Linear Algebra for Wireless Communications

Lecture: 6

Differential equations, Gerschgorin's circle theorem, symmetric/hermitian matrices, and similarity transformations

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Differential equations

Assume that we have a differential equation of the type

$$\frac{d\mathbf{u}}{dt} = \mathbf{A}\mathbf{u}$$

Given an initial value **u**(0) the solution becomes

$$\mathbf{u}(t) = e^{\mathbf{A}t}\mathbf{u}(0)$$

where the matrix exponential is defined as

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{(\mathbf{A}t)^2}{2!} + \ldots = \sum_{k=0}^{\infty} \mathbf{A}^k \frac{t^k}{k!}$$

If **A** can be diagonalized, we have

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \mathbf{A}^k \frac{t^k}{k!} = \mathbf{S}\left(\sum_{k=0}^{\infty} \mathbf{\Lambda}^k \frac{t^k}{k!}\right)\mathbf{S}^{-1} = \mathbf{S}\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_N t} \end{bmatrix}\mathbf{S}^{-1}$$

Differential equations (cont.)

... and the solution can be expressed in terms of "ordinary" exponentials:

$$\mathbf{u}(t) = e^{\mathbf{A}t}\mathbf{u}(0) = \mathbf{S}\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_N t} \end{bmatrix}\mathbf{S}^{-1}\mathbf{u}(0) \qquad \boxed{\mathbf{c} = \mathbf{S}^{-1}\mathbf{u}(0)}$$

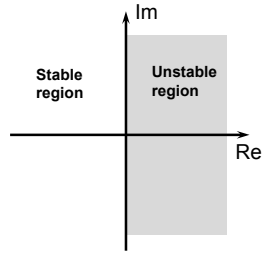
$$= \mathbf{S}\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_N t} \end{bmatrix}\mathbf{c} = c_1 e^{\lambda_1 t}\mathbf{x}_1 + \ldots + c_N e^{\lambda_N t}\mathbf{x}_N$$

Linear combination of *N* "pure solutions" depending only on eigenvalues and corresponding eigenvectors

Differential equations – stability

From previous slide:

$$\mathbf{u}(t) = \mathbf{S}e^{\mathbf{\Lambda}t}\mathbf{S}^{-1}\mathbf{u}_0 = c_1 e^{\lambda_1 t}\mathbf{x}_1 + \ldots + c_N e^{\lambda_N t}\mathbf{x}_N$$



Stability depends on the eigenvalues:

stable if all eigenvalues satisfy $\operatorname{Re}\lambda_k < 0$

neutrally stable if some $\operatorname{Re}\lambda_k = 0$ and all the other $\operatorname{Re}\lambda_m < 0$

unstable if at least one eigenvalue has $\operatorname{Re}\lambda_k > 0$

Matrix exponential – a warning

Many properties of the matrix exponential are analogus to the properties of ordinary exponentials

... but there are differences!

Example:

Ordinary exponential $e^{a+b} = e^a e^b$

Matrix exponential $e^{\mathbf{A}+\mathbf{B}} \neq e^{\mathbf{A}} e^{\mathbf{B}}$

Equality holds, however, if **A** and **B** commute, i.e., if **AB** = **BA**.

Matrix exponential – how to calculate

There are many suggested methods for calculating matrix exponentials.

Diagonalization is one way, if the matrix in the exponent can be diagonalized:

$$e^{\mathbf{A}t} = \mathbf{S}e^{\mathbf{\Lambda}t}\mathbf{S}^{-1} \qquad \left[\mathbf{A} = \mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}\right]$$

If **A** is **nil-potent**, i.e., $\mathbf{A}^m = \mathbf{0}$ for some m, we can use the definition (finite sum):

$$e^{\mathbf{A}t} = \sum_{k=0}^{m-1} \mathbf{A}^k \frac{t^k}{k!}$$

Matrix exponential – how to calculate

If **A** is a **projection matrix**, i.e., if $\mathbf{A}^2 = \mathbf{A}$ the definition gives:

