Study week 3.

3.4.1 Low-Rate QAM-Type of Input Signals

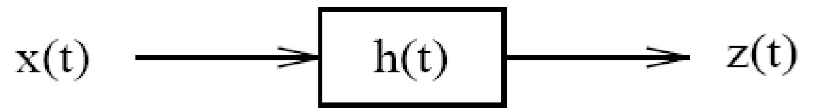


Figure 3.11: Bandpass filtering.

$$x(t) = x_I(t)\cos(\omega_c t) - x_Q(t)\sin(\omega_c t) = Re\{\tilde{x}(t)e^{j\omega_c t}\} \qquad (3.103)$$

$$\tilde{x}(t) = x_I(t) + jx_Q(t) \qquad (3.104)$$

This complex signal contains the information!

$$x(t) = x_I(t)\cos(\omega_c t) - x_Q(t)\sin(\omega_c t) = Re\{\tilde{x}(t)e^{j\omega_c t}\} \quad (3.103)$$

$$z(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)Re\{\tilde{x}(t-\tau)e^{j\omega_c(t-\tau)}\}d\tau =$$

$$= Re\left\{e^{j\omega_c t}\int_{-\infty}^{\infty} h(\tau)\tilde{x}(t-\tau)e^{-j\omega_c\tau}d\tau\right\} \quad (3.105)$$

3 assumptions:

1) The duration of the impulse response $h(t)$ can be considered to be equal to T_h. This means that essentially all the energy in $h(t)$ is assumed to be contained within the time interval $0 \le t \le T_h$.

2) The input signal is assumed to be a QAM-type of signal with duration $T = T_s$:

$$x(t) = \begin{cases} 0 & , \quad t < 0 \\ A\cos(\omega_c t) - B\sin(\omega_c t) = \sqrt{A^2+B^2}\cos(\omega_c t + \nu) & , \quad 0 \le t \le T_s \\ 0 & , \quad t > T_s \end{cases} \quad (3.106)$$

3) $T_s > T_h$ ("low" signaling rate).

$$\tilde{x}(t) = \begin{cases} A + jB = \sqrt{A^2 + B^2}\; e^{j\nu} & , \quad 0 \le t \le T_s \\ 0 & , \quad \text{otherwise} \end{cases} \tag{3.108}$$

$\boldsymbol{T_h \le t \le T_s}$ **:**

$$\begin{aligned} z(t) &= Re\left\{ e^{j\omega_c t} \int_0^{T_h} h(\tau)\sqrt{A^2 + B^2}\; e^{j\nu} e^{-j\omega_c \tau} d\tau \right\} = \\ &= Re\{\sqrt{A^2 + B^2}\; e^{j\nu} \cdot H(f_c) e^{j\omega_c t}\} = \\ &= |H(f_c)|\sqrt{A^2 + B^2} \cos(\omega_c t + \nu + \phi(f_c)) = A_z \cos(\omega_c t) - B_z \sin(\omega_c t) \end{aligned} \tag{3.109}$$

Hence, a QAM-signal at the output in this time interval!

However, **attenuation and rotation** compared with the input!
Compare with the input x(t) in (3.106)!

$$\begin{aligned} A_z + jB_z &= (A + jB)H(f_c) = \sqrt{A^2 + B^2}|H(f_c)|e^{j(\nu + \phi(f_c))} = \\ &= (A + jB)(H_{Re}(f_c) + jH_{Im}(f_c)) \end{aligned} \tag{3.110}$$

$$A_z + jB_z = (A + jB)H(f_c) = \sqrt{A^2 + B^2}|H(f_c)|e^{j(\nu + \phi(f_c))} =$$
$$= (A + jB)(H_{Re}(f_c) + jH_{Im}(f_c)) \qquad (3.110)$$

A COMPACT MODEL WITH A COMPLEX CHANNEL PARAMETER!!

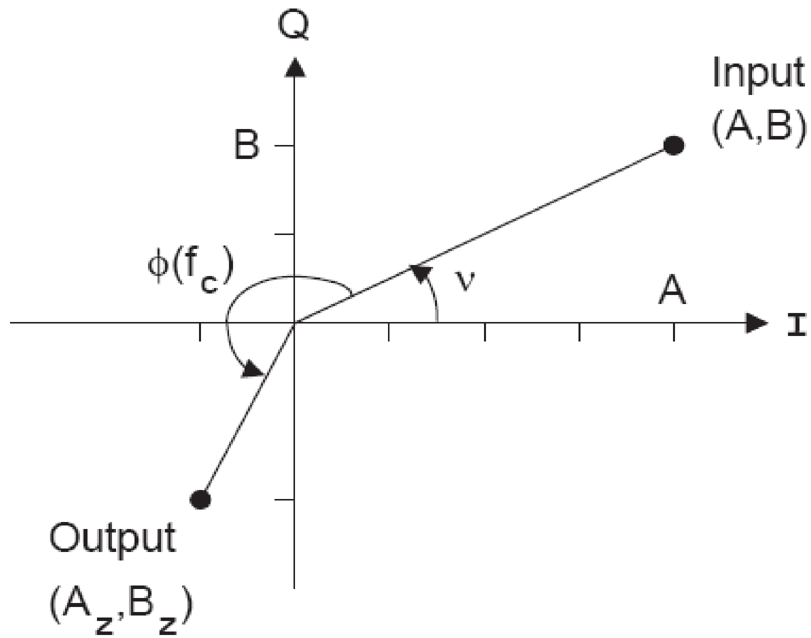


Figure 3.13: Illustrating that the input I-Q amplitudes (A,B) are scaled and rotated by the channel $H(f)$, see (3.109) and (3.110).

$$z(t) = \begin{cases} 0 & , \quad t < 0 \\ \text{“non-stationary transient” starting interval} & , \quad 0 \le t \le T_h \\ |H(f_c)|\sqrt{A^2+B^2}\;\cos(\omega_c t + \nu + \phi(f_c)) & , \quad T_h \le t \le T_s \\ \text{“non-stationary transient” ending interval} & , \quad T_s \le t \le T_s + T_h \\ 0 & , \quad t > T_s + T_h \end{cases}$$

$$\text{and within } T_h \le t \le T_s,\ A_z + jB_z = (A + jB)H(f_c) \tag{3.111}$$

An important result here is that the input QAM signal $x(t)$ in (3.106) is changed to a new QAM signal by $|H(f_c)|$ and $\phi(f_c)$ in the interval $T_h \le t \le T_s$, see also Figure 3.13 and (3.110) how the I-Q components are changed. Furthermore, in OFDM applications the signaling rate $1/T_s$ is low such that $T_s \gg T_h$, and many QAM signals with different carrier frequencies are sent in parallel. ***Due to linearity, the result in (3.111) can be applied to each QAM signal in the OFDM signal by replacing f_c with f_n.*** In OFDM applications the receiver uses the time interval $\Delta_h \le t \le T_s$ for detection of the output QAM signals, and the duration of this observation interval is denoted $T_{obs} = T_s - \Delta_h$ (compare with (2.110) on page 51, and $T_h \le \Delta_h$).

So, the n:th QAM signal constellation in a sent OFDM signal is attenuated and rotated by H(fn) which is the value of the channel transfer function H(f) at the carrier frequency fn.

