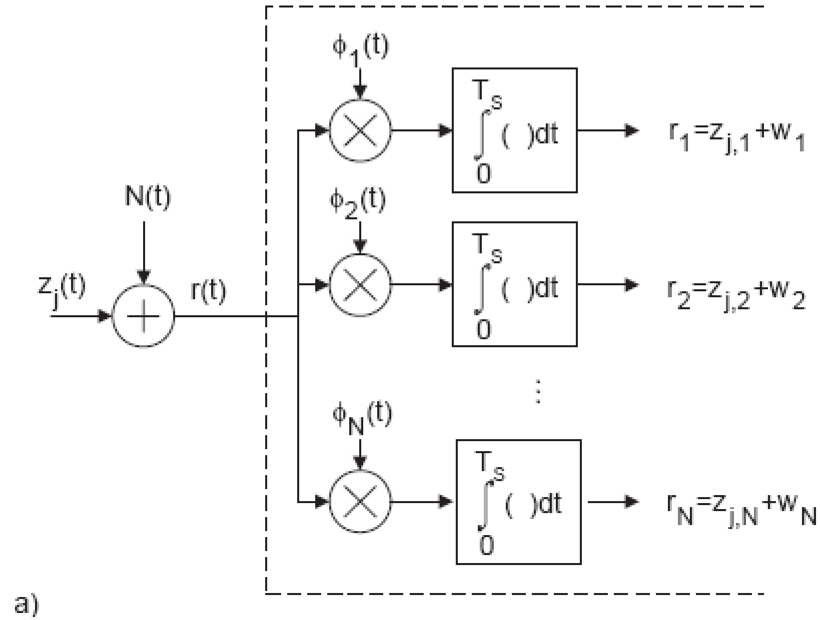


Figure 5.6: a) The first step in the MAP receiver;

$$\int_0^{T_s} z_j(t)\phi_\ell(t)dt = \int_0^{T_s} \sum_{n=1}^{N} z_{j,n}\phi_n(t)\phi_\ell(t)dt = \sum_{n=1}^{N} z_{j,n}\int_0^{T_s}\phi_n(t)\phi_\ell(t)dt = z_{j,\ell} \quad (5.12)$$

After the correlators we obtain a ***received noisy signalpoint*** ***r!***

$$\boxed{\begin{array}{l} E\{w_\ell\}=0 \\ \sigma_\ell^2 = E\{w_\ell^2\} = N_0/2 \\ E\{w_\ell w_m\}=0, \quad \ell \neq m \end{array}} \quad \ell = 1,2,\ldots,N \qquad (5.22)$$