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \mathbf{A}^k \frac{t^k}{k!} = \sum_{k=0}^{\infty} \mathbf{A}^k \frac{t^k}{k!} = \mathbf{I} + \sum_{k=1}^{\infty} \mathbf{A} \frac{t^k}{k!}$$

$$= \mathbf{I} + \mathbf{A}\sum_{k=1}^{\infty} \frac{t^k}{k!} = \mathbf{I} + \mathbf{A}\left(e^t - 1\right)$$

For the full treatment:
C. Moler & C van Loan. **Nineteen Dubious Ways to Compute the Exponential of a Matrix, Twenty-Five Years Later**. SIAM REVIEW (electronic publ. 2003)

Matrix exponential - determinant

The determinant of an matrix exponential is:

$$\det\left(e^{\mathbf{A}t}\right) = e^{\mathrm{tr}(\mathbf{A}t)}$$

Important consequence: A matrix exponential is always invertible, since its determinant cannot be zero!

If **A** can be diagonalized:

$$\det\left(e^{\mathbf{A}t}\right) = \det\left(\mathbf{S}e^{\mathbf{\Lambda}}\mathbf{S}^{-1}\right) = \det\left(\mathbf{S}\right)\det\left(e^{\mathbf{\Lambda}t}\right)\det\left(\mathbf{S}^{-1}\right) = \det\left(e^{\mathbf{\Lambda}t}\right) =$$

$$= \det\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_N t} \end{bmatrix} = e^{\lambda_1 t} e^{\lambda_2 t} \cdots e^{\lambda_N t} = e^{(\lambda_1 + \lambda_2 + \ldots + \lambda_N)t} = e^{\mathrm{tr}(\mathbf{A}t)}$$

EIGENVALUES ARE IMPORTANT!

We have seen that eigenvalues are important in many situations. They determine the stability of both difference and differential equations ... they can help us to find solutions to many problems formulated in terms of matrix equations ... etc.

To get some more feeling for the eigenvalues of matrices, let's take a look at Gershgorin's Cricle Theorem. [Only briefly explained in the textbook.]

Gershgorin's circle theorem

Given an *NxN* matrix **A** define the following sums of magnitudes of off-diagonal elements in each row:

$$R_k = \sum_{\substack{l=1 \\ l \neq k}}^{N} \left| a_{k,l} \right|$$

Now, each eigenvalue of **A** is in at least one of the following (Gershgorin) disks:

$$\left\{ z : \left| z - a_{k,k} \right| \leq R_k \right\}$$

This theorem can be used to quickly bound the magnitude of eigenvalues, in some situations find out if a matrix is non-singular (if all eigenvalues are non-zero), if a system of difference or differential equations are stable (not always possible), etc.

Gershgorin's circle theorem (cont.)

The Gersgorin circle theorem is suprisingly simple to prove.

Let **x** be an eigenvector of an *NxN* matrix **A** = $[a_{m,n}]$, with corresponding eigenvalue λ, i.e., **Ax** = λ**x**.

Furher, assume that the *k*th element x_k of **x** is the one with the ***largest magnitude*** (cannot be zero!).

Ax = λ**x** now gives (look at element *k*):

$$\sum_{l=1}^{N} a_{k,l} x_l = \lambda x_k \quad \Rightarrow \quad \sum_{\substack{l=1 \\ l \neq k}}^{N} a_{k,l} x_l = \lambda x_k - a_{k,k} x_k = \left(\lambda - a_{k,k} \right) x_k$$

$$\Rightarrow \quad \left| \lambda - a_{k,k} \right| \left| x_k \right| \leq \sum_{\substack{l=1 \\ l \neq k}}^{N} \left| a_{k,l} \right| \left| x_l \right| \quad \Rightarrow \quad \left| \lambda - a_{k,k} \right| \leq \sum_{\substack{l=1 \\ l \neq k}}^{N} \left| a_{k,l} \right| \underbrace{\frac{\left| x_l \right|}{\left| x_k \right|}}_{\leq 1}$$

$$\Rightarrow \quad \left| \lambda - a_{k,k} \right| \leq \sum_{\substack{l=1 \\ l \neq k}}^{N} \left| a_{k,l} \right| \qquad \text{DONE!}$$

Gershgorin's circle theorem (cont.)

