Study week 3.

3.4.1 Low-Rate QAM-Type of Input Signals

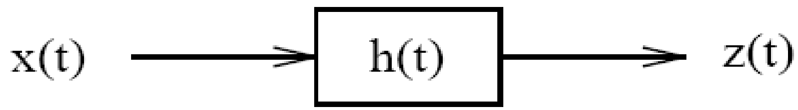


Figure 3.11: Bandpass filtering.

$$x(t) = x_I(t)\cos(\omega_c t) - x_Q(t)\sin(\omega_c t) = Re\{\tilde{x}(t)e^{j\omega_c t}\} \qquad (3.103)$$

$$\tilde{x}(t) = x_I(t) + jx_Q(t) \qquad (3.104)$$

This complex signal contains the information!

$$x(t) = x_I(t)\cos(\omega_c t) - x_Q(t)\sin(\omega_c t) = Re\{\tilde{x}(t)e^{j\omega_c t}\} \quad (3.103)$$

$$z(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)Re\{\tilde{x}(t-\tau)e^{j\omega_c(t-\tau)}\}d\tau =$$

$$= Re\left\{e^{j\omega_c t}\int_{-\infty}^{\infty} h(\tau)\tilde{x}(t-\tau)e^{-j\omega_c\tau}d\tau\right\} \quad (3.105)$$

3 assumptions:

1) The duration of the impulse response $h(t)$ can be considered to be equal to T_h. This means that essentially all the energy in $h(t)$ is assumed to be contained within the time interval $0 \leq t \leq T_h$.

2) The input signal is assumed to be a QAM-type of signal with duration $T = T_s$:

$$x(t) = \begin{cases} 0 & , \quad t < 0 \\ A\cos(\omega_c t) - B\sin(\omega_c t) = \sqrt{A^2+B^2}\cos(\omega_c t + \nu) & , \quad 0 \leq t \leq T_s \\ 0 & , \quad t > T_s \end{cases} \quad (3.106)$$

3) $T_s > T_h$ (“low” signaling rate).

$$\tilde{x}(t) = \begin{cases} A + jB = \sqrt{A^2 + B^2}\, e^{j\nu} & , \quad 0 \le t \le T_s \\ 0 & , \quad \text{otherwise} \end{cases} \tag{3.108}$$

$\boldsymbol{T_h \le t \le T_s}$ **:**

$$\begin{aligned} z(t) &= Re\left\{ e^{j\omega_c t} \int_0^{T_h} h(\tau)\sqrt{A^2+B^2}\, e^{j\nu} e^{-j\omega_c \tau} d\tau \right\} = \\ &= Re\{\sqrt{A^2+B^2}\, e^{j\nu} \cdot H(f_c) e^{j\omega_c t}\} = \\ &= |H(f_c)|\sqrt{A^2+B^2}\cos(\omega_c t + \nu + \phi(f_c)) = A_z \cos(\omega_c t) - B_z \sin(\omega_c t) \end{aligned} \tag{3.109}$$

Hence, a QAM-signal at the output in this time interval!

However, **attenuation and rotation** compared with the input! Compare with the input x(t) in (3.106)!

$$\begin{aligned} A_z + jB_z &= (A + jB)H(f_c) = \sqrt{A^2+B^2}|H(f_c)|e^{j(\nu+\phi(f_c))} = \\ &= (A + jB)(H_{Re}(f_c) + jH_{Im}(f_c)) \end{aligned} \tag{3.110}$$

$$A_z + jB_z = (A + jB)H(f_c) = \sqrt{A^2 + B^2}|H(f_c)|e^{j(\nu+\phi(f_c))} = \\ = (A + jB)(H_{Re}(f_c) + jH_{Im}(f_c)) \quad (3.110)$$

A COMPACT MODEL WITH A COMPLEX CHANNEL PARAMETER!!

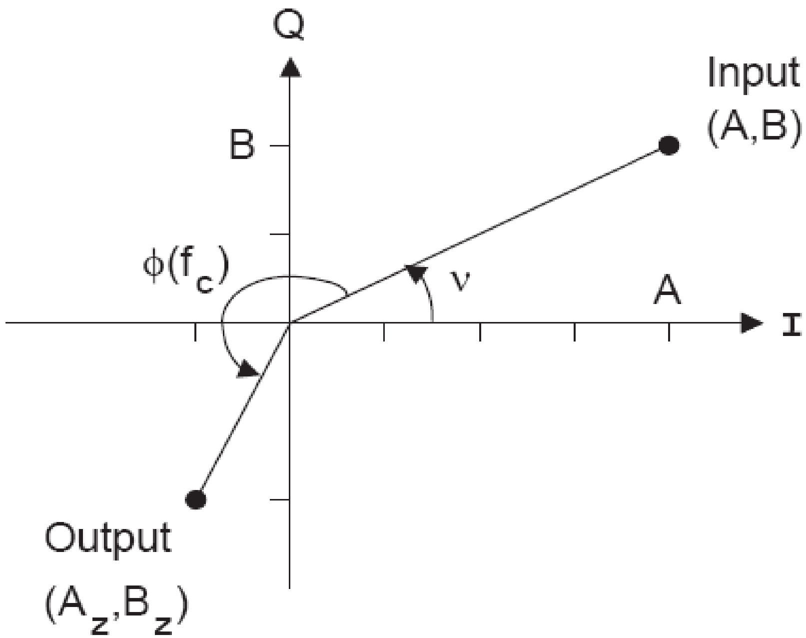


Figure 3.13: Illustrating that the input I-Q amplitudes (A,B) are scaled and rotated by the channel $H(f)$, see (3.109) and (3.110).

$$
z(t) = \begin{cases} 0 & , \quad t < 0 \\ \text{“non-stationary transient” starting interval} & , \quad 0 \leq t \leq T_h \\ |H(f_c)|\sqrt{A^2+B^2}\ \cos(\omega_c t + \nu + \phi(f_c)) & , \quad T_h \leq t \leq T_s \\ \text{“non-stationary transient” ending interval} & , \quad T_s \leq t \leq T_s + T_h \\ 0 & , \quad t > T_s + T_h \end{cases}
$$

and within $T_h \leq t \leq T_s,\ A_z + jB_z = (A + jB)H(f_c)$

(3.111)

An important result here is that the input QAM signal $x(t)$ in (3.106) is changed to a new QAM signal by $|H(f_c)|$ and $\phi(f_c)$ in the interval $T_h \leq t \leq T_s$, see also Figure 3.13 and (3.110) how the I-Q components are changed. Furthermore, in OFDM applications the signaling rate $1/T_s$ is low such that $T_s \gg T_h$, and many QAM signals with different carrier frequencies are sent in parallel. ***Due to linearity, the result in (3.111) can be applied to each QAM signal in the OFDM signal by replacing f_c with f_n.*** In OFDM applications the receiver uses the time interval $\Delta_h \leq t \leq T_s$ for detection of the output QAM signals, and the duration of this observation interval is denoted $T_{obs} = T_s - \Delta_h$ (compare with (2.110) on page 51, and $T_h \leq \Delta_h$).

So, the n:th QAM signal constellation in a sent OFDM signal is attenuated and rotated by H(fn) which is the value of the channel transfer function H(f) at the carrier frequency fn.