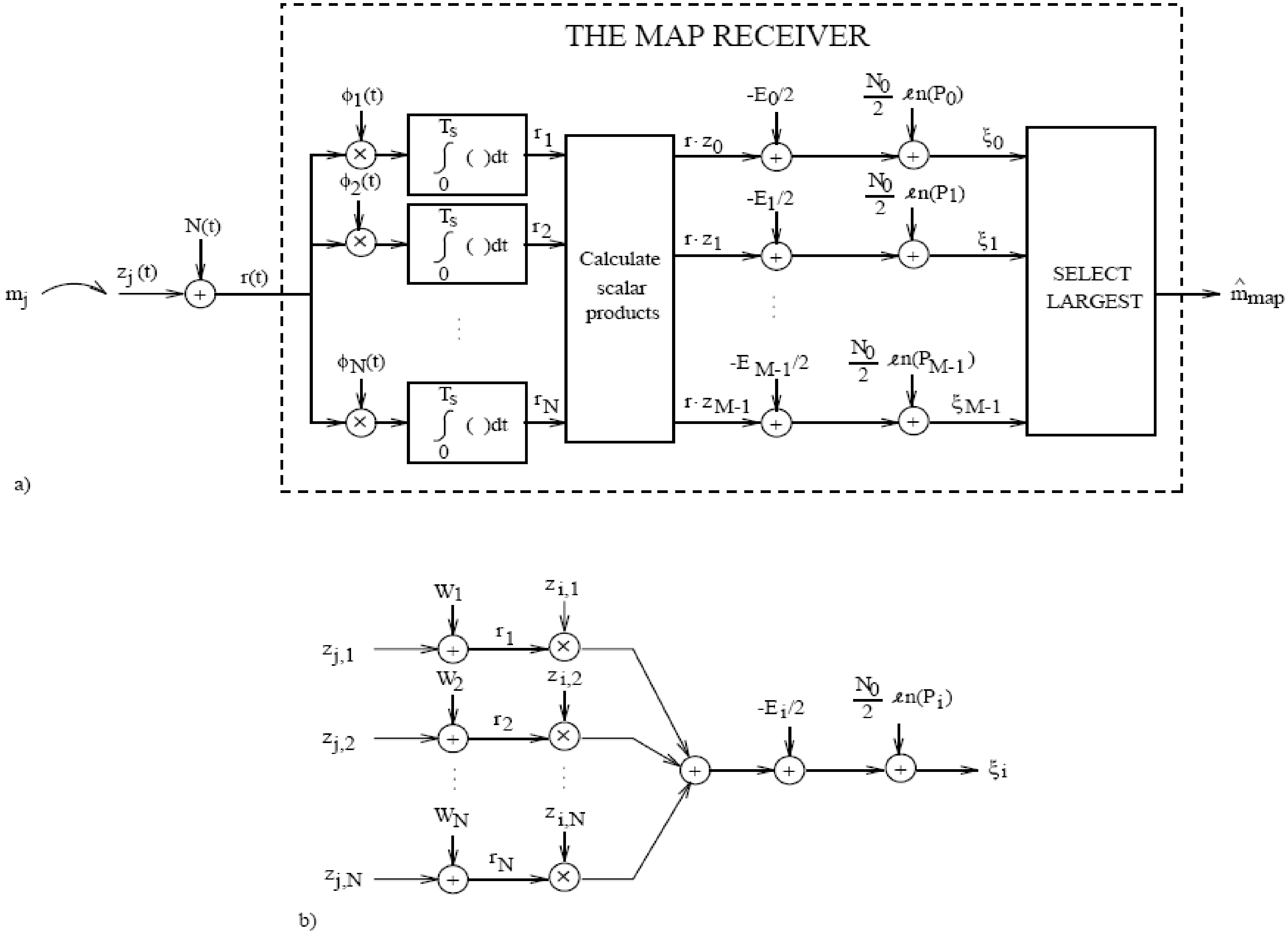


Figure 5.8: a) The MAP receiver; b) A discrete-time model of the decision variable ξ_i.

$$\boxed{\boldsymbol{r} = \boldsymbol{z_j} + \boldsymbol{w}} \tag{5.18}$$

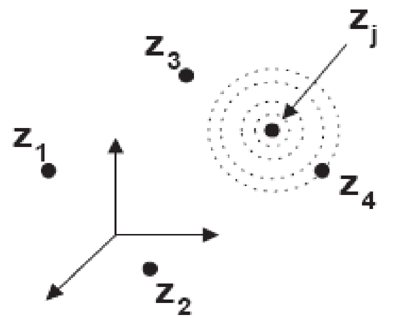


Figure 5.7: Illustrating "the cloud" of noise in $\boldsymbol{r}$ if message m_j is sent.

The distance between the received noisy signal point **r** and the signal point **z**j is:

$$\boxed{D^2_{r,j} = (\boldsymbol{r} - \boldsymbol{z}_j)^{tr}(\boldsymbol{r} - \boldsymbol{z}_j) = \sum_{\ell=1}^{N}(r_\ell - z_{j,\ell})^2} \tag{5.24}$$

The general MAP decision rule in (4.14) can now be formulated as,

$$\boxed{\hat{m}(\boldsymbol{r}) = m_\ell \Leftrightarrow \max_{\{i\}} P_i f_{\boldsymbol{r}|m_i}(\boldsymbol{r}|m_i) = P_\ell f_{\boldsymbol{r}|m_\ell}(\boldsymbol{r}|m_\ell)} \qquad (4.18)$$

$$\boxed{\begin{aligned} & f_{\boldsymbol{r}|m_j}(\boldsymbol{r}|m_j) = \\ & = \frac{e^{-(r_1 - z_{j,1})^2/2\sigma^2}}{\sqrt{2\pi}\ \sigma} \cdot \frac{e^{-(r_2 - z_{j,2})^2/2\sigma^2}}{\sqrt{2\pi}\ \sigma} \ \cdots \ \cdot \frac{e^{-(r_N - z_{j,N})^2/2\sigma^2}}{\sqrt{2\pi}\ \sigma} = \\ & = \frac{e^{-D_{r,j}^2/2\sigma^2}}{(2\pi\sigma^2)^{N/2}}\ , \quad j = 0, 1, \ldots, M-1 \end{aligned}} \qquad (5.23)$$

where $D_{r,j}$ denotes the Euclidean distance in signal space between the received noisy vector $\boldsymbol{r}$ and the message point $\boldsymbol{z}_j$, MAP decision rule:

$$\hat{m}(\boldsymbol{r}) = m_\ell \Leftrightarrow \min_{\{i\}} \{D^2_{r,i} - N_0 \ell n(P_i)\} = D^2_{r,\ell} - N_0 \ell n(P_\ell) \quad (5.25)$$

$$\Updownarrow$$

$$\max_{\{i\}} \{\boldsymbol{r}^{tr} \boldsymbol{z}_i + c_i\} = \boldsymbol{r}^{tr} \boldsymbol{z}_\ell + c_\ell$$