Use the theorem on the following matrix:

$$\mathbf{A} = \begin{bmatrix} 10 & 5 & 4 \\ 3 & 5 & 0 \\ 2 & 7 & -10 \end{bmatrix}$$

Conclusions:
- **A** is invertible (by rows)
- No eigenvalue has magnitude larger than 17 (by columns).
- No eigenvalue has a magnitude less than 1 (by rows).

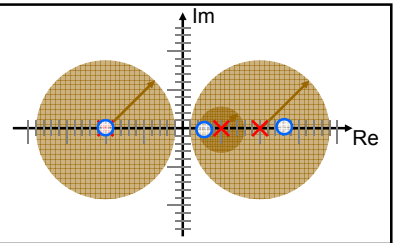
... more ?

× = diagonal elements of **A** ○ = eigenvalue of A

By rows

$$R_k = \sum_{\substack{l=1 \\ l \neq k}}^{N} \left| a_{k,l} \right|$$

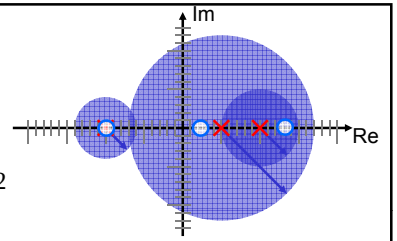
$$R_1 = |5| + |4| = 9$$
$$R_2 = |0| + |3| = 3$$
$$R_3 = |2| + |7| = 9$$



By columns

$$R_l = \sum_{\substack{k=1 \\ k \neq l}}^{N} \left| a_{k,l} \right|$$

$$R_1 = |3| + |2| = 5$$
$$R_2 = |5| + |7| = 12$$
$$R_3 = |4| + |0| = 4$$

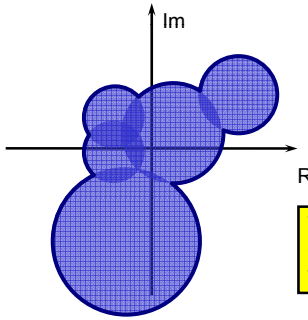


Gershgorin's circle theorem (cont.)

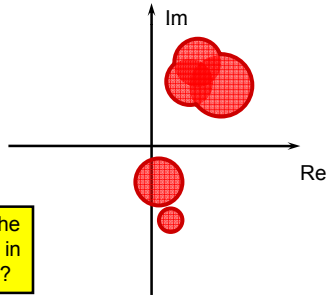
We know that all eigenvaues must be in at least one of the Gershgorin disks, i.e., all eigenvalues are found **SOMEWHERE INSIDE THE UNION** of the Gersgorin disks.

Can we say anything more about where eigenvalues are?

Example 1 (fully joined union):



Example 2 (three disjoint sets)



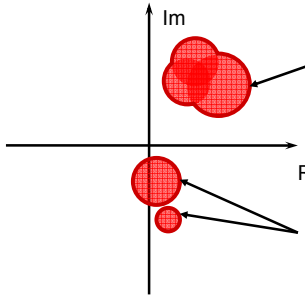
Where are the eigenvalues in these cases?

Gershgorin's circle theorem (cont.)

An extra result concerning the location of eigenvalues in the Gershgorin disks:

If the union of all Gershgorin disks consists of several disjoint subsets, each such subset contains a number of eigenvalues corresponding to the number of disks forming the subset.

Example 2 on previous slide:



Three eigenvalues in this subset!

The eigenvalues can be **anywhere** in the subset!