3.4.3 N-Ray Channel Model

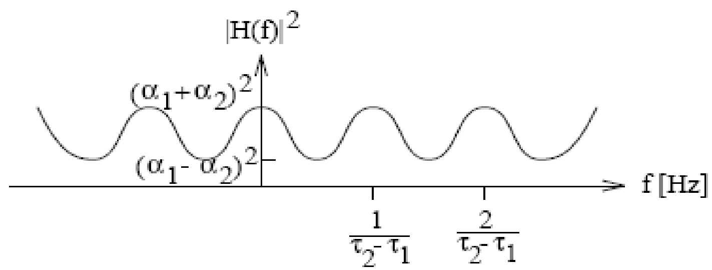
$$z(t) = x(t) * \underbrace{\left(\sum_{i=1}^{N} \alpha_i \delta(t-\tau_i)\right)}_{\text{Impulse response } h(t)} = \sum_{i=1}^{N} \alpha_i x(t-\tau_i) \qquad (3.126)$$

$$H(f) = \mathcal{F}\{h(t)\} = \sum_{i=1}^{N} \alpha_i e^{-j2\pi f \tau_i} \qquad (3.128)$$

So, **H(fc)** is easy to find!

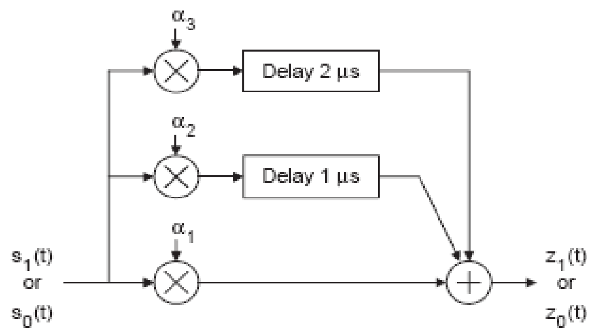
EXAMPLE 3.20

Rough sketch:



It is seen in this figure that the two signal paths add constructively or destructively (fading) depending on the frequency. Furthermore, if $\alpha_1 \approx \alpha_2$ then $|H(f)|$ is very close to zero at certain frequencies (so-called deep fades)!

EXAMPLE 3.19



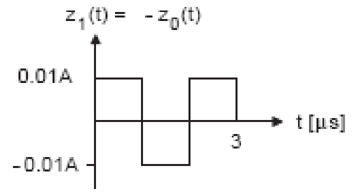
The signal $z_i(t) = s_i(t) * h(t)$ is the output signal corresponding to the input signal $s_i(t)$, $i = 0, 1$. Determine and sketch $z_0(t)$ and $z_1(t)$ if $\alpha_1 = 0.01$, $\alpha_2 = -0.01$, and $\alpha_3 = 0.01$.

Your conclusions concerning choice of bit rate to avoid overlapping signal alternatives after the channel?

Solution:

$$z_\ell(t) = \sum_{i=1}^{3} \alpha_i s_\ell(t - \tau_i) = 0.01 s_\ell(t) - 0.01 s_\ell(t - 10^{-6}) + 0.01 s_\ell(t - 2 \cdot 10^{-6}), \ \ell = 0, 1$$

yields,



Observe that the signal alternatives are changed significantly by the channel (filtering), and that the duration of both signal alternatives is increased from 1 [µs] before the channel, to 3[µs] after the channel!

If the bit rate is reduced to at most $10^6/3$ bps, *then no overlap of signal alternatives will exist after the channel.* □

See (3.109)-(3.110)!

See (3.110)!

5.34 Consider a communication system where N_t M-ary QAM signals are sent simultaneously (from N_t antennas). The n:th transmitted M-ary QAM signal is denoted $s_n(t)$,

$$s_n(t) = A(n)g(t)\cos(\omega_c t) - B(n)g(t)\sin(\omega_c t) \quad (5.133)$$

for $n = 1, 2, \ldots, N_t$. Note that the same carrier frequency is used for all N_t transmitted QAM signals!

The receiver is assumed to have N_r receiving antennas. The received signal $r_k(t)$ at the k:th receiving antenna is here modelled as

$$r_k(t) = \sum_{n=1}^{N_t} \left(\left[H_{k,n}^{Re} A(n) - H_{k,n}^{Im} B(n)\right] g(t)\cos(\omega_c t) - \right.$$
$$\left. - \left[H_{k,n}^{Re} B(n) + H_{k,n}^{Im} A(n)\right] g(t)\sin(\omega_c t) \right) + w_k(t) \quad (5.134)$$

for $k = 1, 2, \ldots, N_r$. The variables $H_{k,n}^{Re}$ and $H_{k,n}^{Im}$ models how the n:th transmitted QAM signal is received at the k:th receiving antenna (attenuation and rotation of the I-Q components).

After I and Q demodulation of $r_k(t)$ to baseband, the receiver obtains the noisy signal space coordinates, here collected in r_k as

$$r_k = \underbrace{\sum_{n=1}^{N_t} \left(H_{k,n}^{Re} A(n) - H_{k,n}^{Im} B(n)\right)}_{\text{received } I \text{ component}} + j \underbrace{\sum_{n=1}^{N_t} \left(H_{k,n}^{Re} B(n) + H_{k,n}^{Im} A(n)\right)}_{\text{received } Q \text{ component}}$$
$$+ \underbrace{(w_k^{Re} + j w_k^{Im})}_{\text{due to AWGN}} \quad (5.135)$$

Note that complex notation ($j^2 = -1$) is used in (5.135)!

Let us now introduce the complex notations:

$$\begin{aligned} d_n &= A(n) + jB(n) \\ \alpha_{k,n} &= H_{k,n}^{Re} + jH_{k,n}^{Im} \\ w_k &= w_k^{Re} + j w_k^{Im} \end{aligned} \quad (5.136)$$

Then (5.135) can be formulated as,

$$r_k = \sum_{n=1}^{N_t} \alpha_{k,n} d_n + w_k \quad ,k = 1, 2, \ldots, N_r \tag{5.137}$$

A compact formulation is now obtained as

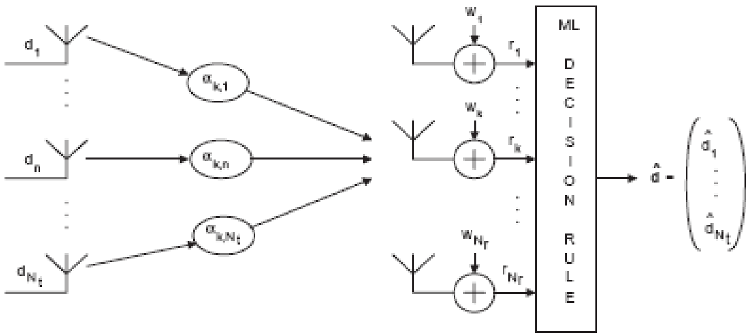
$$\boldsymbol{r} = \begin{pmatrix} r_1 \\ \vdots \\ r_{N_r} \end{pmatrix} = \boldsymbol{A} \begin{pmatrix} d_1 \\ \vdots \\ d_{N_t} \end{pmatrix} + \begin{pmatrix} w_1 \\ \vdots \\ w_{N_r} \end{pmatrix} = \boldsymbol{A}\boldsymbol{d} + \boldsymbol{w} \tag{5.138}$$

Very important!

where the $N_r \times N_t$ matrix $\boldsymbol{A}$ contains the channel coefficients $\{\alpha_{k,n}\}$. The relationship in (5.138) is a basic model in so-called multiple-input multiple-output (MIMO) systems.

