# Detection Theory

Chapter 10: NonGaussian Noise

Reza Meraji

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## Outline

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  - Laplacian pdf
  - Generalized gaussian pdf
- **Detection: Known deterministic signals**
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  - Asymptotic detector
  - Example
- **Detection: Deterministic signals with unknown parameters**
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  - Rao test
  - Example
  - Theorem
  - Example
- **Problems**

# NonGaussian Noise Characteristics

## NonGaussian Noise Characteristics (1/4)

Example:

Gaussian:

$$p(w[n]) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2} w^2[n]\right)$$

Laplacian

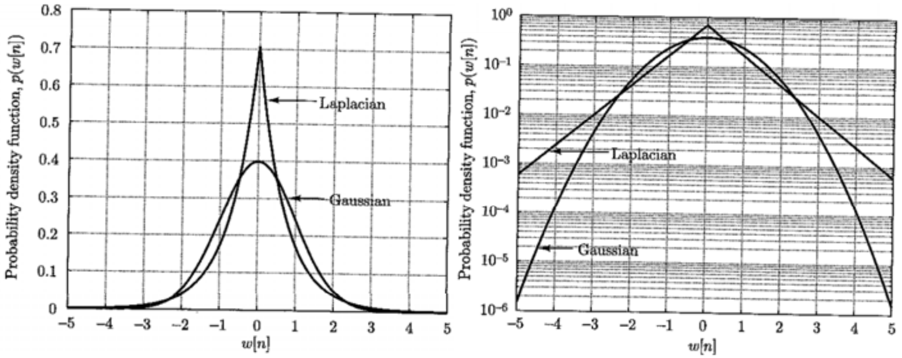
$$p(w[n]) = \frac{1}{\sqrt{2\sigma^2}} \exp\left(-\sqrt{\frac{2}{\sigma^2}}|w[n]|\right)$$

The degree of nonGaussianity of a zero mean PDF:

***Kurtosis:*** $\gamma_2 = \frac{E(w^4[n])}{E^2(w^2[n])} - 3$

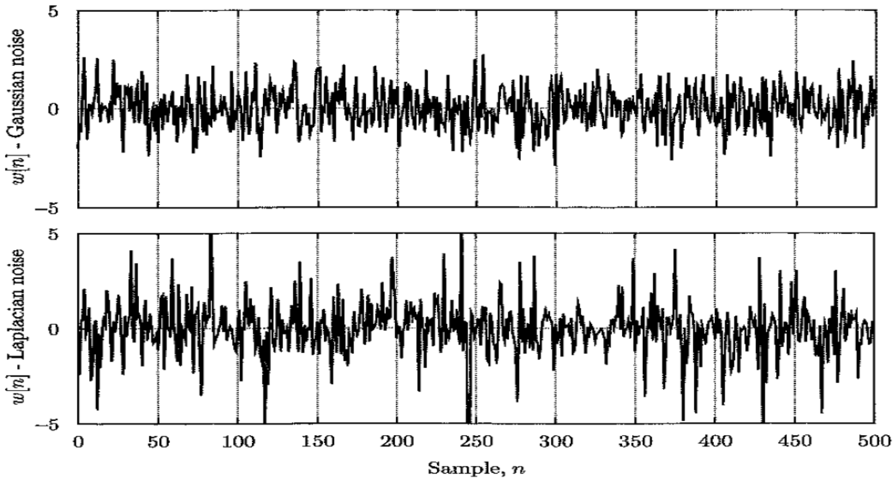
## NonGaussian Noise Characteristics (2/4)

Gaussian versus nonGaussian PDF




## NonGaussian Noise Characteristics (3/4)

Time series realizations of IID noise samples:




## NonGaussian Noise Characteristics (4/4)

A general family of PDFs that encompass the Gaussian, Laplacian, and uniform PDFs  Is the ***generalized Gaussian*** distribution:

Expressed by Gamma function

$$p(w) = \frac{c_1(\beta)}{\sqrt{\sigma^2}} \exp\left(-c_2(\beta)\left|\frac{w}{\sqrt{\sigma^2}}\right|^{\frac{2}{1+\beta}}\right)$$

# Detection:

# Known Deterministic Signals

## Detection: Known Deterministic Signals (1/8)

*Example*: **DC level in IID nonGaussian noise**

$$\mathcal{H}_0 : x[n] = w[n]$$
$$\mathcal{H}_1 : x[n] = A + w[n]$$

A is known with A>0
Noise: IID samples with known pdf

NP detector: decide H1 if

$$\ln L(\mathbf{x}) = \sum_{n=0}^{N-1} \ln \frac{p(x[n]-A)}{p(x[n])} > \ln\gamma = \gamma'$$