ML decision rule = minimum distance decision rule:

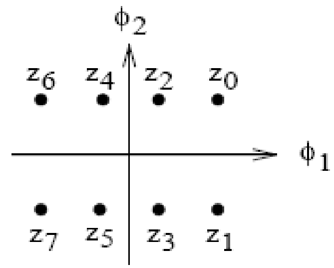
In the MAP decision rule (5.25)–(5.26) we observe that if $P_i = 1/M$, then the terms $N_0 \ell n(P_\ell)$ can be ignored, resulting in the decision rule

$$\hat{m}(r) = m_\ell \Leftrightarrow \min_{\{i\}} D^2_{r,i} = D^2_{r,\ell} \quad (5.28)$$

Hence, ***if $P_i = 1/M$, then the ML decision rule is obtained as the minimum Euclidean distance decision rule.*** Observe also in (5.25) that

EXAMPLE 5.3

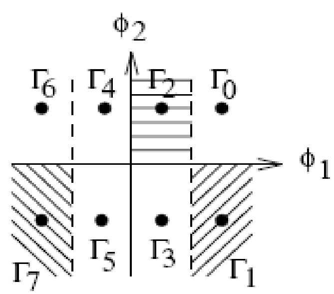
Assume that eight equally likely signal alternatives $\{z_\ell(t)\}_{\ell=0}^{7}$ *are used, and that* $N = 2$*. The possible noiseless values of the received vector* $\boldsymbol{r} = (r_1, r_2)^{tr}$ *in Figure 5.8a are shown below.*



Construct the decision regions used by the MAP-receiver.

Solution:

Since the MAP-receiver in this case is identical with the ML receiver, only the Euclidean distances $D_{r,0}, D_{r,1}, \ldots, D_{r,7}$ *are used in the decision process. The decision boundaries are therefore drawn exactly in the middle between the different signal points (the ML receiver chooses the signal alternative (message) that is closest to the received noisy signal point* $\boldsymbol{r}$*). The result is,*



□

5.1.3 The Symbol Error Probability for M-ary PAM

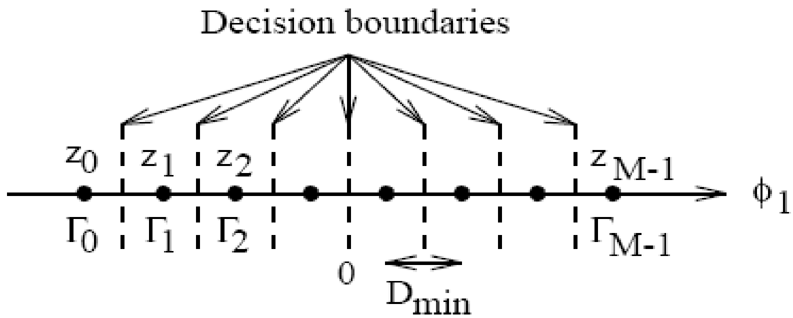


Figure 5.9: The signal space for M-ary PAM with equispaced amplitudes, centered symmetrically around zero (see (5.4)).

$$Prob\{\text{error}|m_0 \text{ sent}\} = Prob\left\{w_1 > \frac{D_{\min}}{2}\right\} =$$

$$= Prob\left\{\frac{w_1}{\sqrt{N_0/2}} > \frac{D_{\min}}{\sqrt{2N_0}}\right\} = Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) \qquad (5.31)$$

$$
\begin{aligned}
Prob\{\text{error}|m_1 \text{ sent}\} &= Prob\left\{w_1 < -\frac{D_{\min}}{2} \text{ or } w_1 > \frac{D_{\min}}{2}\right\} = \\
&= Prob\left\{\frac{w_1}{\sqrt{N_0/2}} < -\frac{D_{\min}}{\sqrt{2N_0}}\right\} + Prob\left\{\frac{w_1}{\sqrt{N_0/2}} > \frac{D_{\min}}{\sqrt{2N_0}}\right\} = \\
&= 2Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) \qquad (5.32)
\end{aligned}
$$

$$
P_s = \sum_{j=0}^{M-1} P_j Prob\{\text{error}|m_j \text{ sent}\}
$$

$$
\boxed{P_s = \frac{2}{M}\,(M-1)Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right)}, \quad \text{M-ary PAM} \qquad (5.35)
$$

P_s is shown in Figure 5.13 on page 362.