The MIMO model is illustrated in the figure below,

MIMO MODEL



SISO
SIMO
MISO
MIMO
Diversity gain
Spatial multiplexing gain

$$r_k = \sum_{n=1}^{N_t} \alpha_{k,n} d_n + w_k$$

$$\boldsymbol{r} = \begin{pmatrix} r_1 \\ \vdots \\ r_{N_r} \end{pmatrix} = \boldsymbol{A} \begin{pmatrix} d_1 \\ \vdots \\ d_{N_t} \end{pmatrix} + \begin{pmatrix} w_1 \\ \vdots \\ w_{N_r} \end{pmatrix} = \boldsymbol{A}\boldsymbol{d} + \boldsymbol{w}$$

z=Ad is the received signalpoint and w is the additive noise vector.

64-QAM+Nt=8 (48bits): ML symbol decision rule ....?

Chapter 8

Trellis-coded Signals

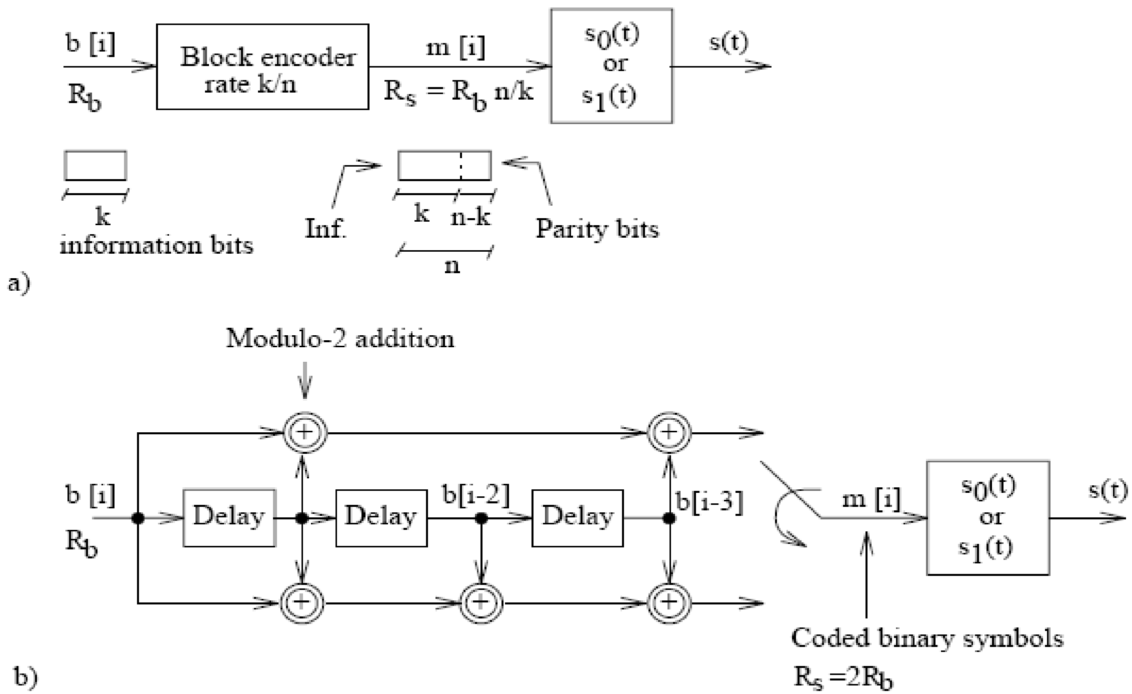


Figure 8.1: a) Block coding, $r_c = k/n$. b) Convolutional coding, $r_c = 1/2$.

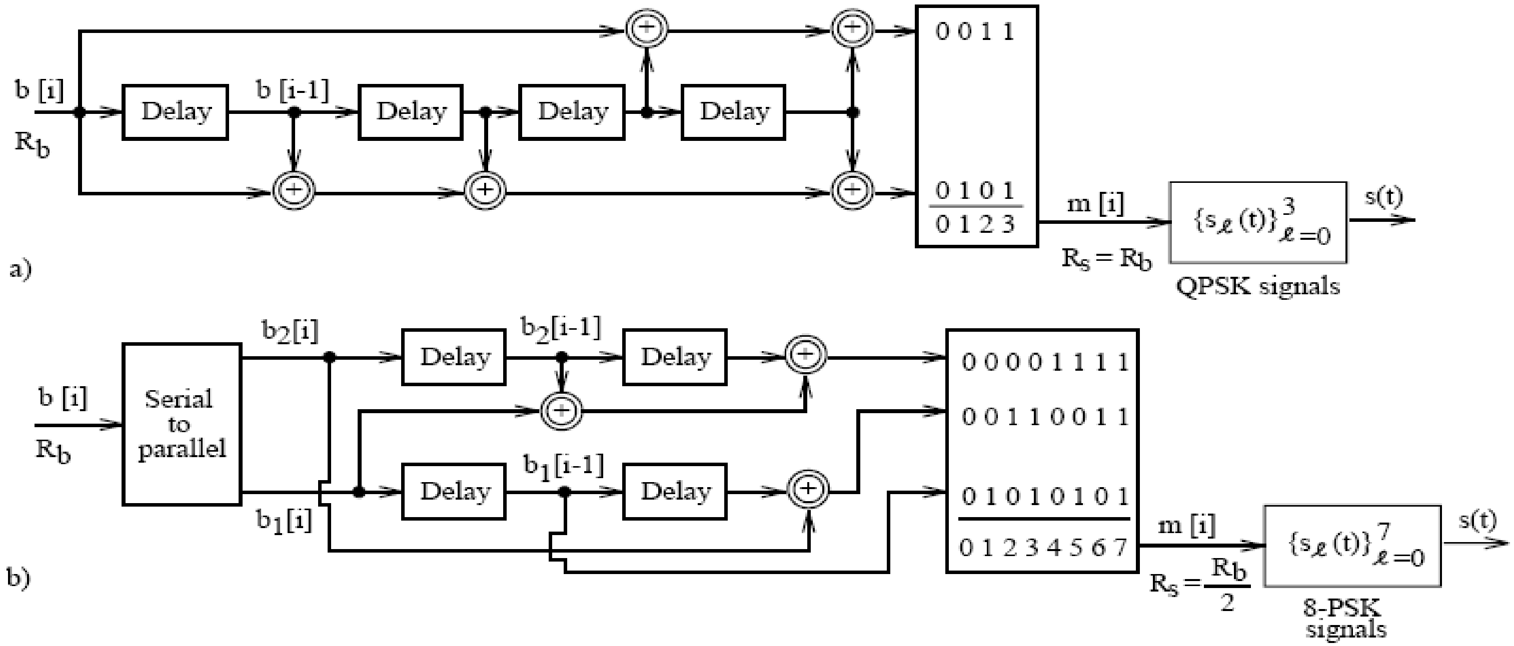
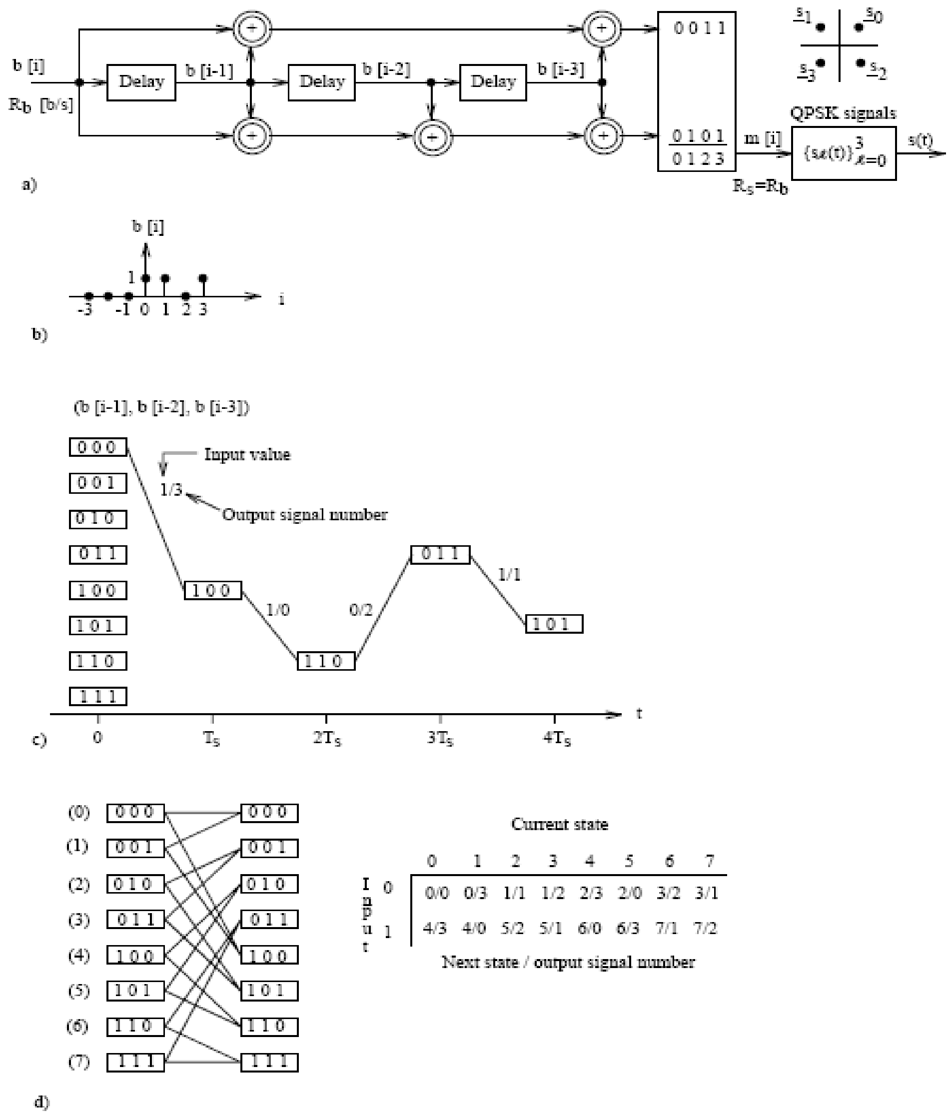
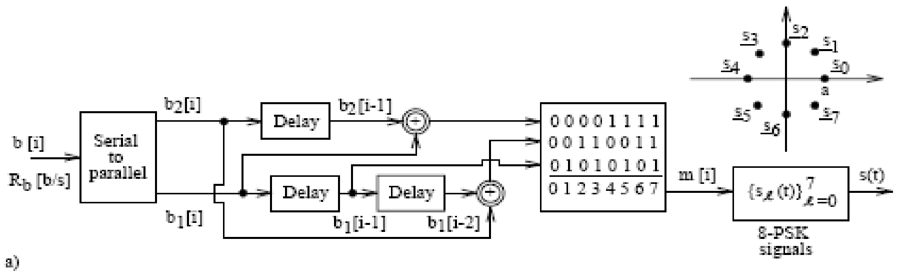