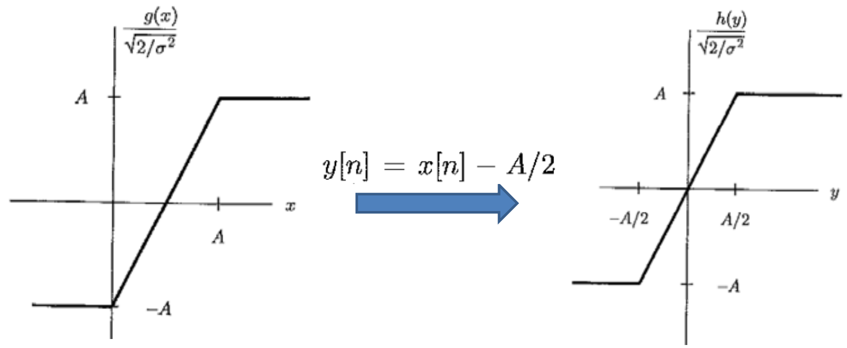
$\ln \frac{p(x[n]-A)}{p(x[n])}$ → **g(x[n])**

## Detection: Known Deterministic Signals (2/8)

**Guassian noise**: g(x) is linear → Sample mean statistics $\bar{x}$

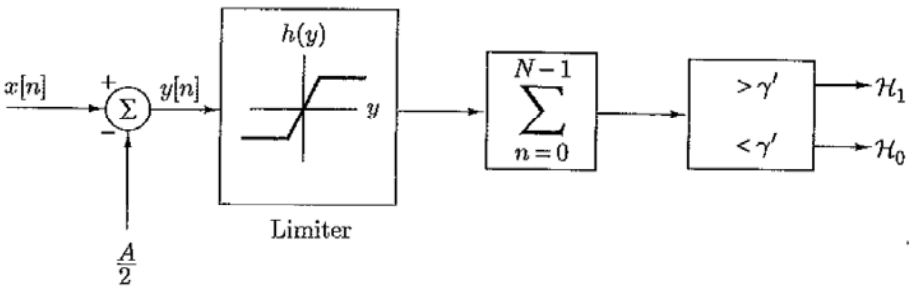
**nonGuassian noise**: g(x) is nonlinear

*Example*: **Laplacian noise** → $g(x) = \sqrt{\frac{2}{\sigma^2}}(|x| - |x-A|)$




## Detection: Known Deterministic Signals (3/8)

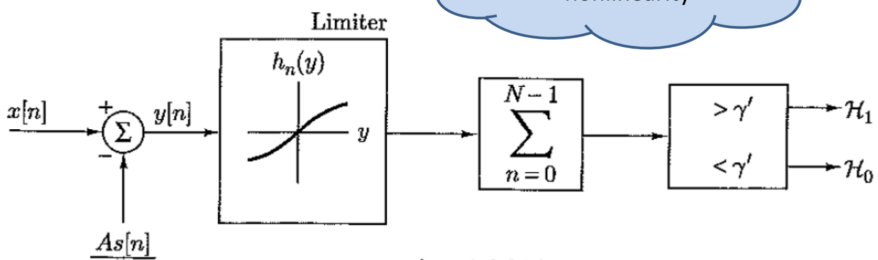
NP detector for DC level in IID **Laplacian noise**




## Detection: Known Deterministic Signals (4/8)

More generally, for the detection of a known deterministic signal s[n] in IID nonGassian noise:

Determination of PD and PFA is difficult due to nonlinearity

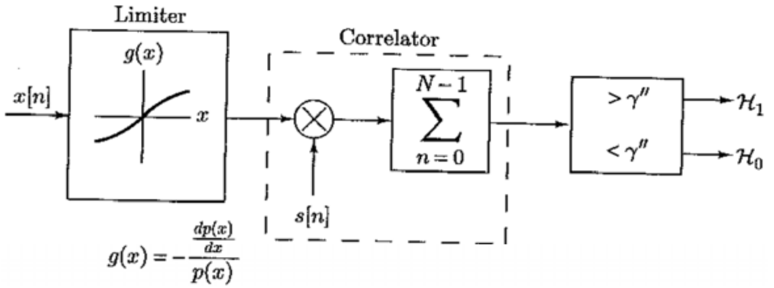



$$h_n(y) = \ln \frac{p(y - As[n]/2)}{p(y + As[n]/2)}$$

## Detection: Known Deterministic Signals (5/8)

**Suggestion -> Asymptotic detector:**
NP detector as A -> 0 or when the signal is weak

NP detector for **known weak deterministic signal** can be realized By taking the first-order Taylor series expansion of the nonlinear function about A=0