3.4.3 N-Ray Channel Model

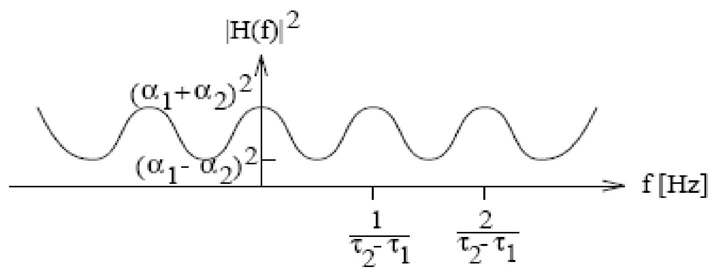
$$z(t) = x(t) * \underbrace{\left(\sum_{i=1}^{N} \alpha_i \delta(t - \tau_i) \right)}_{\text{Impulse response } h(t)} = \sum_{i=1}^{N} \alpha_i x(t - \tau_i) \qquad (3.126)$$

$$H(f) = \mathcal{F}\{h(t)\} = \sum_{i=1}^{N} \alpha_i e^{-j2\pi f \tau_i} \qquad (3.128)$$

So, **H(fc)** is easy to find!

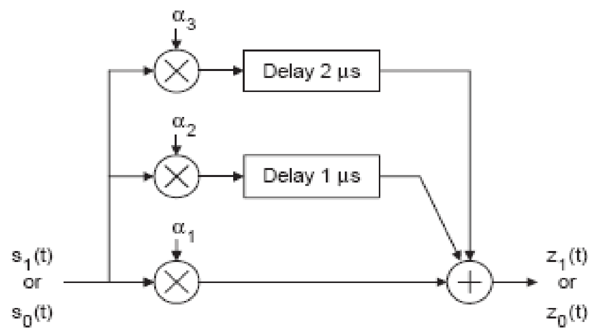
EXAMPLE 3.20

Rough sketch:



It is seen in this figure that the two signal paths add constructively or destructively (fading) depending on the frequency. Furthermore, if $\alpha_1 \approx \alpha_2$ *then* $|H(f)|$ *is very close to zero at certain frequencies (so-called deep fades)!*

EXAMPLE 3.19



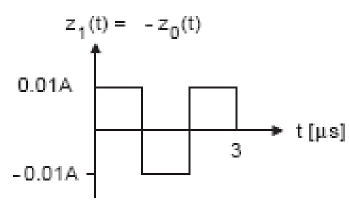
The signal $z_i(t) = s_i(t) * h(t)$ is the output signal corresponding to the input signal $s_i(t)$, $i = 0, 1$. Determine and sketch $z_0(t)$ and $z_1(t)$ if $\alpha_1 = 0.01$, $\alpha_2 = -0.01$, and $\alpha_3 = 0.01$.

Your conclusions concerning choice of bit rate to avoid overlapping signal alternatives after the channel?

Solution:

$$z_\ell(t) = \sum_{i=1}^{3} \alpha_i s_\ell(t - \tau_i) = 0.01 s_\ell(t) - 0.01 s_\ell(t - 10^{-6}) + 0.01 s_\ell(t - 2 \cdot 10^{-6}), \;\; \ell = 0, 1$$

yields,



Observe that the signal alternatives are changed significantly by the channel (filtering), and that the duration of both signal alternatives is increased from 1 [µs] before the channel, to 3[µs] after the channel!

If the bit rate is reduced to at most $10^6/3$ bps, *then no overlap of signal alternatives will exist after the channel.* □

5.34 Consider a communication system where N_t M-ary QAM signals are sent simultaneously (from N_t antennas). The n:th transmitted M-ary QAM signal is denoted $s_n(t)$,

$$s_n(t) = A(n)g(t)\cos(\omega_c t) - B(n)g(t)\sin(\omega_c t) \qquad (5.133)$$

for $n = 1, 2, \ldots, N_t$. Note that the same carrier frequency is used for all N_t transmitted QAM signals!

The receiver is assumed to have N_r receiving antennas. The received signal $r_k(t)$ at the k:th receiving antenna is here modelled as

See (3.109)-(3.110)!

$$r_k(t) = \sum_{n=1}^{N_t} \Big(\left[H^{Re}_{k,n} A(n) - H^{Im}_{k,n} B(n)\right] g(t)\cos(\omega_c t) - \\ - \left[H^{Re}_{k,n} B(n) + H^{Im}_{k,n} A(n)\right] g(t)\sin(\omega_c t)\Big) + w_k(t) \qquad (5.134)$$

for $k = 1, 2, \ldots, N_r$. The variables $H^{Re}_{k,n}$ and $H^{Im}_{k,n}$ models how the n:th transmitted QAM signal is received at the k:th receiving antenna (attenuation and rotation of the I-Q components).

After I and Q demodulation of $r_k(t)$ to baseband, the receiver obtains the noisy signal space coordinates, here collected in r_k as

$$r_k = \underbrace{\sum_{n=1}^{N_t} \left(H^{Re}_{k,n} A(n) - H^{Im}_{k,n} B(n)\right)}_{\text{received } I \text{ component}} + j \underbrace{\sum_{n=1}^{N_t} \left(H^{Re}_{k,n} B(n) + H^{Im}_{k,n} A(n)\right)}_{\text{received } Q \text{ component}} \\ + \underbrace{(w^{Re}_k + j w^{Im}_k)}_{\text{due to AWGN}} \qquad (5.135)$$

Note that complex notation ($j^2 = -1$) is used in (5.135)!

Let us now introduce the complex notations:

See (3.110)!

$$\begin{aligned} d_n &= A(n) + jB(n) \\ \alpha_{k,n} &= H^{Re}_{k,n} + jH^{Im}_{k,n} \\ w_k &= w^{Re}_k + jw^{Im}_k \end{aligned} \qquad (5.136)$$

Then (5.135) can be formulated as,

$$r_k = \sum_{n=1}^{N_t} \alpha_{k,n} d_n + w_k \quad ,k = 1, 2, \ldots, N_r \tag{5.137}$$

A compact formulation is now obtained as

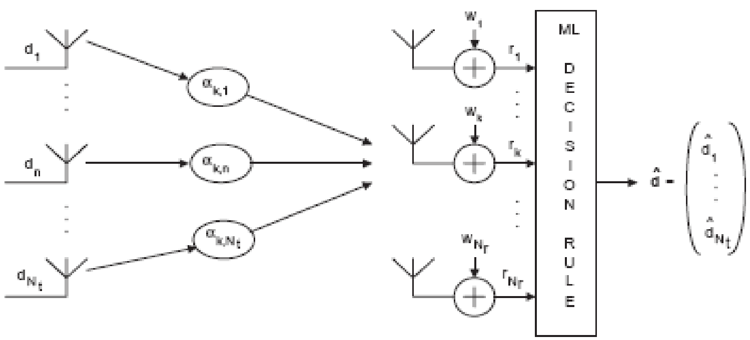
$$\boldsymbol{r} = \begin{pmatrix} r_1 \\ \vdots \\ r_{N_r} \end{pmatrix} = \boldsymbol{A} \begin{pmatrix} d_1 \\ \vdots \\ d_{N_t} \end{pmatrix} + \begin{pmatrix} w_1 \\ \vdots \\ w_{N_r} \end{pmatrix} = \boldsymbol{A}\boldsymbol{d} + \boldsymbol{w} \tag{5.138}$$

Very important!

where the $N_r \times N_t$ matrix $\boldsymbol{A}$ contains the channel coefficients $\{\alpha_{k,n}\}$. The relationship in (5.138) is a basic model in so-called multiple-input multiple-output (MIMO) systems.