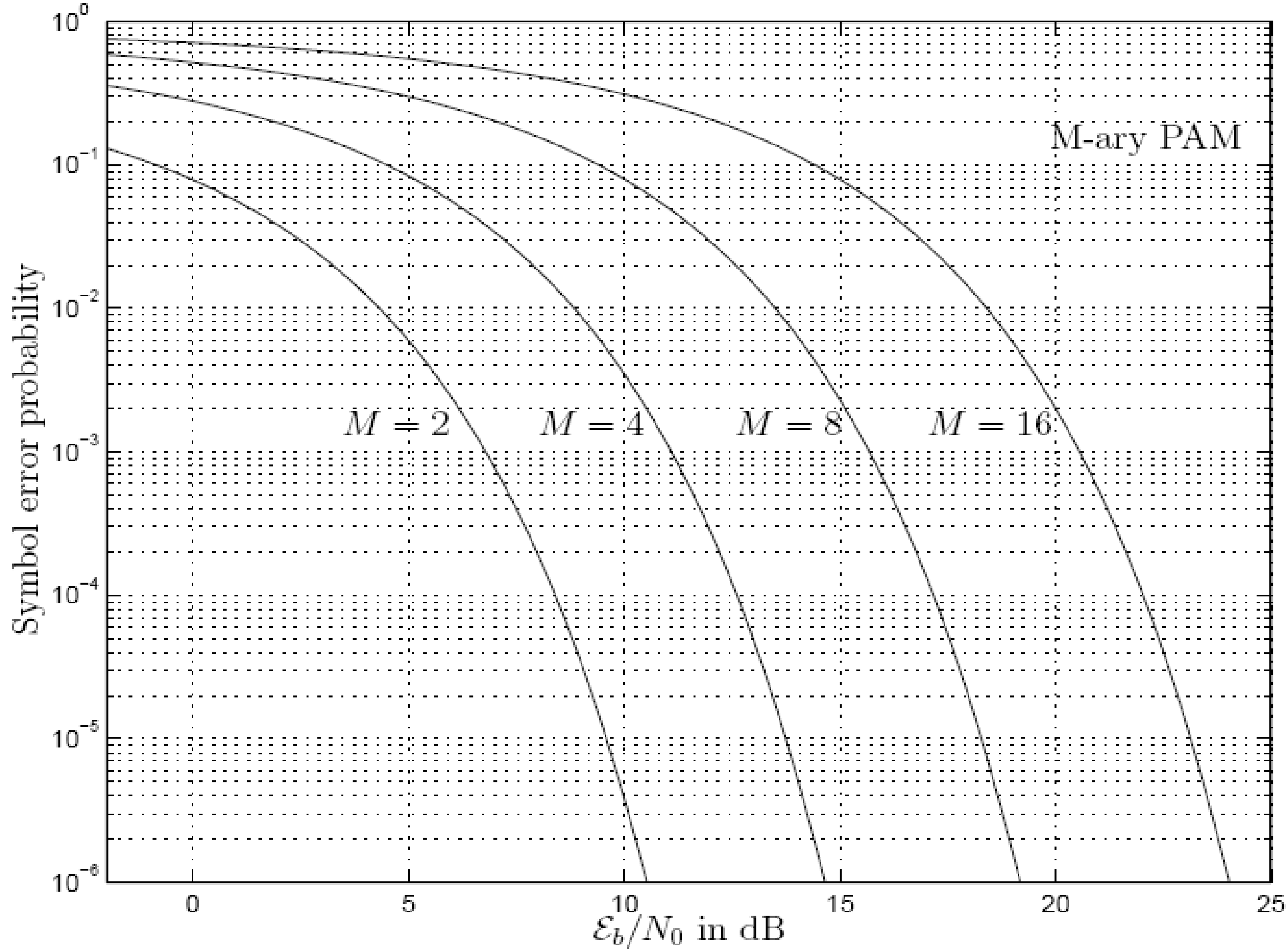
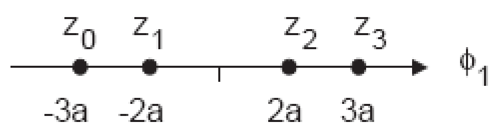


Figure 5.13: The symbol error probability for M-ary PAM, $M = 2, 4, 8, 16$, see Table 5.1. The specific assumptions are given in Subsection 2.4.1.1, and in Subsection 5.1.3.