One eigenvalue in each of these!

Gershgorin's circle theorem (cont.)

A simple illustration of how to prove the result on the previous slide:

An arbitrary square matrix **A** can be written in the form:

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{NN} \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & & \\ & \ddots & \\ & & a_{NN} \end{bmatrix}}_{\mathbf{D}\text{ = diagonal entries}} + \underbrace{\begin{bmatrix} 0 & \cdots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \cdots & 0 \end{bmatrix}}_{\mathbf{B}\text{ = off-diagonal entries}}$$

Now, define a matrix **G**(α) as

$$\mathbf{G}(\alpha) = \mathbf{D} + \alpha\mathbf{B}$$

where α is a real number in the interval [0,1].

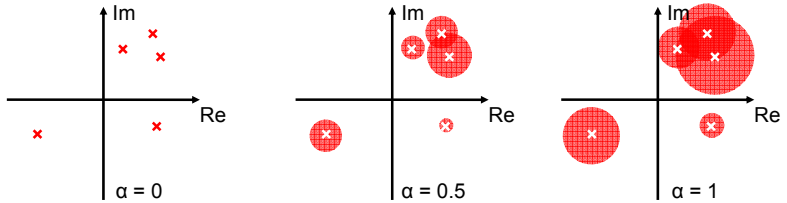
We have: $\mathbf{G}(0) = \mathbf{D}$ $\quad$ $\mathbf{G}(1) = \mathbf{A}$

Gershgorin's circle theorem (cont.)

We know three things:

1. The eigenvalues of **G**(0) are exactly the diagonal entries (of **A**).
2. The eigenvalues of any matrix are continuous functions of the matrix entries (or α in this case). If matrix elements change smoothly, there are no "jumps" in the eigenvalues.
3. For each value on α, Gershgorins circle theorem tells us in which set all eigenvalues of G(α) are.

Now, the trick: A smooth transition from α = 0 to α = 1



The eigenvalues that start at the diagonal entries (α = 0) cannot escape to another disjoint subset of the Gershgorin disks, since they are continuous functions of α. **DONE!**

Symmetric/hermitian matrices

Hermitian matrices

A Hermitian matrix has the property

$$\mathbf{A} = \mathbf{A}^H$$

Hermitian (or real symmetric) matrices have the following properties:

1. $\mathbf{x}^H\mathbf{A}\mathbf{x}$ is real for all complex vectors $\mathbf{x}$.
2. every eigenvalue is real.
3. two eigenvectors coming from different eigenvalues are orthogonal.

Hermitian matrices [cont.]

1. $\mathbf{x}^H\mathbf{A}\mathbf{x}$ is real for all complex vectors $\mathbf{x}$.

Let's see what we get when we take the complex conjugate of the scalar $\mathbf{x}^H\mathbf{A}\mathbf{x}$:

$$\overline{\mathbf{x}^H\mathbf{A}\mathbf{x}} = \left(\mathbf{x}^H\mathbf{A}\mathbf{x}\right)^H = \mathbf{x}^H\mathbf{A}^H\mathbf{x} = \mathbf{x}^H\mathbf{A}\mathbf{x}$$

$\mathbf{x}^H\mathbf{A}\mathbf{x}$ equals its own complex conjugate => REAL!

Hermitian matrices [cont.]

2. every eigenvalue is real.

We have

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

multiplying by $\mathbf{x}^H$ from the left gives

$$\underbrace{\mathbf{x}^H\mathbf{A}\mathbf{x}}_{\text{Real Prop 1.}} = \lambda\underbrace{\mathbf{x}^H\mathbf{x}}_{\text{Real}}$$

which forces λ to be real.

Hermitian matrices [cont.]

3. two eigenvectors coming from different eigenvalues are orthogonal.

