{s} = \mathrm{H}\theta \sim N(0, \mathrm{HC}_\theta\mathrm{H}^\mathrm{T})$

$$\mathrm{T(x)} = \mathrm{x}^\mathrm{T}\mathrm{C}_s(\mathrm{C}_s + \sigma^2\mathrm{I})^{-1}\mathrm{x} = \mathrm{x}^\mathrm{T}\mathrm{HC}_\theta\mathrm{H}^\mathrm{T}(\mathrm{HC}_\theta\mathrm{H}^\mathrm{T} + \sigma^2\mathrm{I})^{-1}\mathrm{x}$$
$$= \mathrm{x}^\mathrm{T}\mathrm{H}\hat{\theta}$$

where $\hat{\theta}$ is the MMSE estimator of θ

Linear Model: example

- Rayleigh fading sinusoid

$$x[n] = A\cos(2\pi f_0 n + \phi) + w[n] \qquad n = 0, 1, \ldots N-1$$

Received signal changed both in amplitude and phase

Then

$$s[n] = A\cos(\phi)\cos(2\pi f_0 n) - A\sin(\phi)\sin(2\pi f_0 n)$$

due to the central limit theorem, we assume $\theta = \begin{bmatrix} A\cos(\phi) \\ -A\sin(\phi) \end{bmatrix} \sim N(0, \sigma_s^2 \mathrm{I})$

$$H = \begin{bmatrix} 1 & 0 \\ \cos(2\pi f_0) & \sin(2\pi f_0) \\ \vdots & \vdots \\ \cos(2\pi f_0 (N-1)) & \sin(2\pi f_0 (N-1)) \end{bmatrix}$$

So the NP detector as

$$\mathrm{T(x)} = \mathrm{x}^{\mathrm{T}} \mathrm{H} \mathrm{C}_\theta \mathrm{H}^{\mathrm{T}} (\mathrm{H} \mathrm{C}_\theta \mathrm{H}^{\mathrm{T}} + \sigma^2 \mathrm{I})^{-1} \mathrm{x} = \sigma_s^2 \mathrm{x}^{\mathrm{T}} \mathrm{H} \mathrm{H}^{\mathrm{T}} (\sigma_s^2 \mathrm{H} \mathrm{H}^{\mathrm{T}} + \sigma^2 \mathrm{I})^{-1} \mathrm{x}$$

Linear Model: example

Finally

$$T^{'}(x) = \frac{1}{N}\left\|\begin{bmatrix}\sum_{n=0}^{N-1} x[n]\cos(2\pi f_0 n)\\ \sum_{n=0}^{N-1} x[n]\sin(2\pi f_0 n)\end{bmatrix}\right\|^2$$

or

$$T^{'}(x) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x[n]\exp(-j2\pi f_0 n)\right|^2$$

Quadrature match filter

Periodogram detector

- Pilot observations in AWGN – synchronization problem, not channel estimation

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Estimator-correlator for large data records

- When N is large, the estimator-correlator can be approximated as a detector based on PSD of s[n]
- The T(x) is changed as:

$$\mathrm{T(x)} = \mathrm{N}\int_{-1/2}^{1/2} \frac{P_{ss}(f)}{P_{ss}(f)+\sigma^2} I(f)df > \gamma'', I(f) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x[n]\exp(-j2\pi fn)\right|^2$$

then the Wiener filter frequency response is

$$H(f) = \frac{P_{ss}(f)}{P_{ss}(f)+\sigma^2},$$ and it is a real function of frequency

Hence,

$$\mathrm{T(x)} = \int_{-1/2}^{1/2} H(f)X(f)X^*(f)df = \int_{-1/2}^{1/2} X(f)\hat{S}^*(f)df$$

where $\hat{S}(f) = H(f)X(f)$ is the estimator of the Fourier transform of the signal.