The MIMO model is illustrated in the figure below,

MIMO MODEL



SISO
SIMO
MISO
MIMO
Diversity gain
Spatial multiplexing gain

$$r_k = \sum_{n=1}^{N_t} \alpha_{k,n} d_n + w_k$$

$$\boldsymbol{r} = \begin{pmatrix} r_1 \\ \vdots \\ r_{N_r} \end{pmatrix} = \boldsymbol{A} \begin{pmatrix} d_1 \\ \vdots \\ d_{N_t} \end{pmatrix} + \begin{pmatrix} w_1 \\ \vdots \\ w_{N_r} \end{pmatrix} = \boldsymbol{A}\boldsymbol{d} + \boldsymbol{w}$$

z=Ad is the received signalpoint and w is the additive noise vector.

64-QAM+Nt=8 (48bits): ML symbol decision rule ....?

Chapter 8

Trellis-coded Signals

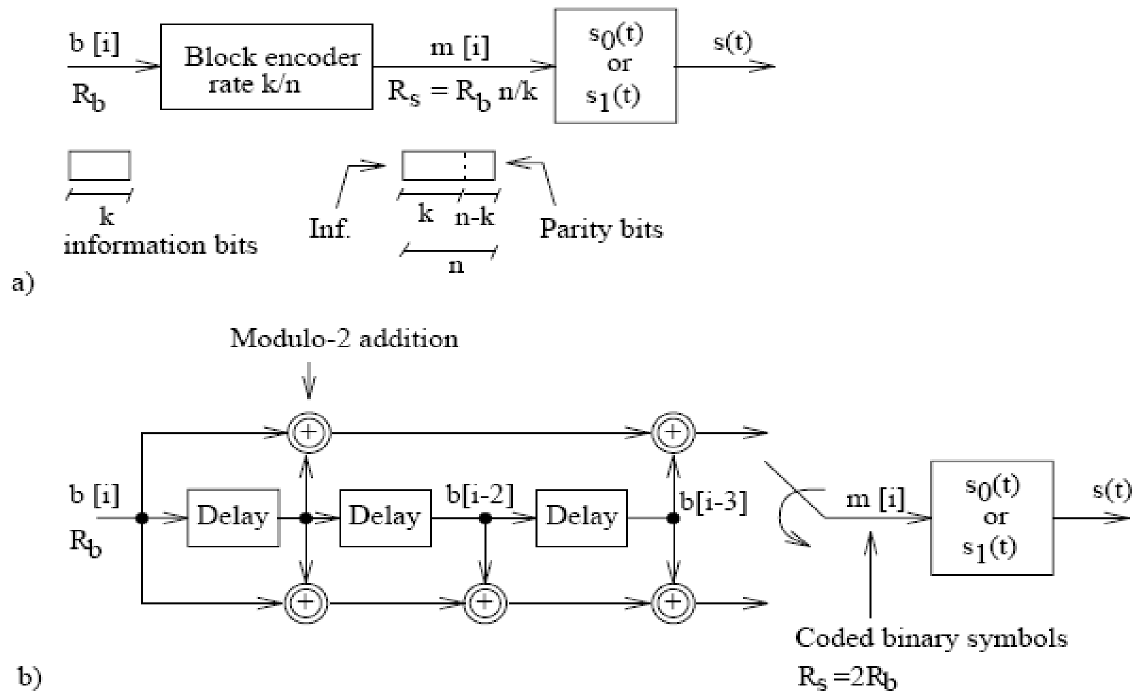


Figure 8.1: a) Block coding, $r_c = k/n$. b) Convolutional coding, $r_c = 1/2$.

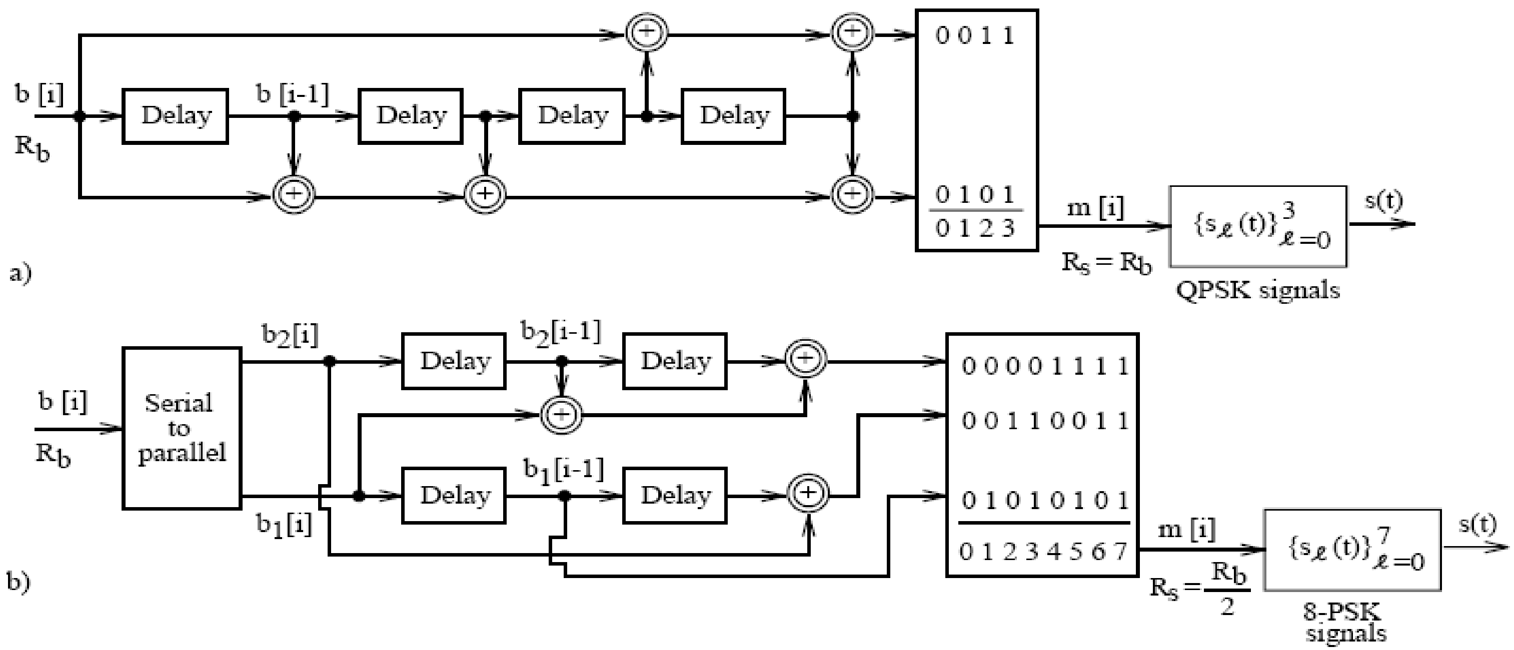
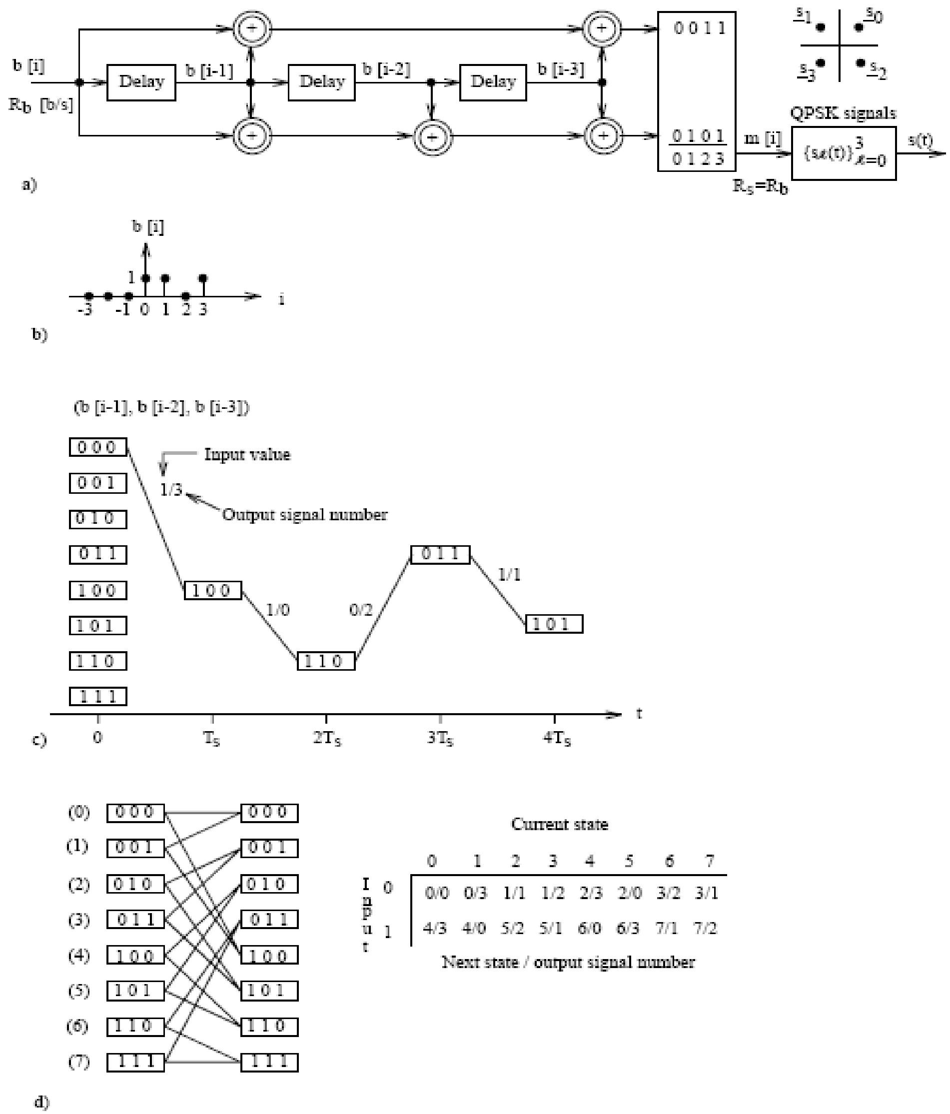
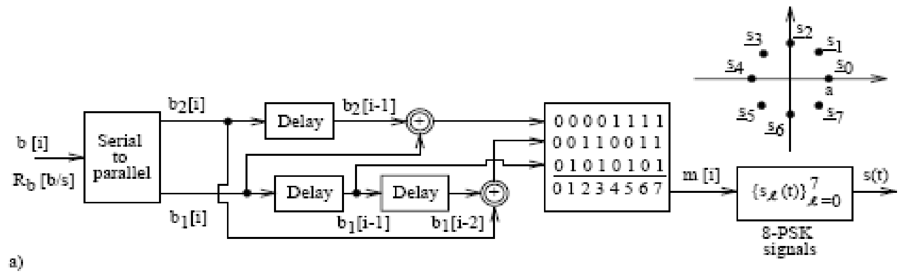