5.14 $\Pr\{\text{error}|m_0 \text{ sent}\} = \Pr\{w_1 > a/2\} =$

$$= \Pr\left\{\frac{w_1}{\sqrt{N_0/2}} > \frac{a/2}{\sqrt{N_0/2}}\right\} = Q\left(\sqrt{\frac{a^2}{2N_0}}\right)$$

$\Pr\{\text{error}|m_1 \text{ sent}\} = \Pr\{w_1 < -a/2 \text{ or } w_1 > 2a\}$.
So,we obtain

$$P_s = \sum_{j=0}^{3} P_j \Pr\{\text{error}|m_j \text{ sent}\} = Q\left(\sqrt{\frac{a^2}{2N_0}}\right) + \frac{1}{2}\, Q\left(\sqrt{\frac{8a^2}{N_0}}\right)$$

5.1.4 The Symbol Error Probability for QPSK

$$r(t) = z_j(t) + N(t), \quad 0 \leq t \leq T_s, \quad j = 0, 1, \ldots, M-1 \tag{5.13}$$

$$r_1 = z_{j,1} + w_1 \tag{5.36}$$

$$r_2 = z_{j,2} + w_2 \tag{5.37}$$

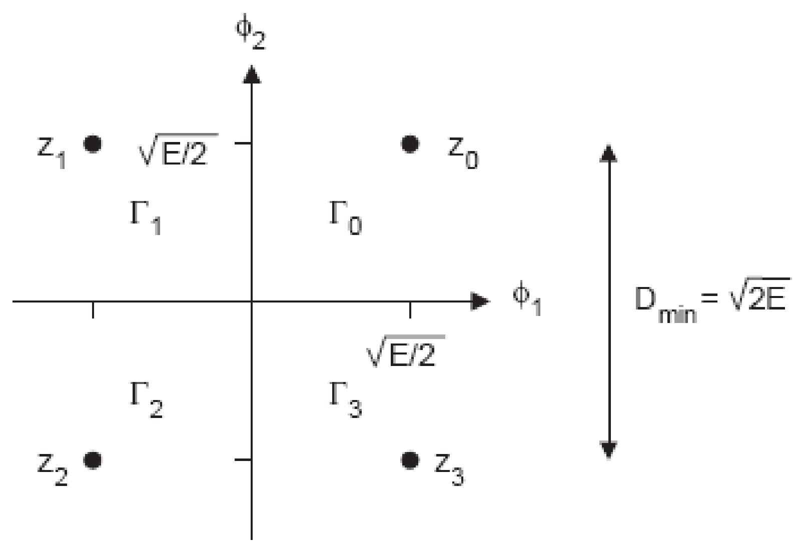


Figure 5.10: The signal space for QPSK if $\nu_\ell = (2\pi \frac{\ell}{M} + \pi/4)$ (see (5.4)).

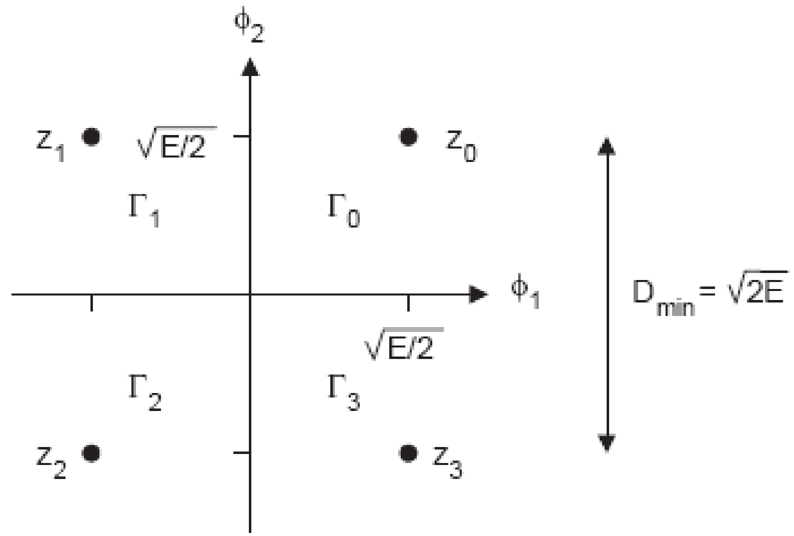


Figure 5.10: The signal space for QPSK if $\nu_\ell = (2\pi \frac{\ell}{M} + \pi/4)$ (see (5.4)).

$$
\begin{aligned}
& Prob\{\text{error}|m_0 \text{ sent}\} = 1 - Prob\{\text{correct decision}|m_0 \text{ sent}\} = \\
& = 1 - Prob\left\{ w_1 \geq -\frac{D_{\min}}{2} \; , \; w_2 \geq -\frac{D_{\min}}{2} \right\} = \\
& = 1 - Prob\left\{ w_1 \geq -\frac{D_{\min}}{2} \right\} \; Prob\left\{ w_2 \geq -\frac{D_{\min}}{2} \right\} = \\
& = 1 - \left[1 - Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}} \right) \right]^2 = 2Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}} \right) - Q^2\left(\sqrt{\frac{D_{\min}^2}{2N_0}} \right) \overset{\text{symmetry}}{=} \\
& = Prob\{\text{error}|m_j \text{ sent}\} \; , j = 0, 1, 2, 3
\end{aligned}
\tag{5.38}
$$

$$\boxed{P_s = 2Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) - Q^2\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right)}, \quad QPSK \qquad (5.39)$$