## Detection: Known Deterministic Signals (6/8)

**Example: Weak signal detection in Laplacian noise**

$$\mathcal{H}_0 : x[n] = w[n]$$
$$\mathcal{H}_1 : x[n] = As[n] + w[n]$$

In this example: Signal is a known DC level with **A>0** and **S[n] = 1**

The weak signal NP detector decides H1 if:

$$T(\mathbf{x}) = \sum_{n=0}^{N-1} g(x) > \gamma''$$

$$g(x) = -\frac{\frac{dp(x)}{dx}}{p(x)} = -\frac{d\ln p(x)}{dx} = \sqrt{\frac{2}{\sigma^2}}\frac{d|x|}{dx}$$

sgn(x)

The weak signal detector simply adds the signs of the data samples together

## Detection: Known Deterministic Signals (7/8)

**Example: continue ...**

The asymptotic detection performance is:

$$P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{d^2}\right)$$

In this example: S[n] = 1

$$d^2 = A^2 i(A) \sum_{n=0}^{N-1} s^2[n] \qquad i(A) = \int_{-\infty}^{\infty} \frac{\left(\frac{dp(w)}{dw}\right)^2}{p(w)} dw$$

**deflection coefficient** ($d^2$)

**Fisher information** ($i(A)$)

## Detection: Known Deterministic Signals (8/8)

Notes:

In comparing two detectors for large data records we can use the *ratio* of the deflection coefficients.

The effect of the noise PDF on the asymptotic detection performance is only via i(A): the Fisher information.

The PDF that yields the smallst i(A) and hence the poorest detection performance is the Gaussian PDF.

Detection:

Deterministic Signals with Unknown Parameters

## Deterministic Signals with Unknown Parameters (1/10)

The problem that will be addressed here is:

$$\mathcal{H}_0 : x[n] = w[n]$$
$$\mathcal{H}_1 : x[n] = As[n] + w[n]$$

A is unknown, s[n] is known
w[n] is IID nonGaussian noise with known pdf

**A>0:** ⇨ One-sided hypothesis test

As A → 0:
- Optimal NP detector of the previous section
- Or
- LMP test as in ch.6 which is asymptotically optimal

**-∞<A<∞ :**
- GLRT: **MLE can be difficult to obtain in nonGaussian**
- Or
- Asymptotically equivalent Rao test

## Deterministic Signals with Unknown Parameters (2/10)

**GLRT:**

**The GLRT decides H1 if:** $L_G(\mathbf{x}) = \dfrac{p(\mathbf{x}; \hat{A}, \mathcal{H}_1)}{p(\mathbf{x}; \mathcal{H}_0)} > \gamma$

$$2 \ln L_G(\mathbf{x}) = 2 \max_A \sum_{n=0}^{N-1} \ln \frac{p(x[n] - As[n])}{p(x[n])}$$

Asymptotic performance:

$$2 \ln L_G(\mathbf{x}) \overset{a}{\sim} \begin{cases} \chi_1^2 & \text{under } \mathcal{H}_0 \\ \chi_1'^2(\lambda) & \text{under } \mathcal{H}_1 \end{cases}$$

$\lambda = A^2 I(A = 0)$

I(A): Fisher information

## Deterministic Signals with Unknown Parameters (3/10)

**Rao Test:**

**Decides H1 if:**

$$T_R(\mathbf{x}) = \frac{\left( \displaystyle\sum_{n=0}^{N-1} - \frac{\frac{dp(x[n])}{dx[n]}}{p(x[n])} s[n] \right)^2}{i(A) \displaystyle\sum_{n=0}^{N-1} s^2[n]} > \gamma'$$

The asympotic detection performance is the same as that of the GLRT.

## Deterministic Signals with Unknown Parameters (4/10)

**Example: Rao test for DC level in IID Laplacian noise, A unknown**

Since s[n]=1 and $i(A) = 2/\sigma^2$

$$T_R(\mathbf{x}) = \frac{\left(\sum_{n=0}^{N-1} -\frac{\frac{dp(x[n])}{dx[n]}}{p(x[n])}\right)^2}{2N/\sigma^2} > \gamma'$$

$$-\frac{\frac{dp(x[n])}{dx[n]}}{p(x[n])} = \sqrt{\frac{2}{\sigma^2}}\mathrm{sgn}(x[n])$$

Then: $$T_R(\mathbf{x}) = N\left(\frac{1}{N}\sum_{n=0}^{N-1}\mathrm{sgn}(x[n])\right)^2$$

To within a scale factor, the Rao test averages the signs of the samples and squares the result.