Figure 8.2: a) Rate $r_c = 1/2$ convolutional encoder combined with QPSK; b) Rate $r_c = 2/3$ convolutional encoder combined with 8-PSK, from [63], [64].



Input	0	1	2	3	4	5	6	7
0	0/0	0/3	1/1	1/2	2/3	2/0	3/2	3/1
1	4/3	4/0	5/2	5/1	6/0	6/3	7/1	7/2

Figure 8.4: a) A rate 1/2 convolutional encoder combined with QPSK signal alternatives; b) A specific input sequence $b[i]$; c) The corresponding path in the trellis; d) A trellis section, and a table containing all relevant parameters.



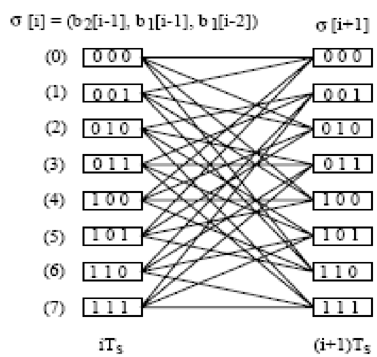
a)

Current state σ [i]

INPUT $\binom{b_2[i]}{b_1[i]}$	(000) 0	(001) 1	(010) 2	(011) 3	(100) 4	(101) 5	(110) 6	(111) 7
$\binom{0}{0}$	0/0	0/2	1/1	1/3	0/4	0/6	1/5	1/7
$\binom{0}{1}$	2/4	2/6	3/5	3/7	2/0	2/2	3/1	3/3
$\binom{1}{0}$	4/2	4/0	5/3	5/1	4/6	4/4	5/7	5/5
$\binom{1}{1}$	6/6	6/4	7/7	7/5	6/2	6/0	7/3	7/1

σ [i+1] / m [i]

b)



c)

Figure 8.6: a) An example of TCM, from [63]–[64]; b) The mappings $F(\cdot,\cdot)$ and $G(\cdot,\cdot)$; c) A trellis section.

<u>Memory (redundancy, dependancy) is introduced among the sent signal alternatives!</u>

This gives us some new properties like, e.g.,:

Which of the following signal sequences are impossible?

1. $s_3(t), s_2(t-T_b), s_1(t-2T_b), s_1(t-3T_b)$
2. $s_3(t), s_2(t-T_b), s_2(t-2T_b), s_1(t-3T_b)$
3. $s_3(t), s_1(t-T_b), s_0(t-2T_b), s_2(t-3T_b)$
4. $s_3(t), s_1(t-T_b), s_3(t-2T_b), s_1(t-3T_b)$

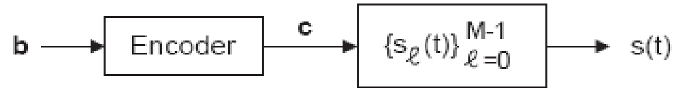
Note: In the uncoded case all signal sequences are possible.

Find the "missing" signal, in the sequence below,

$$s_1(t), s_3(t-T_b),\ ?\ , s_2(t-3T_b), s_3(t-4T_b), s_0(t-5T_b)$$

Note: This is not possible to do in the uncoded case!

2.32 Let us here study adaptive coding and modulation according to the block diagram below.



$$\bar{E}_{sent} = r_c \log_2(M) E_{b,sent} = \frac{k}{n} \log_2(M) E_{b,sent} \quad (8.4)$$

$$R_s = 1/T_s = \frac{1}{r_c} \cdot \frac{1}{\log_2(M)} \cdot R_b = \frac{1}{k/n} \cdot \frac{1}{\log_2(M)} \cdot R_b \quad (8.5)$$

$$W = c \cdot R_s \quad (8.6)$$

Typically, the bandwidth W is fixed and given but:

the rate of the encoder

the number of signal alternatives

and the bit rate can be **ADAPTIVE**, see (8.5)-(8.6)!

We have memory in the sequence of sent signal alternatives!

Some sequences are impossible, see problem!