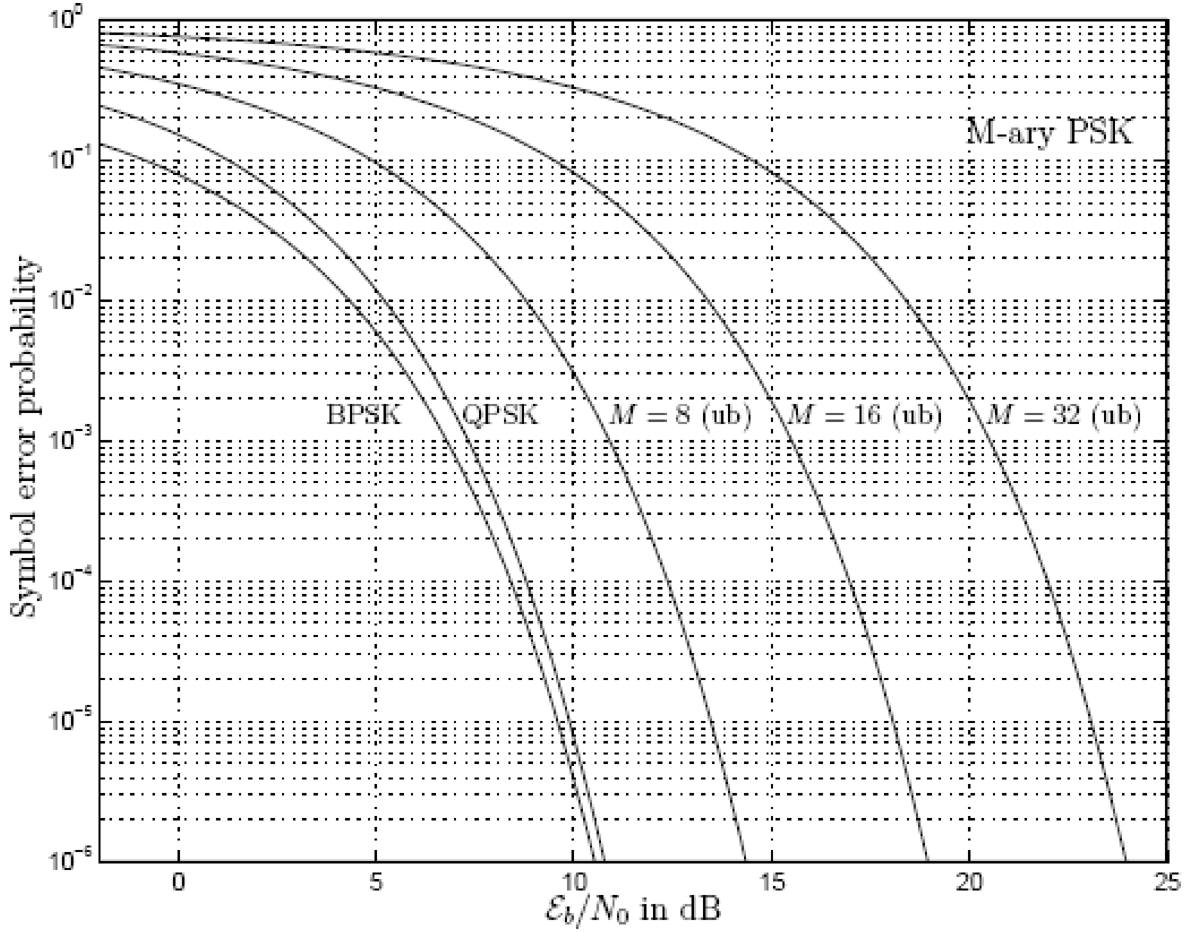


Figure 5.14: The symbol error probability for M-ary PSK, $M = 2, 4, 8, 16, 32$, see Table 5.1. In this figure upper bounds are denoted (ub). See also Subsection 5.1.5.

The bit error probability for QPSK?