Figure 8.2: a) Rate $r_c = 1/2$ convolutional encoder combined with QPSK; b) Rate $r_c = 2/3$ convolutional encoder combined with 8-PSK, from [63], [64].



Input	0	1	2	3	4	5	6	7
0	0/0	0/3	1/1	1/2	2/3	2/0	3/2	3/1
1	4/3	4/0	5/2	5/1	6/0	6/3	7/1	7/2

Figure 8.4: a) A rate 1/2 convolutional encoder combined with QPSK signal alternatives; b) A specific input sequence $b[i]$; c) The corresponding path in the trellis; d) A trellis section, and a table containing all relevant parameters.

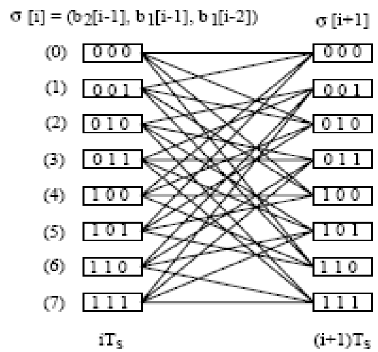


a)

INPUT $\begin{pmatrix} b_2[i] \\ b_1[i] \end{pmatrix}$ \ Current state σ [i]	(000) 0	(001) 1	(010) 2	(011) 3	(100) 4	(101) 5	(110) 6	(111) 7
$\binom{0}{0}$	0/0	0/2	1/1	1/3	0/4	0/6	1/5	1/7
$\binom{0}{1}$	2/4	2/6	3/5	3/7	2/0	2/2	3/1	3/3
$\binom{1}{0}$	4/2	4/0	5/3	5/1	4/6	4/4	5/7	5/5
$\binom{1}{1}$	6/6	6/4	7/7	7/5	6/2	6/0	7/3	7/1

σ [i+1] / m [i]

b)



c)

Figure 8.6: a) An example of TCM, from [63]–[64]; b) The mappings $F(\cdot,\cdot)$ and $G(\cdot,\cdot)$; c) A trellis section.

<u>Memory (redundancy, dependancy) is introduced among the sent signal alternatives!</u>

This gives us some new properties like, e.g.,:

Which of the following signal sequences are impossible?

1. $s_3(t), s_2(t-T_b), s_1(t-2T_b), s_1(t-3T_b)$
2. $s_3(t), s_2(t-T_b), s_2(t-2T_b), s_1(t-3T_b)$
3. $s_3(t), s_1(t-T_b), s_0(t-2T_b), s_2(t-3T_b)$
4. $s_3(t), s_1(t-T_b), s_3(t-2T_b), s_1(t-3T_b)$

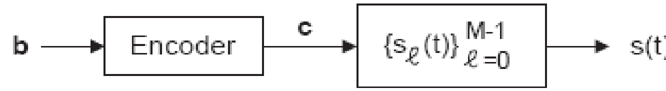
Note: In the uncoded case all signal sequences are possible.

Find the "missing" signal, in the sequence below,

$$s_1(t), s_3(t-T_b), \ ? \ , s_2(t-3T_b), s_3(t-4T_b), s_0(t-5T_b)$$

Note: This is not possible to do in the uncoded case!

2.32 Let us here study adaptive coding and modulation according to the block diagram below.



$$\bar{E}_{sent} = r_c \log_2(M) E_{b,sent} = \frac{k}{n} \log_2(M) E_{b,sent} \tag{8.4}$$

$$R_s = 1/T_s = \frac{1}{r_c} \cdot \frac{1}{\log_2(M)} \cdot R_b = \frac{1}{k/n} \cdot \frac{1}{\log_2(M)} \cdot R_b \tag{8.5}$$

$$W = c \cdot R_s \tag{8.6}$$

Typically, the bandwidth W is fixed and given but:
the rate of the encoder
the number of signal alternatives
and the bit rate can be **ADAPTIVE**, see (8.5)-(8.6)!