Only ”good” sequences are sent!

OFDM - INTRO

OFDM signal(t) = $g_{rec}(t)\sum_{k=0}^{K-1} Re\{a_k e^{j2\pi f_k t}\} = g_{rec}(t) Re\{\sum_{k=0}^{K-1} a_k e^{j2\pi f_k t}\} =$

$$= g_{rec}(t) Re\{\sum_{k=0}^{K-1} a_k e^{j2\pi(f_0+kf_\Delta)t}\} = g_{rec}(t) Re\{(\sum_{k=0}^{K-1} a_k e^{j2\pi(g_0+k)f_\Delta t}) e^{j2\pi f_{rc} t}\} =$$

$$= g_{rec}(t) Re\{(\sum_{k=0}^{K-1} a_k e^{j2\pi g_k f_\Delta t}) e^{j2\pi f_{rc} t}\} \quad (1.13)$$

Equation (1.13) shows that an OFDM signal can be viewed as the sum of K QAM signals.

$$a_k = a_{k,I} + j a_{k,Q}, \quad k = 0,1,\dots,K-1 \quad (1.5)$$

$$T_{obs} = T_s - T_{CP} \quad (1.18)$$

$$f_\Delta = 1/T_{obs} \quad (2.1)$$

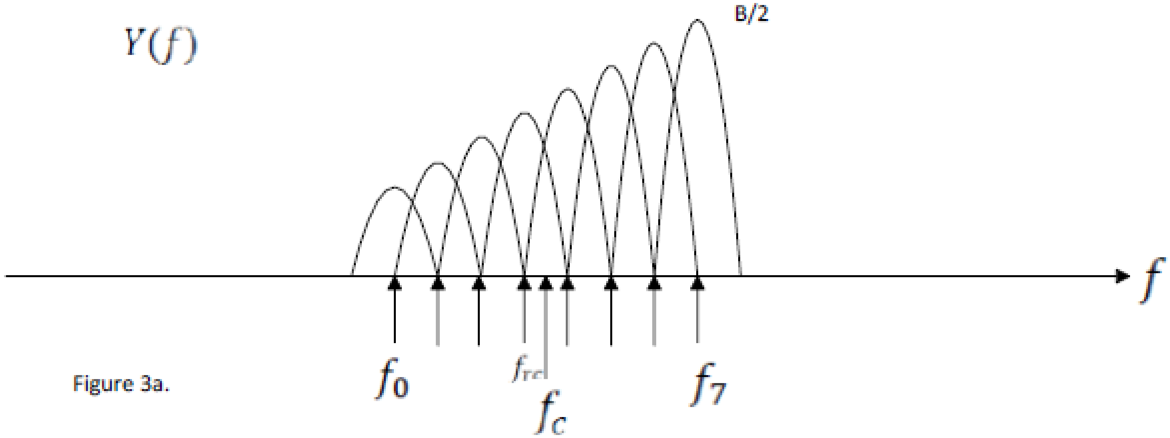


Figure 3a.

$$f_k = f_0 + kf_\Delta, \quad k = 0,1,\ldots,K-1 \qquad (1.1)$$

$$W_{OFDM} \approx Kf_\Delta \text{ (Hz)} \qquad (1.3)$$

The transmitted ***information bit rate***, denoted R_b, then equals,

$$R_b = \frac{r_c \sum_{k=0}^{K-1} log_2(M_k)}{T_s} \text{ (bps)} \qquad (1.16)$$

Furthermore, assuming also that K is >> 1 the ***bandwidth efficiency***, denoted ρ, is

$$\rho = \frac{R_b}{W_{OFDM}} = \frac{r_c \sum_{k=0}^{K-1} log_2(M_k)}{T_s K f_\Delta} \text{ (bps/Hz)} \qquad (1.17)$$

As an **example**: Let us consider the WLAN standard IEEE 802.11n (see ref. [11]). In this system OFDM is used with K=117, $f_\Delta = 312.5$ kHz and $T_s = 4$ µs (normal). Of the 117 subcarriers, the 3 center subcarriers are set to zero, 108 subcarriers are used for data transmission, and 6 subcarriers are used as pilots. In case of $r_c = 5/6$, and 64-QAM on each of the 108 subcarriers, the information bit rate equals 135 Mbps. Furthermore, for this scheme $W_{OFDM} \approx$ 36.6 MHz (the nominal bandwidth is 40 MHz).

As an example: Let us consider LTE-systems (Long-Term Evolution). In LTE (from ref. [9]), OFDM is used and $f_\Delta = \frac{1}{T_{obs}} = 15$ kHz (which means that $T_{obs} = 66.67$ µs, see Equation (2.1) in section 2). A typical OFDM symbol interval T_s in LTE is 71.36 µs, and 14 consecutive OFDM signals are then generated every ms. Furthermore, a so-called **resource block** in LTE typically consists of 12 consecutive sub-carrier frequencies (covering 180 kHz) and 7 consecutive OFDM symbol intervals (covering 0.5 ms). Hence, such a resource block contains 84 resource units (i.e. 84 QAM signal points). Within a 20 Mhz bandwidth typically 110 such resource blocks are defined, covering 19.8 MHz and corresponding to K=1320.

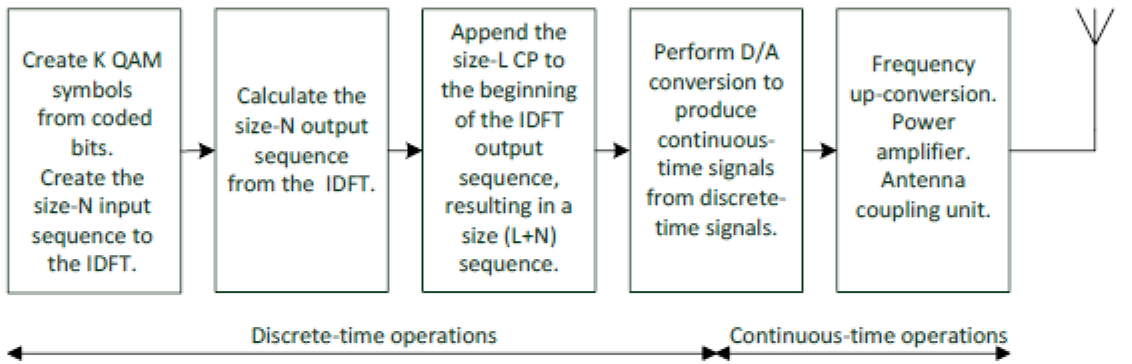


Figure 1. Illustrates the over-all structure and operations performed by an OFDM transmitter within an OFDM symbol interval T_s.

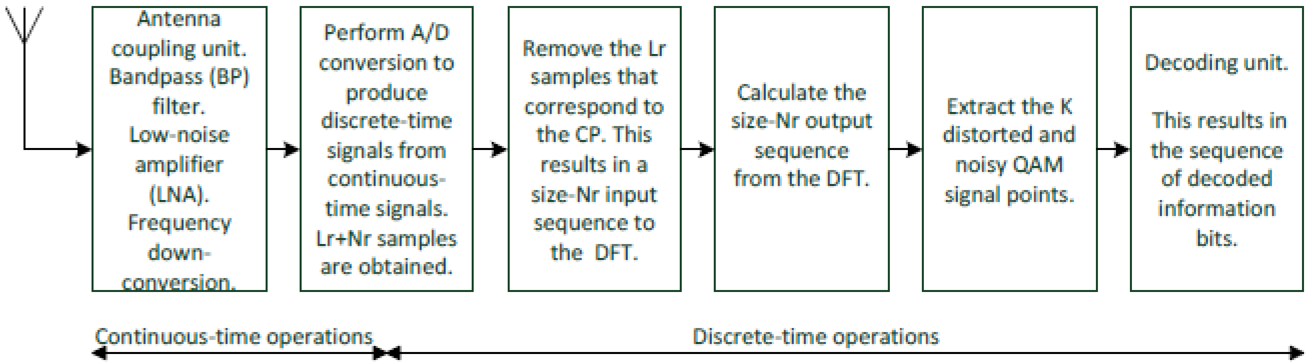


Figure 2. Illustrates the over-all structure and operations performed by an OFDM receiver within an OFDM symbol interval T_s.