We have

$$\mathbf{A}\mathbf{x}_1 = \lambda_1 \mathbf{x}_1 \quad \text{and} \quad \mathbf{A}\mathbf{x}_2 = \lambda_2 \mathbf{x}_2$$

We can show that

$$\left(\lambda_1 \mathbf{x}_1\right)^H \mathbf{x}_2 = \left(\mathbf{A}\mathbf{x}_1\right)^H \mathbf{x}_2 = \mathbf{x}_1{}^H \mathbf{A}^H \mathbf{x}_2$$

$$= \mathbf{x}_1{}^H \mathbf{A}\mathbf{x}_2 = \mathbf{x}_1{}^H \lambda_2 \mathbf{x}_2$$

or (eigenvalues are real, Prop 2.)

$$\lambda_1 \mathbf{x}_1{}^H \mathbf{x}_2 = \lambda_2 \mathbf{x}_1{}^H \mathbf{x}_2$$

which for different eigenvalues can only be true of $\mathbf{x}_1$ and $\mathbf{x}_2$ are othogonal.

Unitary matrices

Unitary matrices are the complex equivalents of real orthogonal matrices.

A real orthogonal matrix **Q** has orthonormal columns:

$$\mathbf{Q}^T \mathbf{Q} = \mathbf{I}$$

A complex unitary matrix **U** has orthonormal columns:

$$\mathbf{U}^H \mathbf{U} = \mathbf{I}$$

Spectral theorem

The previous properties for complex Hermitian matrices lead to one of the most important results in linear algebra – the spectral theorem:

REAL CASE

A real symmetric matrix **A** can be factored into $\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T$. The ortonormal eigenvectors of **A** are in the orthogonal matrix **Q** and the corresponding eigenvalues in the diagonal matrix **Λ**.

COMPLEX CASE

A Hermitian matrix **A** can be factored into $\mathbf{U}\mathbf{\Lambda}\mathbf{U}^H$. The ortonormal eigenvectors of **A** are in the unitary matrix **U** and the corresponding eigenvalues in the diagonal matrix **Λ**.

The above are ”simply” special cases of the general results on matrix diagonalization when eigenvalues are distinct. They can, however, be proven true also for repeated eigenvalues.

Spectral theorem [cont.]

The spectral theorem implies that a Hermitian matrix

$$\mathbf{A} = \begin{bmatrix} | & & | \\ \mathbf{u}_1 & \cdots & \mathbf{u}_N \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} - & \mathbf{u}_1^H & - \\ & \vdots & \\ - & \mathbf{u}_N^H & - \end{bmatrix}$$

can be written as a sum of N rank-1 projection matrices:

$$\mathbf{A} = \sum_{n=1}^{N} \lambda_n \mathbf{u}_n \mathbf{u}_n^H$$

If the eigenvalues of **A** (nonnegative, real) are sorted in decreasing order, we can make ”good” low-rank approximations of **A** by limiting the number of terms we use in the above sum.

Similarity transformations

Similarity transformations

We have discussed several forms of factoring matrices based on eigenvalues and eigenvectors:

$$\mathbf{A} = \mathbf{S}\boldsymbol{\Lambda}\mathbf{S}^{-1} \qquad \boldsymbol{\Lambda} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} \qquad \text{[Diagonalization]}$$

$$\mathbf{A} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^{-1} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^{T} \qquad \boldsymbol{\Lambda} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} \qquad \text{[}\mathbf{A}\text{ symetric]}$$

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{-1} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{H} \qquad \boldsymbol{\Lambda} = \mathbf{U}^{-1}\mathbf{A}\mathbf{U} \qquad \text{[}\mathbf{A}\text{ Hermitian]}$$

These are all are all on the form:

$$\mathbf{M}^{-1}\mathbf{A}\mathbf{M}$$

which we call a *similarity transformation* of **A**.

We say that $\mathbf{M}^{-1}\mathbf{A}\mathbf{M}$ is similar to **A**, but in what way?

Similarity transformations [cont.]

If $\mathbf{B} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}$, then **A** and **B** have the same eigenvalues and every eigenvector **x** of **A** corresponds to an eigenvector $\mathbf{M}^{-1}\mathbf{x}$ of **B**.