We have memory in the sequence of sent signal alternatives!

Some sequences are impossible, see problem!

Only ”good” sequences are sent!

OFDM - INTRO

OFDM signal(t) $= g_{rec}(t)\sum_{k=0}^{K-1} Re\{a_k e^{j2\pi f_k t}\} = g_{rec}(t) Re\{\sum_{k=0}^{K-1} a_k e^{j2\pi f_k t}\} =$

$$= g_{rec}(t) Re\{\sum_{k=0}^{K-1} a_k e^{j2\pi(f_0 + k f_\Delta)t}\} = g_{rec}(t) Re\{(\sum_{k=0}^{K-1} a_k e^{j2\pi(g_0+k) f_\Delta t}) e^{j2\pi f_{rc} t}\} =$$

$$= g_{rec}(t) Re\{(\sum_{k=0}^{K-1} a_k e^{j2\pi g_k f_\Delta t}) e^{j2\pi f_{rc} t}\} \qquad (1.13)$$

Equation (1.13) shows that an OFDM signal can be viewed as the sum of K QAM signals.

$$a_k = a_{k,I} + j a_{k,Q}, \quad k = 0, 1, \dots, K-1 \qquad (1.5)$$

$$T_{obs} = T_s - T_{CP} \qquad (1.18)$$

$$f_\Delta = 1/T_{obs} \qquad (2.1)$$

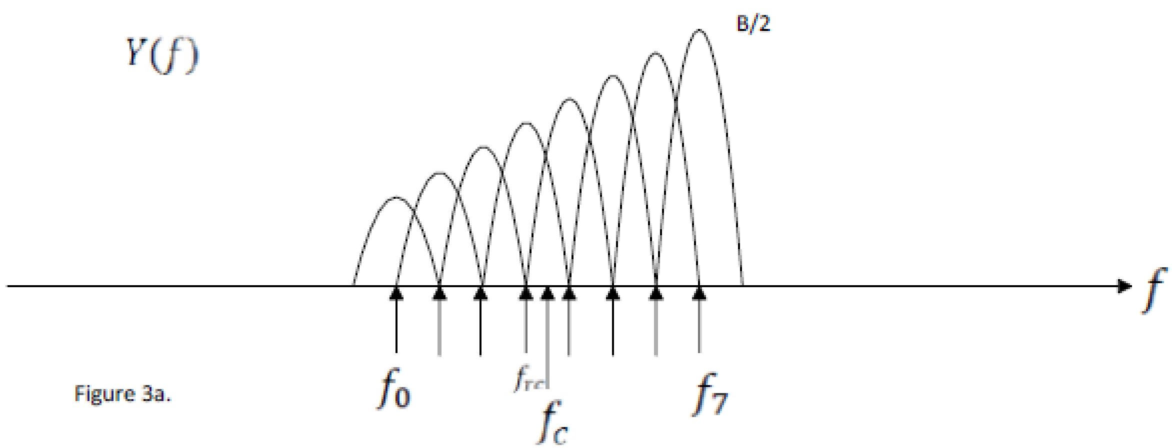


Figure 3a.

$$f_k = f_0 + kf_\Delta, \;\; k = 0,1,\dots,K-1 \qquad (1.1)$$

$$W_{OFDM} \approx Kf_\Delta \text{ (Hz)} \qquad (1.3)$$

The transmitted ***information bit rate***, denoted R_b, then equals,

$$R_b = \frac{r_c \sum_{k=0}^{K-1} log_2(M_k)}{T_s} \text{ (bps)} \qquad (1.16)$$

Furthermore, assuming also that K is >> 1 the ***bandwidth efficiency***, denoted ρ, is

$$\rho = \frac{R_b}{W_{OFDM}} = \frac{r_c \sum_{k=0}^{K-1} log_2(M_k)}{T_s K f_\Delta} \text{ (bps/Hz)} \qquad (1.17)$$

As an **example**: Let us consider the WLAN standard IEEE 802.11n (see ref. [11]). In this system OFDM is used with K=117, $f_\Delta = 312.5$ kHz and $T_s = 4$ µs (normal). Of the 117 subcarriers, the 3 center subcarriers are set to zero, 108 subcarriers are used for data transmission, and 6 subcarriers are used as pilots. In case of $r_c = 5/6$, and 64-QAM on each of the 108 subcarriers, the information bit rate equals 135 Mbps. Furthermore, for this scheme $W_{OFDM} \approx$ 36.6 MHz (the nominal bandwidth is 40 MHz).

As an example: Let us consider LTE-systems (Long-Term Evolution). In LTE (from ref. [9]), OFDM is used and $f_\Delta = \frac{1}{T_{obs}} = 15$ kHz (which means that $T_{obs} = 66.67$ µs, see Equation (2.1) in section 2). A typical OFDM symbol interval T_s in LTE is 71.36 µs, and 14 consecutive OFDM signals are then generated every ms. Furthermore, a so-called **resource block** in LTE typically consists of 12 consecutive sub-carrier frequencies (covering 180 kHz) and 7 consecutive OFDM symbol intervals (covering 0.5 ms). Hence, such a resource block contains 84 resource units (i.e. 84 QAM signal points). Within a 20 Mhz bandwidth typically 110 such resource blocks are defined, covering 19.8 MHz and corresponding to K=1320.

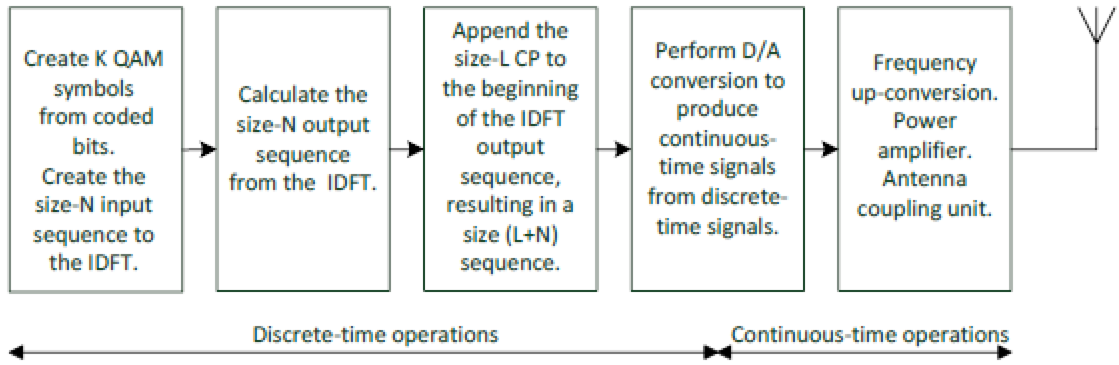


Figure 1. Illustrates the over-all structure and operations performed by an OFDM transmitter within an OFDM symbol interval T_s.