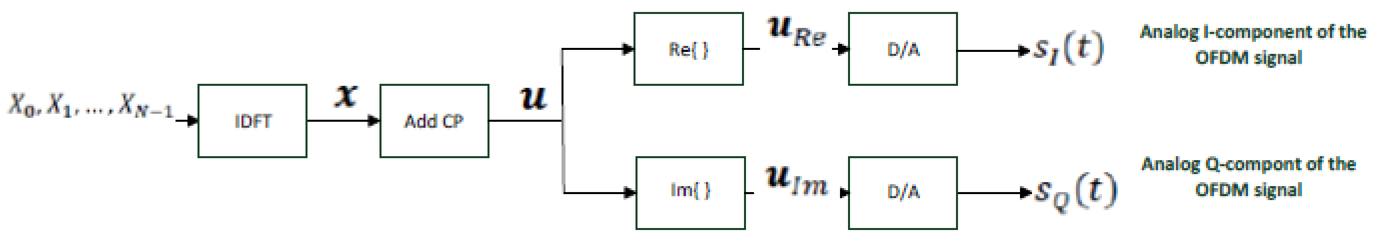


Figure 7. Block diagram illustrating the operations in the digital domain, and the transition to the analog domain. The IDFT is given in Equation (2.30) (and in Equation (2.18)).

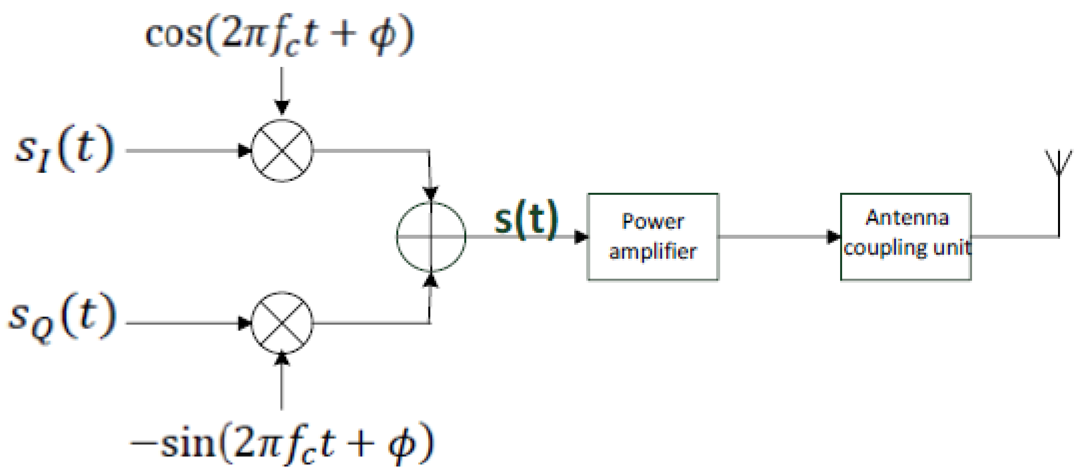


Figure 8. Block diagram illustrating frequency up-conversion (mixer stage) to the carrier frequency (K is odd), the power amplifier, and the antenna coupling unit. The OFDM signal $s(t)$ is given in equation (4.1).

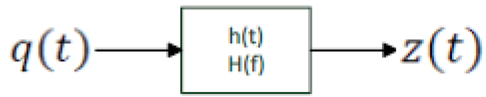


Figure 9. Illustrating a linear time-invariant filter channel.

INPUT OFDM: $As(t) = ARe\{\sum_{k=0}^{K-1} a_k \, e^{j(2\pi f_k t + \theta_k)}\}$, $\quad 0 \leq t \leq T_s \qquad (5.11)$

OUTPUT OFDM: $z(t) = ARe\{\sum_{k=0}^{K-1} a_k H(f_k) \, e^{j(2\pi f_k t + \theta_k)}\}$, $\quad T_{CP} \leq t \leq T_s \qquad (5.13)$

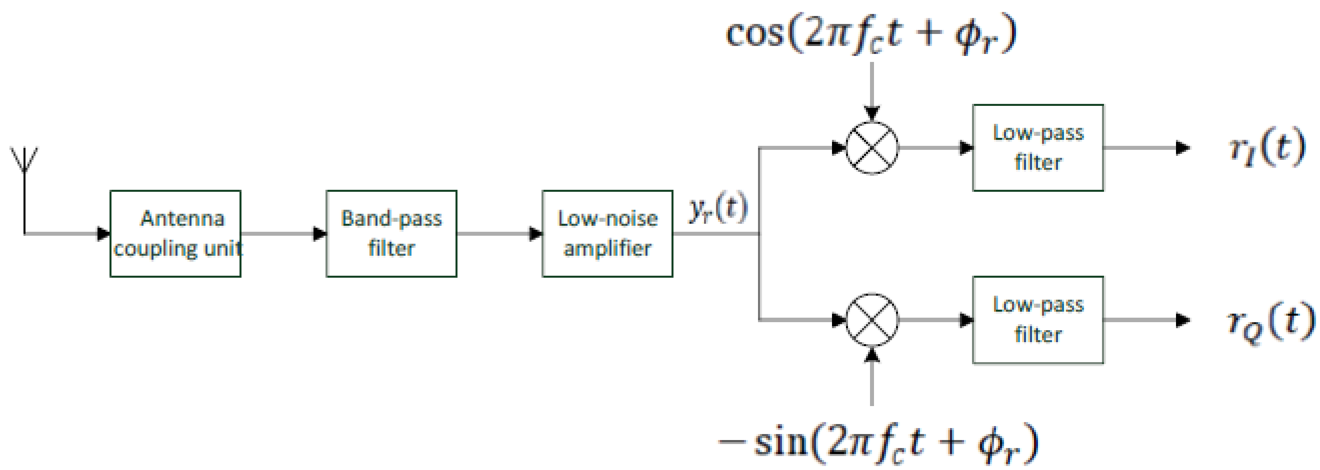


Figure 10. Illustrating the first part of the receiver: the antenna coupling unit, band-pass filter (wide), low-noise amplifier (LNA) and a homodyne unit for frequency down-conversion and extracting the baseband signals $r_I(t)$ and $r_Q(t)$. It is here assumed that K is odd for which $f_{rc} = f_c$.