Assume $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ and $\mathbf{A} = \mathbf{M}\mathbf{B}\mathbf{M}^{-1}$ then

B has same eigenvalue as **A** ...

$$\mathbf{A}\mathbf{x} = \mathbf{M}\mathbf{B}\mathbf{M}^{-1}\mathbf{x} = \lambda\mathbf{x} \qquad \Rightarrow \mathbf{B}\underline{\mathbf{M}^{-1}\mathbf{x}} = \lambda\underline{\mathbf{M}^{-1}\mathbf{x}}$$

... but a new eigenvector

There is more in the textbook about this if you are interested!

Some notes on applications of Hermitian matrices

Correlation

Assume that **x** is a zero-mean stochastic vector.

The autocorrelation is defined as the expectation:

$$\mathbf{R}_{\mathbf{xx}} = E\left\{\mathbf{x}\mathbf{x}^H\right\}$$

This autocorrelation matrix is Hermitian and can therefore be factorized as

$$\mathbf{R}_{\mathbf{xx}} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^H$$

Given this spectral factorization we can make the following transformation

$$\mathbf{y} = \mathbf{U}^H\mathbf{x}$$

This is called the *Hotelling transform* of **x**. (c.f. the Karhunen-Loewe transform)

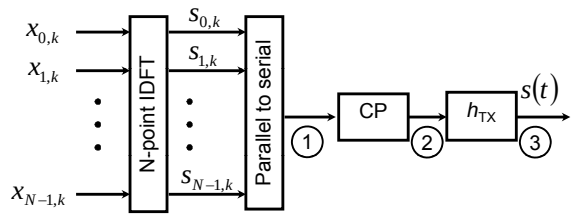
which results in a new, zero-mean, stochastic vector **y** with autocorrelation matrix

$$\mathbf{R}_{\mathbf{yy}} = E\left\{\mathbf{y}\mathbf{y}^H\right\} = E\left\{\mathbf{U}^H\mathbf{x}\mathbf{x}^H\mathbf{U}\right\} = \mathbf{U}^H E\left\{\mathbf{x}\mathbf{x}^H\right\}\mathbf{U}$$

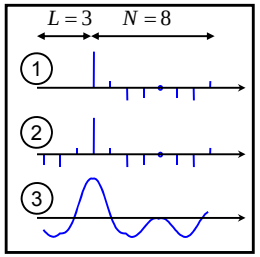
$$= \mathbf{U}^H\mathbf{R}_{\mathbf{xx}}\mathbf{U} = \mathbf{U}^H\mathbf{U}\mathbf{\Lambda}\mathbf{U}^H\mathbf{U} = \mathbf{\Lambda}$$

Elements of **y** are uncorrelated!

OFDM System model



k	– symbol
m	– sample
n	– subcarrier
L	– CP length
T_{samp}	– sampling period
h_{TX}	– TX filter

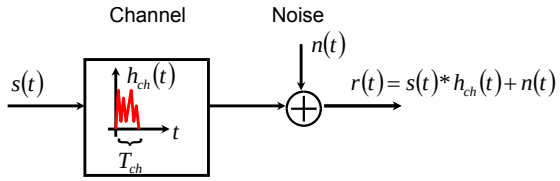


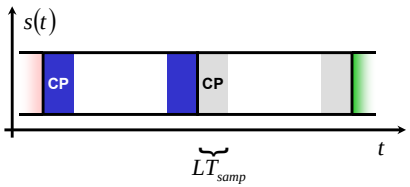
N-point IDFT: $s_{m,k} = \frac{1}{N}\sum_{n=0}^{N-1} x_{n,k}\exp\left(j2\pi\frac{mn}{N}\right)$ for $0 \le m \le N-1$

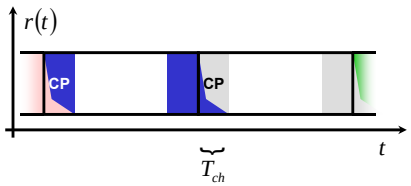
Adding CP: $s_{m,k} = s_{N+m,k}$ for $-L \le m \le -1$

TX filtering: $s(t) = h_{TX}(t) * \left(\sum_k \sum_{m=-L}^{N-1} s_{m,k}\,\delta\left(t - k(N+L)T_{samp} - mT_{samp}\right)\right)$

OFDM System model [cont.]