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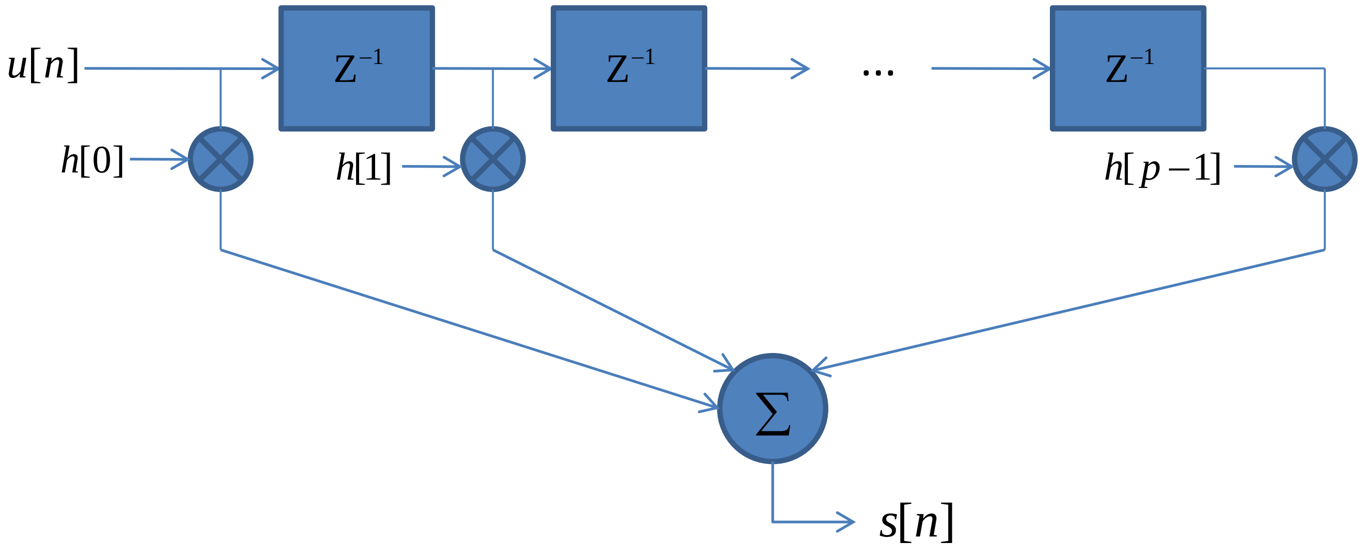
Example: Tapped delay line channel model



Assume transmit signal $u[n]$ to be a pseudorandom noise sequence.

The weights to be random variables with zero mean and $\mathrm{var}(h[k]) = \sigma_k^2$,

uncorrelated scatters, $\mathrm{cov}(h[i],h[j]) = 0$, for $i \neq j$.

$$\mathrm{h} = \begin{bmatrix} h[0] \\ h[1] \\ \vdots \\ h[p-1] \end{bmatrix} \sim N(0, \mathrm{C}_h),\ \mathrm{C}_h = diag(\sigma_0^2, \sigma_1^2, \cdots \sigma_{p-1}^2)$$

$$x[n] = \sum_{k=0}^{p-1} h[k]u[n-k] + w[n]$$
$$= \mathrm{H}\theta + \mathrm{w}$$

Example: Tapped delay line channel model

example: $u[n]$ only have nonzero values in interval [0, 1], e.g. $u[0],\ u[1]$

$p = 4$, only 4 delay taps

hence in the linear model,

$$\mathrm{H} = \begin{bmatrix} u[0] & 0 & 0 & 0 \\ u[1] & u[0] & 0 & 0 \\ 0 & u[1] & u[0] & 0 \\ 0 & 0 & u[1] & u[0] \\ 0 & 0 & 0 & u[1] \end{bmatrix}, \qquad \theta = \begin{bmatrix} h[0] \\ h[1] \\ h[2] \\ h[3] \end{bmatrix} \sim N(0, \mathrm{C}_h)$$

So the NP detector decides H_1 if

$$\mathrm{T}(\mathrm{x}) = \mathrm{x}^{\mathrm{T}}\mathrm{H}\mathrm{C}_h\mathrm{H}^{\mathrm{T}}(\mathrm{H}\mathrm{C}_h\mathrm{H}^{\mathrm{T}} + \sigma^2\mathrm{I})^{-1}\mathrm{x} > \gamma''$$

Example: Tapped delay line channel model

Since we assume the $u[n]$ is a PRN sequence, then the T(x) can be changed as

$$\mathrm{T(x)} = \sum_{k=0}^{p-1} \frac{\varepsilon\sigma_k^2}{\varepsilon\sigma_k^2+\sigma^2}\left(\frac{z[k]}{\sqrt{\varepsilon}}\right)^2 > \gamma''$$

where $z[k]=\sum_{n=k}^{K-1+k} x[n]u[n-k]$, K is the length of the nonzero input signal,

ε is the input signal energy.

For the random TDL channel with PRN signal inout, the optimal multipath detector is implemented as:

Correlateors

Wiener filter weights

Problems

- 5.2, 5.3, 5.6, 5.9, 5.12, 5.17, 5.24