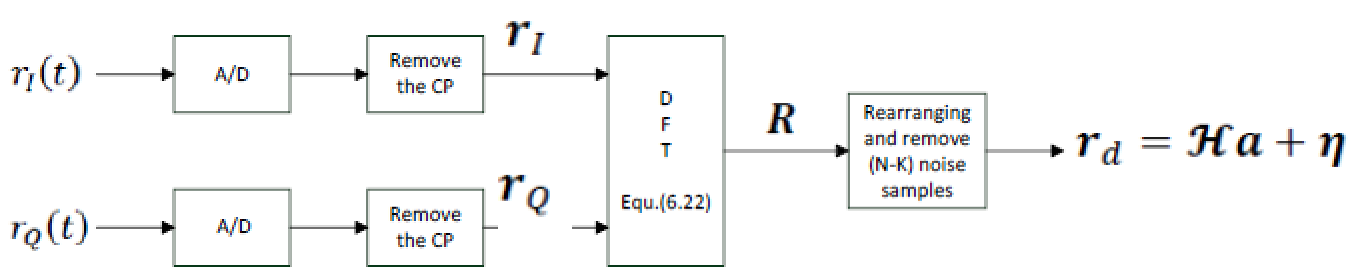


Figure 11. Illustrating sampling, removal of the CP, and the size-N DFT in the receiver to extract the K received distorted and noisy signal points collected in the size-K vector $\boldsymbol{r}_d$.

$$\text{OFDM signal(t)} = g_{rec}(t)\sum_{k=0}^{K-1} Re\{a_k e^{j2\pi f_k t}\} = g_{rec}(t) Re\{\sum_{k=0}^{K-1} a_k e^{j2\pi f_k t}\} =$$

$$= g_{rec}(t) Re\{\sum_{k=0}^{K-1} a_k e^{j2\pi(f_0 + k f_\Delta)t}\} = g_{rec}(t) Re\{(\sum_{k=0}^{K-1} a_k e^{j2\pi(g_0+k) f_\Delta t}) e^{j2\pi f_{rc} t}\} =$$
$$= g_{rec}(t) Re\{(\sum_{k=0}^{K-1} a_k e^{j2\pi g_k f_\Delta t}) e^{j2\pi f_{rc} t}\} \qquad (1.13)$$

$$f_k = f_{rc} + g_k f_\Delta, \quad k = 0,1,\ldots,K-1 \qquad (1.8)$$

The numbers g_k range from g_0 to g_{K-1},

$$g_k: -\frac{K-1}{2} = g_0, \ldots, -1,0,1, \ldots, \frac{K-1}{2} = g_{K-1} \quad \text{if K is odd} \qquad (1.11)$$

$$g_k: -\frac{K-2}{2} = g_0, \ldots, -1,0,1, \ldots, \frac{K}{2} = g_{K-1} \quad \text{if K is even} \qquad (1.12)$$

$$y(t) = Re\{(\sum_{k=0}^{K-1} a_k e^{j2\pi g_k f_\Delta t})e^{j2\pi f_{rc} t}\} = Re\{x(t)e^{j2\pi f_{rc} t}\} \qquad (2.2)$$

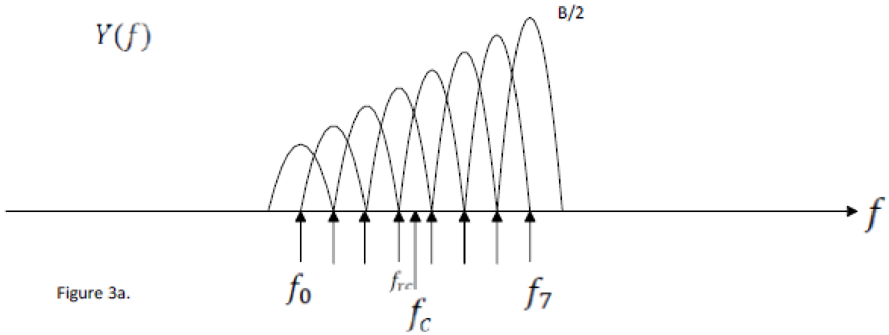


Figure 3a.

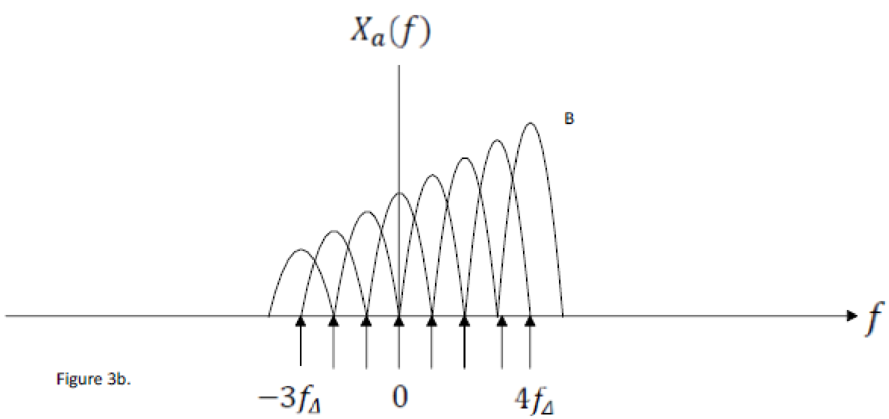


Figure 3b.

$$x(t) = x_{Re}(t) + jx_{Im}(t) = \sum_{k=0}^{K-1} a_k e^{j2\pi g_k f_\Delta t}, \qquad 0 \leq t \leq T_{obs} \qquad (2.3)$$

Observe in Equation (2.3) that the QAM symbol a_k (k=0,1,…,(K-1)), is carried by *the baseband sub-carrier frequency* $g_k f_\Delta$ in the complex baseband OFDM signal $x(t)$!

The high-frequency OFDM signal $y(t)$ in Equation (2.2) can be written as,

$$y(t) = Re\{x(t)e^{j2\pi f_{rc}t}\} = x_{Re}(t)\cos(2\pi f_{rc}t) - x_{Im}(t)\sin(2\pi f_{rc}t) \qquad (2.5)$$

Equation (2.5) is an important relationship since it shows that the OFDM-signal $y(t)$ is ***easily implemented*** as soon as we have created the real part $x_{Re}(t)$ and the imaginary part $x_{Im}(t)$ of $x(t)$.

We should therefore focus on creating $\boldsymbol{x(t)}$***, since*** $\boldsymbol{x_{Re}(t)}$ ***and*** $\boldsymbol{x_{Im}(t)}$ ***then are easy to find.***

Let us now consider sampling the complex signal $x(t)$ in Equation (2.3) every $\frac{T_{obs}}{N}$ second, i.e. with N samples within the time-interval $0 \le t < T_{obs}$. This corresponds to a sampling frequency f_{samp} equal to,

$$f_{samp} = N/T_{obs} = Nf_{\Delta} > Kf_{\Delta} \qquad (2.12)$$

samples per second, and **N should be chosen larger than K**, and large enough such that the sampling theorem can be considered to be sufficiently fulfilled.

Let the column vector (or discrete-time signal) $\boldsymbol{x}$ contain the N time-domain complex samples $x_0, x_1, \dots, x_{N-1}$, of the signal $x(t)$ in Equation (2.3). This means that the sample x_n is,

$$x_n = x(nT_{obs}/N) = \sum_{k=0}^{K-1} a_k e^{j2\pi g_k n/N} \qquad n = 0,1 \dots, (N-1) \qquad (2.13)$$

Observe that the right hand side of Equation (2.13) actually gives us a way to construct the desired samples $x_0, x_1, \dots, x_{N-1}$ **of the complex baseband OFDM signal** $\boldsymbol{x(t)}$ **(i.e. without actually sample the signal** $\boldsymbol{x(t)}$**)!**

The **DFT** is closely connected to the Fourier transform $X(\nu)$ of the discrete-time signal $\boldsymbol{x}$ in Equation (2.13). $X(\nu)$ is defined by, see ref. [1],

$$X(\nu) = \sum_{n=0}^{N-1} x_n e^{-j2\pi\nu n} \qquad (2.14)$$

Note in Equation (2.14) that the Fourier transform $X(\nu)$ is ***periodic in ν with period 1***. Furthermore, the variable ν can be viewed as a **normalized frequency variable**, $\nu = f/f_{samp}$. The periodicity in ν is illustrated in Figure 4.