5.1.5 The Symbol Error Probability for M-ary PSK

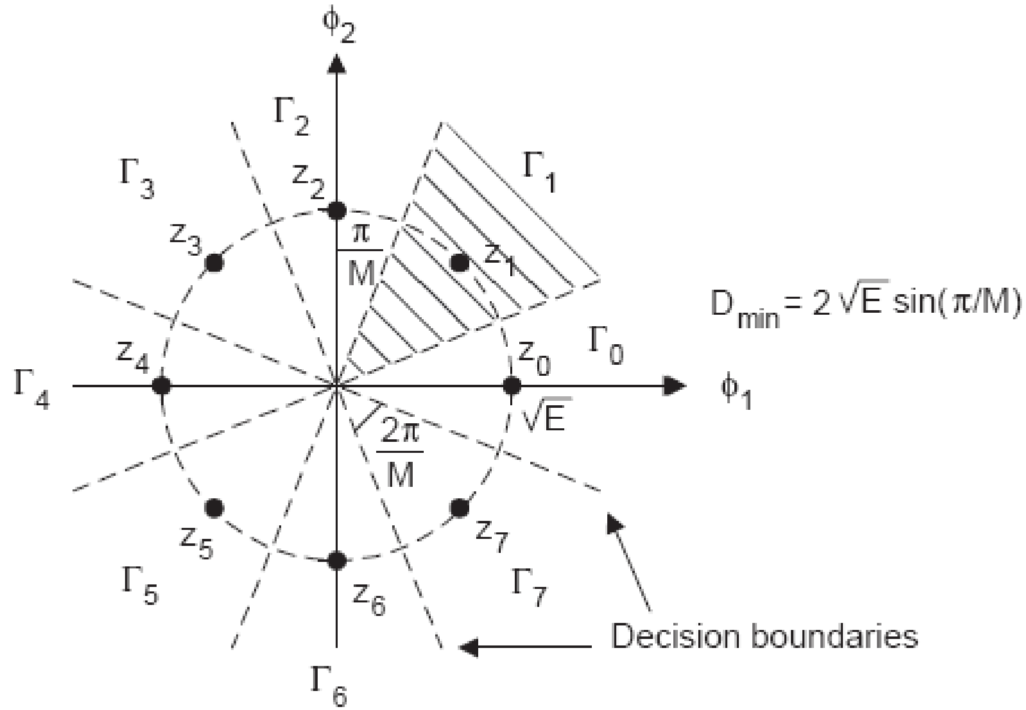


Figure 5.11: The signal space for M-ary PSK if $\nu_\ell = 2\pi\ell/M$ (see (5.4)). $M = 8$ in this figure.

$$\boxed{\begin{array}{l} Q\left(\sqrt{\frac{D^2_{\min}}{2N_0}}\right) \leq P_s < 2Q\left(\sqrt{\frac{D^2_{\min}}{2N_0}}\right) \\ D^2_{\min} = 4E\sin^2(\pi/M) \end{array}}\,, \text{ M-ary PSK} \qquad (5.43)$$

5.1.6 The Symbol Error Probability for M-ary QAM

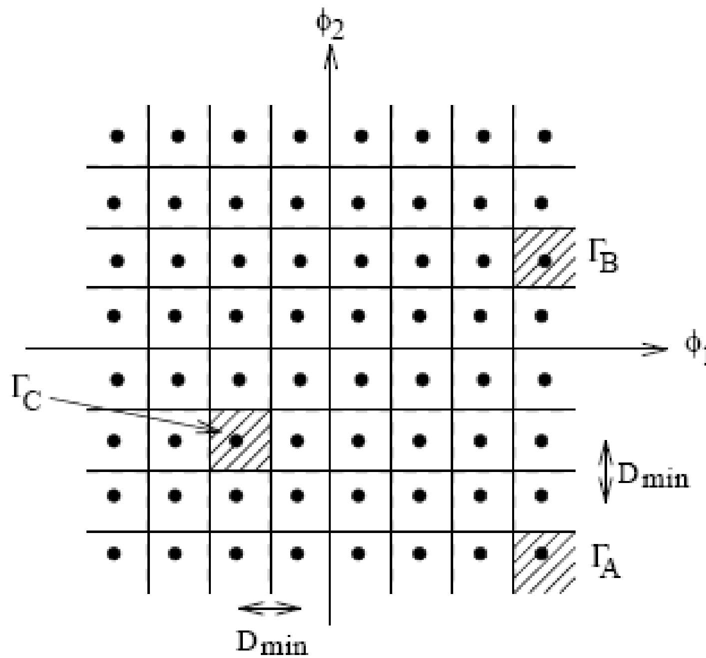


Figure 5.12: The signal space for M-ary QAM (compare with (5.4), see also Subsection 2.4.5.1). M=64 in this figure.