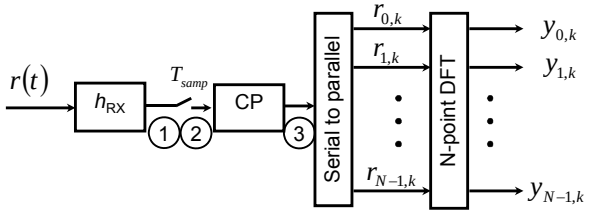




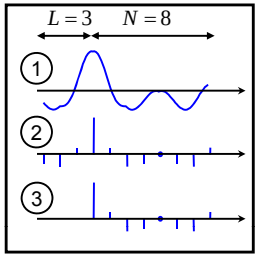
As long as the CP is longer than the delay spread of the channel, $LT_{samp} > T_{ch}$, it will absorb the ISI.

By removing the CP in the receiver, the transmission becomes ISI free.

OFDM System model [cont.]



q	– symbol
p	– sample
n	– subcarrier
L	– CP length
T_{samp}	– sampling period
h_{RX}	– RX filter



RX filtering: $z'(t) = h_{RX}(t) * r(t)$

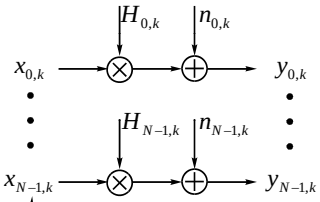
Sampling: $z'_k = z'(kT_{samp})$

Removing CP: $r_{p,q} = z'_{q(N+L)+p}$ for $0 \le p \le N-1$

N-point DFT: $y_{n,q} = \sum_{p=0}^{N-1} r_{p,q}\exp\left(-j2\pi\frac{np}{N}\right)$ for $0 \le n \le N-1$

OFDM
System model [cont.]

Simplified model under ideal conditions (slow enough fading and sufficient CP)



Total filter in the signal path:

$$h_{signal}(t) = h_{TX}(t) * h(t) * h_{RX}(t)$$

$$H_{signal}(f) = H_{TX}(f) * H(f) * H_{RX}(f)$$

OFDM systems have *N* between 64 (WLAN) and 8192 (DigTV).

Given that subcarrier *n* is transmitted at frequency f_n the attenuations become:

$$H_{n,k} = H_{signal}(f_n)$$

OFDM
System model [cont.]

We have ended up with an OFDM matrix model:

$$\mathbf{y} = \mathbf{X}\mathbf{h} + \mathbf{n}$$

where **y** is the received vector, **X** a diagonal matrix with the transmitted constellation points on its diagonal, **h** a vector of channel attenuations, and a vector **n** of receiver noise.

For the purpose of channel estimation, assume that all ones are transmitted, i.e., that **X = I**. We now have a simplified model:

$$\mathbf{y} = \mathbf{h} + \mathbf{n}$$

Further assume that the channel is zero-mean and has autocorrelation $\mathbf{R}_{\mathbf{hh}}$ while the noise is i.i.d zero mean complex Gaussian with autocorrelation $\mathbf{R}_{\mathbf{nn}} = \sigma^2\mathbf{I}$. We also assume that **h** and **n** are independent.