## Deterministic Signals with Unknown Parameters (5/10)

***Theorem 1*: Rao test for linear model signal in IID nonGaussian noise**

*Assume the data have the form $\mathbf{x} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w}$, where $\mathbf{H}$ is a known $N \times p$ $(N > p)$ observation matrix of rank $p$, $\boldsymbol{\theta}$ is a $p \times 1$ vector of parameters, and $\mathbf{w}$ is an $N \times 1$ noise vector whose elements are IID random variables with known PDF $p(w[n])$. The Rao test for the hypothesis testing problem*

$$\mathcal{H}_0 : \boldsymbol{\theta} = \mathbf{0}$$
$$\mathcal{H}_1 : \boldsymbol{\theta} \neq \mathbf{0}$$

Is to decide H1 if:

$$T_R(\mathbf{x}) = \frac{\mathbf{y}^T\mathbf{H}(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{y}}{i(A)} > \gamma'$$

Where: $\mathbf{y} = [y[0]\, y[1] \ldots y[N-1]]^T$ with $y[n] = g(x[n])$

## Deterministic Signals with Unknown Parameters (6/10)

The asymptotic detection performance is given by:

$$P_{FA} = Q_{\chi^2_p}(\gamma')$$
$$P_D = Q_{\chi'^2_p(\lambda)}(\gamma')$$

where $$\lambda = i(A)\boldsymbol{\theta}_1^T\mathbf{H}^T\mathbf{H}\boldsymbol{\theta}_1$$

The Rao test for a signal known except for amplitude is a special case of the linear model

## Deterministic Signals with Unknown Parameters (7/10)

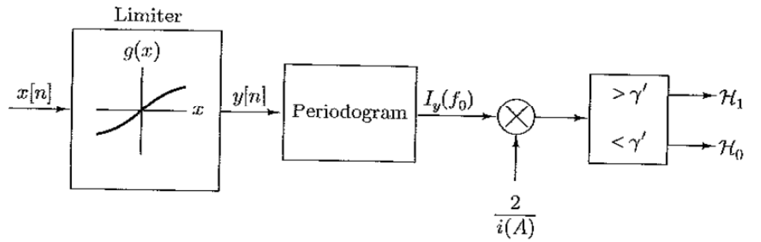
**Example: Detection of a sinusoid of unknown amplitude and phase in IID nonGaussian noise**

$$\mathcal{H}_0 : x[n] = w[n]$$
$$\mathcal{H}_1 : x[n] = A\cos(2\pi f_0 n + \phi) + w[n]$$

$w[n]$ → Generalized Gaussian

$$p(w) = \frac{c_1(\beta)}{\sqrt{\sigma^2}}\exp\left(-c_2(\beta)\left|\frac{w}{\sqrt{\sigma^2}}\right|^{\frac{2}{1+\beta}}\right)$$

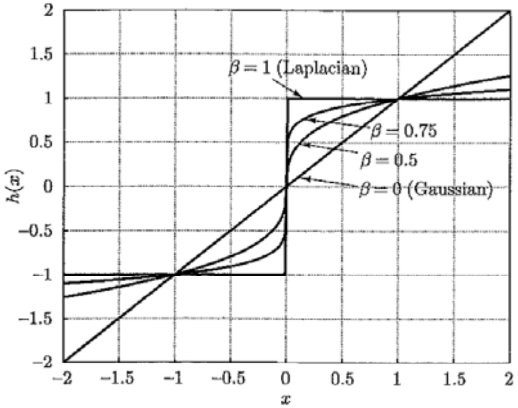
The Rao detector by applying Theorem 1 is:



## Deterministic Signals with Unknown Parameters (8/10)

**Example: continue...**

Limiter for generalized Gaussian noise:




## Deterministic Signals with Unknown Parameters (9/10)

**Example: continue...**

According to theorem 1 asymp. detection performance is:

$$P_D = Q_{\chi_2'^2(\lambda)}(\gamma')$$

PD is monotonically increasing with λ

Gaussian noise (β =0): $\lambda = NA^2/(2\sigma^2)$

Generalized Gaussian noise: $\lambda = NA^2 i(A)/2$

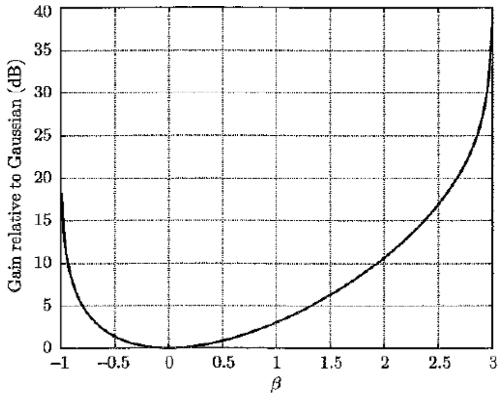
function of β

## Deterministic Signals with Unknown Parameters (10/10)

**Example: continue...**

The gain in performance in dB:

$$\left[\frac{\lambda_{nonGaussian}}{\lambda_{Gaussian}}\right]_{dB}$$




## Problems

**10.1, 10.4, 10.7, 10.8, 10.13**