Γ_A: Compare with (5.39).

$$Prob\{\text{error}|m_A \text{ sent}\} = 2Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) - Q^2\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) \tag{5.46}$$

Γ_B:

$$\begin{aligned} &Prob\{\text{error}|m_B \text{ sent}\} = \\ &= 1 - Prob\left\{w_1 > -\frac{D_{\min}}{2}, -\frac{D_{\min}}{2} \le w_2 \le \frac{D_{\min}}{2}\right\} = \\ &= 1 - \left(1 - Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right)\right)\left(1 - 2Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right)\right) = \\ &= 3Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) - 2Q^2\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) \end{aligned} \tag{5.47}$$

Γ_C:

$$\begin{aligned} &Prob\{\text{error}|m_C \text{ sent}\} = \\ &= 1 - Prob\left\{-\frac{D_{\min}}{2} \le w_1 \le \frac{D_{\min}}{2}, -\frac{D_{\min}}{2} \le w_2 \le \frac{D_{\min}}{2}\right\} = \\ &= 1 - \left(1 - 2Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right)\right)^2 = \\ &= 4Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) - 4Q^2\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) \end{aligned} \tag{5.48}$$

$$\boxed{P_s = \frac{4}{\sqrt{M}}\,(\sqrt{M}\text{-}1)\; Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) - \frac{4}{M}\,(\sqrt{M}\text{-}1)^2\; Q^2\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right)}, \quad \text{M-ary QAM} \tag{5.50}$$

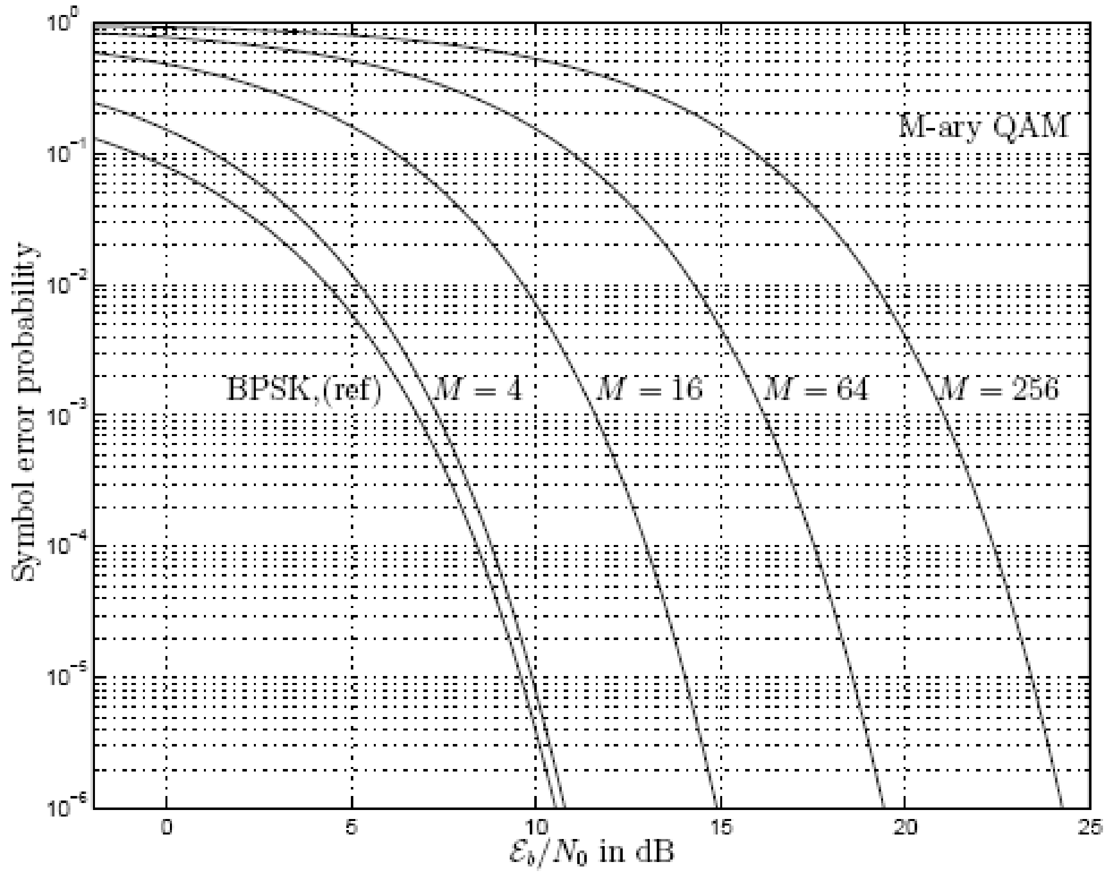


Figure 5.15: The symbol error probability for M-ary QAM, $M = 4, 16, 64, 256$, see Table 5.1. The specific assumptions are given in Subsection 2.4.5.1 and in Subsection 5.1.6. The bit error probability for BPSK is also given as a reference $(= Q(\sqrt{2\mathcal{E}_b/N_0}))$.