OFDM
Channel estimation

When we receive **y**, we want to apply a linear eastimator

$$\hat{\mathbf{h}} = \mathbf{A}\mathbf{y}$$

that minimizes the mean-squared error

$$MSE = E\left\{\left\|\hat{\mathbf{h}} - \mathbf{h}\right\|^2\right\}$$

This gives the minimizing matrix

$$\mathbf{A} = \mathbf{R}_{\mathbf{hy}}\mathbf{R}_{\mathbf{yy}}^{-1} = \mathbf{R}_{\mathbf{hh}}\mathbf{R}_{\mathbf{yy}}^{-1} = \mathbf{R}_{\mathbf{hh}}\left(\mathbf{R}_{\mathbf{hh}} + \sigma^2\mathbf{I}\right)^{-1}$$

and a resulting linear MMSE estimator

$$\hat{\mathbf{h}} = \mathbf{R}_{\mathbf{hh}}\left(\mathbf{R}_{\mathbf{hh}} + \sigma^2\mathbf{I}\right)^{-1}\mathbf{y}$$

Even with known $\mathbf{R}_{\mathbf{hh}}$ and σ^2, making a precalculated matrix possible, each channel estimation requires N^2 operations.

OFDM
Channel estimation [cont.]

Can we simplify the calculation?

Let's use the fact that $\mathbf{R}_{\mathbf{hh}}$ is Hermitian and can be factored as $\mathbf{U\Lambda U}^H$:

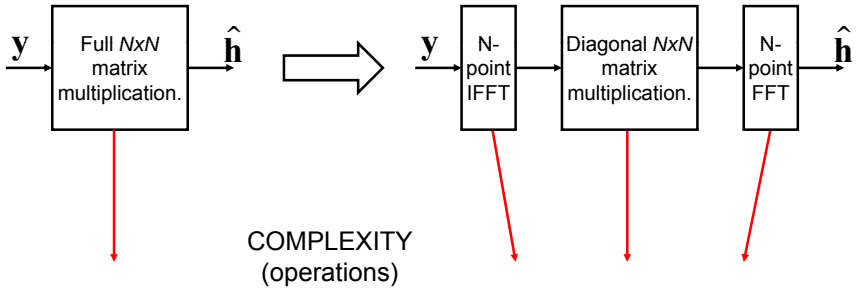
$$\begin{aligned}
\hat{\mathbf{h}} &= \mathbf{U\Lambda U}^H\left(\mathbf{U\Lambda U}^H + \sigma^2\mathbf{I}\right)^{-1}\mathbf{y} \\
&= \mathbf{U\Lambda U}^H\left(\mathbf{U\Lambda U}^H + \sigma^2\mathbf{U}\mathbf{U}^H\right)^{-1}\mathbf{y} \\
&= \mathbf{U\Lambda U}^H\left(\mathbf{U}\left(\mathbf{\Lambda} + \sigma^2\mathbf{I}\right)\mathbf{U}^H\right)^{-1}\mathbf{y} \\
&= \mathbf{U\Lambda U}^H\mathbf{U}\left(\mathbf{\Lambda} + \sigma^2\mathbf{I}\right)^{-1}\mathbf{U}^H\mathbf{y} \\
&= \mathbf{U}\underbrace{\mathbf{\Lambda}\left(\mathbf{\Lambda} + \sigma^2\mathbf{I}\right)^{-1}}_{\text{Diagonal matrix}}\mathbf{U}^H\mathbf{y}
\end{aligned}$$

In reality, it turns out that **U** is very close to the FFT matrix.

OFDM
Channel estimation [cont.]

What have we obtained?

$$\mathbf{R}_{\mathbf{hh}}\left(\mathbf{R}_{\mathbf{hh}}+\sigma^2\mathbf{I}\right)^{-1} \qquad \mathbf{U}^H \quad \mathbf{\Lambda}\left(\mathbf{\Lambda}+\sigma^2\mathbf{I}\right)^{-1} \quad \mathbf{U}$$



COMPLEXITY (operations)

$$N^2$$

$$N\log_2 N + N + N\log_2 N$$
$$= 2N\log_2 N + N$$

OFDM
Channel estimation summary