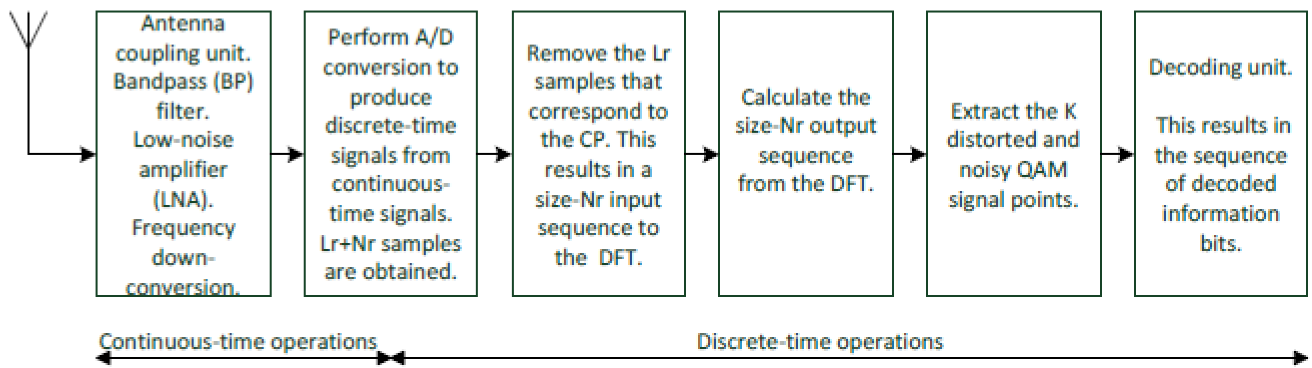


Figure 2. Illustrates the over-all structure and operations performed by an OFDM receiver within an OFDM symbol interval T_s.

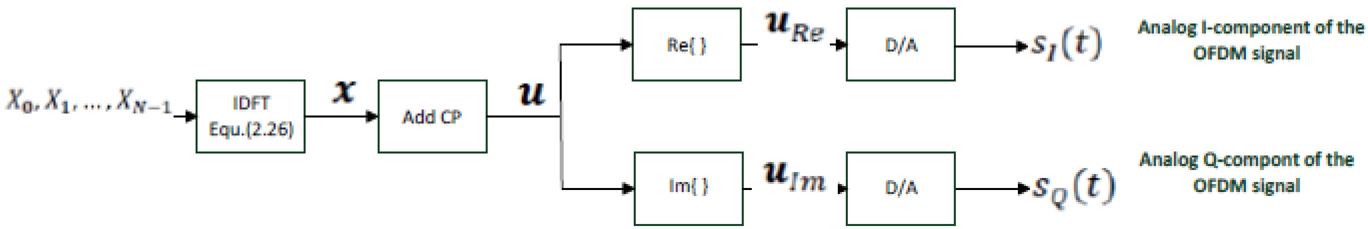


Figure 7. Block diagram illustrating the operations in the digital domain, and the transition to the analog domain.

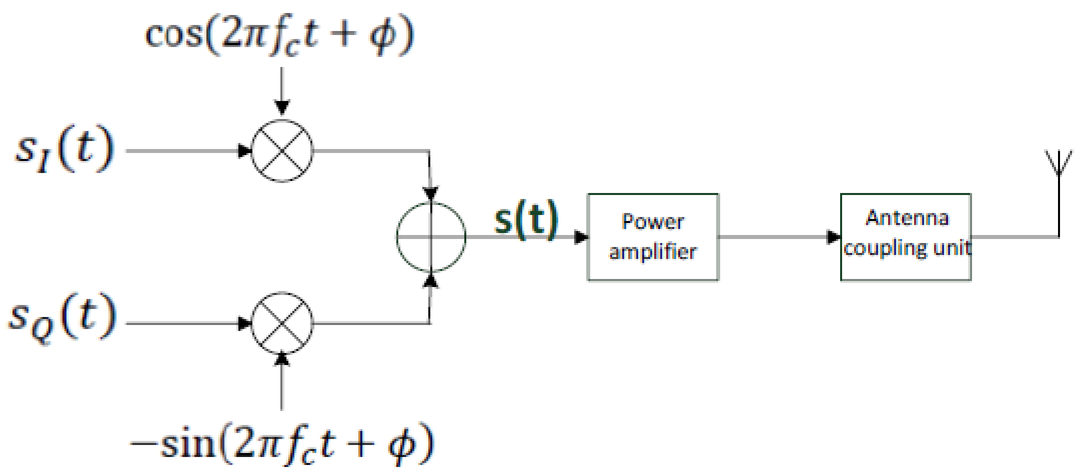


Figure 8. Block diagram illustrating frequency up-conversion (mixer stage) to the carrier frequency (K is odd), the power amplifier, and the antenna coupling unit. The OFDM signal $s(t)$ is given in equation (4.1).

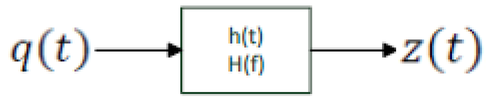


Figure 9. Illustrating a linear time-invariant filter channel.

INPUT OFDM: $As(t) = ARe\left\{\sum_{n=0}^{K-1} a_n \, e^{j(2\pi f_n t+\theta_n)}\right\},\qquad 0 \leq t \leq T_s$

OUTPUT OFDM: $z(t) = ARe\left\{\sum_{n=0}^{K-1} a_n H(f_n) \, e^{j(2\pi f_n t+\theta_n)}\right\},\qquad T_{CP} \leq t \leq T_s \quad (5.13)$

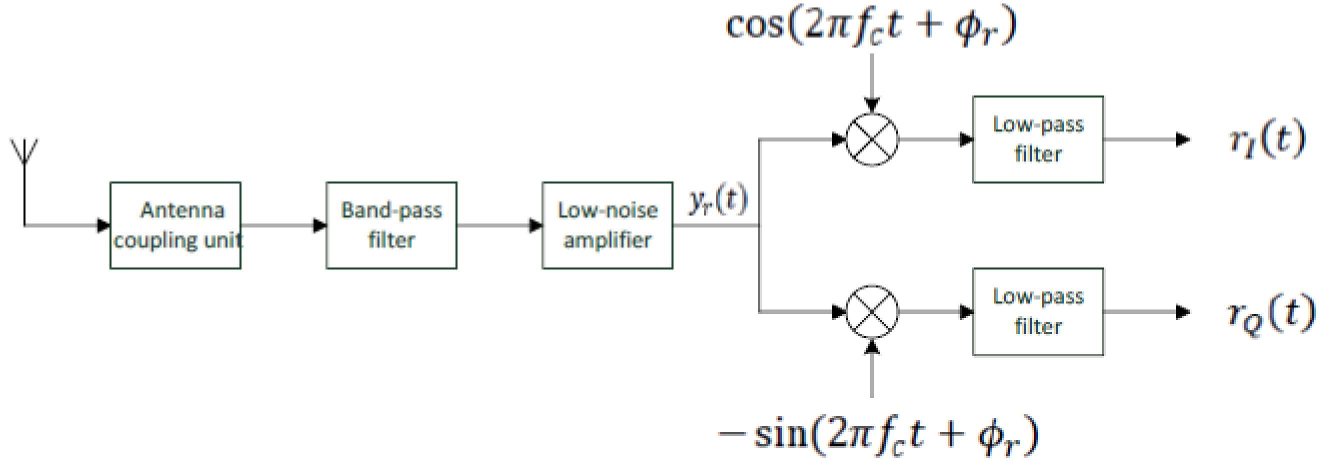


Figure 10. Illustrating the first part of the receiver: the antenna coupling unit, band-pass filter (wide), low-noise amplifier (LNA) and a homodyne unit for frequency down-conversion and extracting the baseband signals $r_I(t)$ and $r_Q(t)$. It is here assumed that K is odd for which $f_{rc} = f_c$.