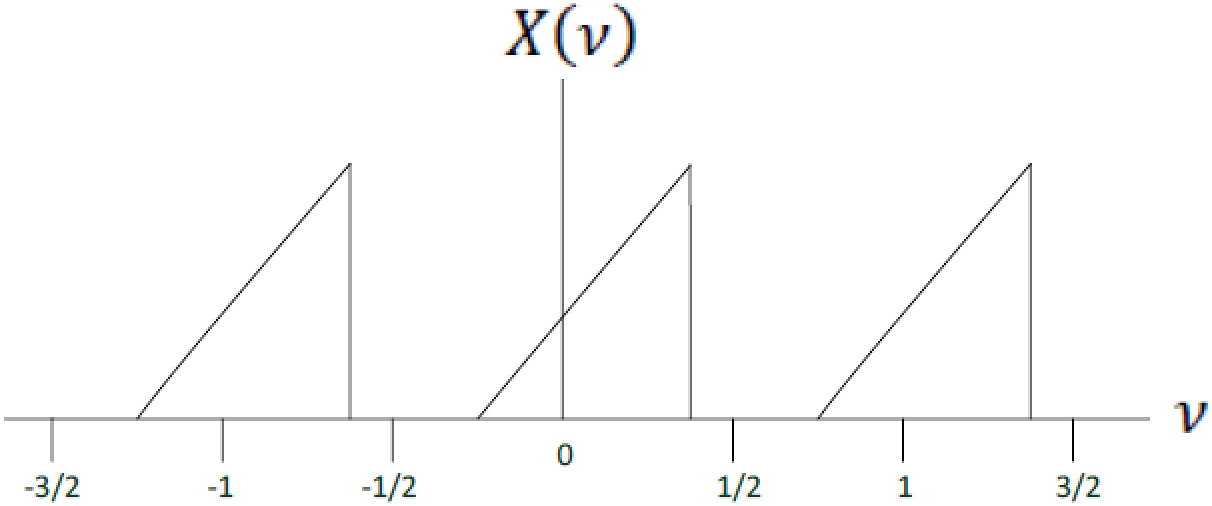


Figure 4. Illustrating that $X(\nu)$ is periodic in ν with period 1. The shape of $X(\nu)$ in this figure is an example of a Fourier transform of a discrete-time complex signal.

$$x_n = x(nT_{obs}/N) = \sum_{k=0}^{K-1} a_k e^{j2\pi g_k n/N} \qquad n = 0,1 \dots, (N-1) \qquad (2.13)$$

$$X(\nu) = \sum_{n=0}^{N-1} x_n e^{-j2\pi \nu n} \qquad (2.14)$$

Furthermore, let X_m denote the **frequency-domain sample** of $X(\nu)$ at $\nu = m/N$, defined by

$$X_m = X(\nu = m/N) = \sum_{n=0}^{N-1} x_n e^{-j2\pi mn/N}, \quad m = 0,1, \dots, N-1 \qquad \textbf{(DFT)} \qquad (2.15)$$

This is the definition (see ref. [1]) of the size-N **DFT** (Discrete Fourier Transform) of the sequence $\boldsymbol{x}$.

However, for the moment we are particularly interested in the size-N **IDFT** (Inverse Discrete Fourier transform) which is defined by (see ref. [1]),

$$x_n = \frac{1}{N} \sum_{m=0}^{N-1} X_m e^{j2\pi mn/N}, \qquad n = 0,1, \dots, N-1 \qquad \textbf{(IDFT)} \qquad (2.16)$$

Hence, as soon as we have determined the samples in the frequency domain $\boldsymbol{X_0, X_1, \dots, X_{N-1}}$ we should use them in the size-N IDFT in Equation (2.16) to create the desired sequence of time-domain samples $\boldsymbol{x}$! *The values X_m will be determined in step 3.*

In practice, **N is chosen to be a power of 2** since fast Fourier transform (FFT) algorithms then can be used to significantly speed up the calculations in Equations (2.15) - (2.16).

Step 3: The relation between the sequences $a_0, a_1, \dots, a_{K-1}$ and $X_0, X_1, \dots, X_{N-1}$.

Let us use Equation (2.13) to establish the connection between the sequences $a_0, a_1, \dots, a_{K-1}$ and $X_0, X_1, \dots, X_{N-1}$. We rewrite Equation (2.13) in the following way,

$$x_n = x\left(\frac{nT_{obs}}{N}\right) = \sum_{k=0}^{K-1} a_k e^{\frac{j2\pi g_k n}{N}} = \sum_{k=0}^{K-1} a_k e^{j2\pi(g_0+k)n/N} =$$

$$= \sum_{k=0}^{-g_0-1} a_k e^{j2\pi(g_0+k+N)n/N} + \sum_{k=-g_0}^{K-1} a_k e^{j2\pi(g_0+k)n/N} =$$

$$= \sum_{m=g_0+N}^{N-1} a_{m-(g_0+N)} e^{j2\pi mn/N} + \sum_{m=0}^{g_{K-1}} a_{m-g_0} e^{j2\pi mn/N} =$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} X_m e^{j2\pi mn/N} \qquad (2.18)$$

Inspection of Equation (2.18) yields the relationships below:

$$X_m = N a_{m-g_0}, \quad \text{if } 0 \le m \le g_{K-1} \qquad (2.19)$$

$$X_m = 0, \quad \text{if } g_{K-1}+1 \le m \le g_0 + N - 1 \qquad (2.20)$$

$$X_m = N a_{m-(g_0+N)}, \text{ if } g_0 + N \le m \le N-1 \qquad (2.21)$$

The last expression in Equation (2.18) is identical to the size-N IDFT in Equation (2.16). The relation between the sequences $a_0, a_1, \dots, a_{K-1}$ and $X_0, X_1, \dots, X_{N-1}$ are given by Equations (2.19) – (2.21).

As an **example** consider a situation with K=53, N=64. In this case $n_{rc} = -g_0 = \frac{K-1}{2} = 26$ and $g_{K-1} = \frac{K-1}{2} = 26$. From Equations (2.19) – (2.21) it is then concluded that the sub-sequence $X_0, X_1, \dots, X_{26}$ contains the QAM signal points $a_{26}, a_{27}, \dots, a_{52}$, the sub-sequence $X_{27}, X_{28}, \dots, X_{37}$ contains only zero values, and the sub-sequence $X_{38}, X_{39}, \dots, X_{63}$ contains the QAM signal points $a_0, a_1, \dots, a_{25}$.

Let us consider another **example** with K=8, N=12. In this case $n_{rc} = -g_0 = \frac{K-2}{2} = 3$ and $g_{K-1} = \frac{K}{2} = 4$. From Equations (2.19) – (2.21) it is then concluded that the sub-sequence $X_0, X_1, X_2 X_3, X_4$ contains the QAM signal points a_3, a_4, a_5, a_6, a_7, the sub-sequence X_5, X_6, X_7, X_8 contains only zero values, and the sub-sequence X_9, X_{10}, X_{11} contains the QAM signal points a_0, a_1, a_2.