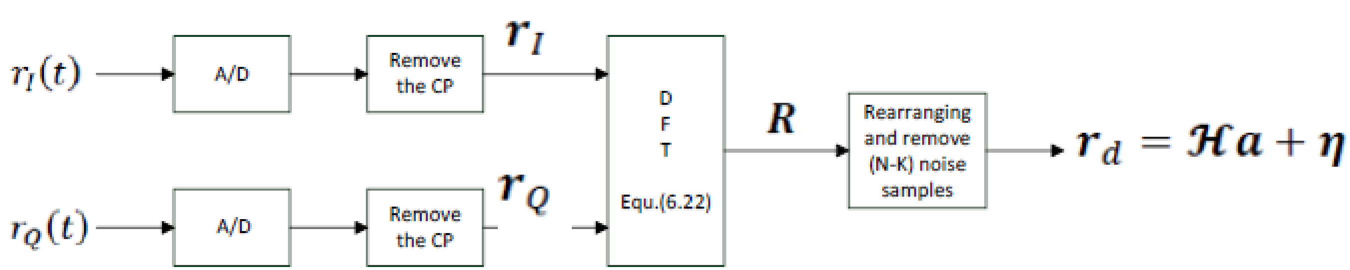


Figure 11. Illustrating sampling, removal of the CP, and the size-N DFT in the receiver to extract the K received distorted and noisy signal points collected in the size-K vector $\boldsymbol{r}_d$.

OFDM signal(t) $= g_{rec}(t)\sum_{k=0}^{K-1} Re\{a_k e^{j2\pi f_k t}\} = g_{rec}(t) Re\{\sum_{k=0}^{K-1} a_k e^{j2\pi f_k t}\} =$

$$= g_{rec}(t) Re\{\sum_{k=0}^{K-1} a_k e^{j2\pi(f_0+kf_\Delta)t}\} = g_{rec}(t) Re\{(\sum_{k=0}^{K-1} a_k e^{j2\pi(g_0+k)f_\Delta t}) e^{j2\pi f_{rc} t}\} =$$
$$= g_{rec}(t) Re\{(\sum_{k=0}^{K-1} a_k e^{j2\pi g_k f_\Delta t}) e^{j2\pi f_{rc} t}\} \qquad (1.13)$$

$$f_k = f_{rc} + g_k f_\Delta, \quad k = 0,1,\ldots,K-1 \qquad (1.8)$$

The numbers g_k range from g_0 to g_{K-1},

$$g_k: -\frac{K-1}{2} = g_0, \ldots, -1,0,1, \ldots, \frac{K-1}{2} = g_{K-1} \quad \text{if K is odd} \qquad (1.11)$$

$$g_k: -\frac{K-2}{2} = g_0, \ldots, -1,0,1, \ldots, \frac{K}{2} = g_{K-1} \quad \text{if K is even} \qquad (1.12)$$

$$y(t) = Re\{(\sum_{k=0}^{K-1} a_k e^{j2\pi g_k f_\Delta t})e^{j2\pi f_{rc} t}\} = Re\{x(t)e^{j2\pi f_{rc} t}\} \qquad (2.2)$$

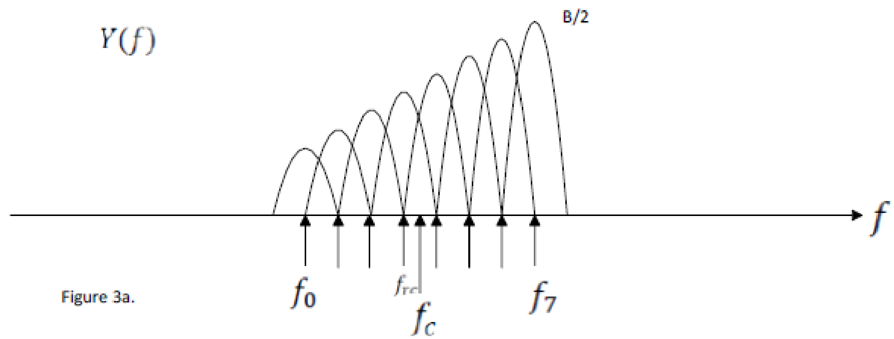


Figure 3a.

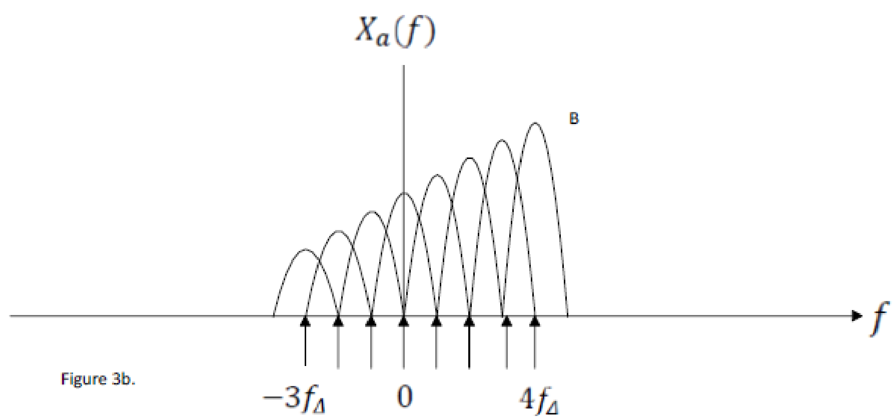


Figure 3b.

$$x(t) = x_{Re}(t) + jx_{Im}(t) = \sum_{k=0}^{K-1} a_k e^{j2\pi g_k f_\Delta t}, \quad 0 \le t \le T_{obs} \qquad (2.3)$$

Observe in Equation (2.3) that the QAM symbol a_k (k=0,1,…,(K-1)), is carried by *the baseband sub-carrier frequency $g_k f_\Delta$* in the complex baseband OFDM signal $x(t)$!

The high-frequency OFDM signal $y(t)$ in Equation (2.2) can be written as,

$$y(t) = Re\{x(t)e^{j2\pi f_{rc}t}\} = x_{Re}(t)\cos(2\pi f_{rc}t) - x_{Im}(t)\sin(2\pi f_{rc}t) \qquad (2.5)$$

Equation (2.5) is an important relationship since it shows that the OFDM-signal $y(t)$ is ***easily implemented*** as soon as we have created the real part $x_{Re}(t)$ and the imaginary part $x_{Im}(t)$ of $x(t)$.

We should therefore focus on creating $\boldsymbol{x(t)}$, since $\boldsymbol{x_{Re}(t)}$ and $\boldsymbol{x_{Im}(t)}$ then are easy to find.

Let us now sample the complex signal $x(t)$ in Equation (2.3) every $\frac{T_{obs}}{N}$ second, i.e. with N samples within the time-interval $0 \le t < T_{obs}$. This corresponds to a sampling frequency f_{samp} equal to,

$$f_{samp} = N/T_{obs} = Nf_{\Delta} > Kf_{\Delta} \qquad (2.12)$$

samples per second, and **N should be chosen larger than K**, and large enough such that the sampling theorem can be considered to be sufficiently fulfilled.

$$x_n = x(nT_{obs}/N) = \sum_{k=0}^{K-1} a_k e^{j2\pi g_k n/N} \qquad n = 0,1 \ldots, (N-1) \qquad (2.13)$$

Observe that the right hand side of equation (2.13) actually gives us a way to create the desired samples $x_0, x_1, \ldots, x_{N-1}$ of the complex baseband OFDM signal $x(t)$!

the Fourier transform $X(\nu)$ of the discrete-time signal $\boldsymbol{x}$ in Equation

(2.13). $X(\nu)$ is defined by, see ref. [1],

$$X(\nu) = \sum_{n=0}^{N-1} x_n e^{-j2\pi\nu n} \qquad (2.14)$$

Note in Equation (2.14) that the Fourier transform $X(\nu)$ is *periodic in ν with period 1*. Furthermore, the variable ν can be viewed as a **normalized frequency variable**, $\nu = f/f_{samp}$. The periodicity in ν is illustrated in Figure 4 on the next page.

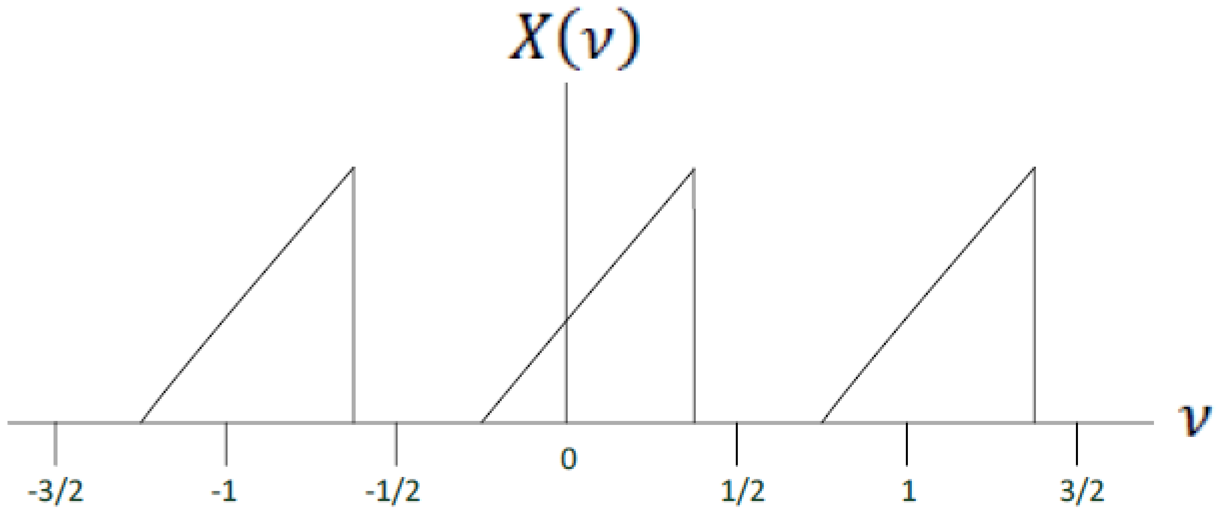


Figure 4. Illustrating that $X(\nu)$ is periodic in ν with period 1. The shape of $X(\nu)$ in this figure is an example of a Fourier transform of a discrete-time complex signal.

$$x_n = x(nT_{obs}/N) = \sum_{k=0}^{K-1} a_k e^{j2\pi g_k n/N} \qquad n = 0,1\ldots,(N-1) \qquad (2.13)$$

$$X(\nu) = \sum_{n=0}^{N-1} x_n e^{-j2\pi\nu n} \qquad (2.14)$$

Furthermore, let X_m denote the **frequency-domain sample** of $X(\nu)$ at $\nu = m/N$, defined by

$$X_m = X(\nu = m/N) = \sum_{n=0}^{N-1} x_n e^{-j2\pi mn/N}, \quad m = 0,1,\ldots,N-1 \qquad \textbf{(DFT)} \qquad (2.15)$$

This is the definition (see ref. [1]) of the size-N **DFT** (Discrete Fourier Transform) of the sequence $\boldsymbol{x}$.

However, for the moment we are particularly interested in the size-N **IDFT** (Inverse Discrete Fourier transform) which is defined by (see ref. [1]),

$$x_n = \frac{1}{N}\sum_{m=0}^{N-1} X_m e^{j2\pi mn/N}, \qquad n = 0,1,\ldots,N-1 \qquad \textbf{(IDFT)} \qquad (2.16)$$

Hence, as soon as we have determined the samples in the frequency domain $\boldsymbol{X_0, X_1, \ldots, X_{N-1}}$ we should use them in the size-N IDFT in Equation (2.16) to create the desired sequence of time-domain samples x! *The values X_m will be determined in step 3.*

In practice, **N is chosen to be a power of 2** since fast Fourier transform (FFT) algorithms then can be used to significantly speed up the calculations in Equations (2.15) - (2.16).

Step 3: The relation between the sequences $a_0, a_1, \dots, a_{K-1}$ and $X_0, X_1, \dots, X_{N-1}$.

Let us use Equation (2.13) to establish the connection between the sequences $a_0, a_1, \dots, a_{K-1}$ and $X_0, X_1, \dots, X_{N-1}$. We rewrite Equation (2.13) in the following way,

$$x_n = x\left(\frac{nT_{obs}}{N}\right) = \sum_{k=0}^{K-1} a_k e^{\frac{j2\pi g_k n}{N}} = \sum_{k=0}^{K-1} a_k e^{j2\pi(g_0+k)n/N} =$$

$$= \sum_{k=0}^{-g_0-1} a_k e^{j2\pi(g_0+k+N)n/N} + \sum_{k=-g_0}^{K-1} a_k e^{j2\pi(g_0+k)n/N} =$$

$$= \sum_{m=g_0+N}^{N-1} a_{m-(g_0+N)} e^{j2\pi mn/N} + \sum_{m=0}^{g_{K-1}} a_{m-g_0} e^{j2\pi mn/N} =$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} X_m e^{j2\pi mn/N} \qquad (2.18)$$

Inspection of Equation (2.18) yields the relationships below:

$$X_m = N a_{m-g_0}, \quad \text{if } \ 0 \le m \le g_{K-1} \qquad (2.19)$$

$$X_m = 0, \quad \text{if } \ g_{K-1} + 1 \le m \le g_0 + N - 1 \qquad (2.20)$$

$$X_m = N a_{m-(g_0+N)}, \text{ if } \ g_0 + N \le m \le N - 1 \qquad (2.21)$$

The last expression in Equation (2.18) is identical to the size-N IDFT in Equation (2.16). The relation between the sequences $a_0, a_1, \dots, a_{K-1}$ and $X_0, X_1, \dots, X_{N-1}$ are given by Equations (2.19) – (2.21).

As an **example** consider a situation with K=53, N=64. In this case $n_{rc} = -g_0 = \frac{K-1}{2} = 26$ and $g_{K-1} = \frac{K-1}{2} = 26$. From Equations (2.19) – (2.21) it is then concluded that the sub-sequence $X_0, X_1, \ldots, X_{26}$ contains the QAM signal points $a_{26}, a_{27}, \ldots, a_{52}$, the sub-sequence $X_{27}, X_{28}, \ldots, X_{37}$ contains only zero values, and the sub-sequence $X_{38}, X_{39}, \ldots, X_{63}$ contains the QAM signal points $a_0, a_1, \ldots, a_{25}$.

Let us consider another **example** with K=8, N=12. In this case $n_{rc} = -g_0 = \frac{K-2}{2} = 3$ and $g_{K-1} = \frac{K}{2} = 4$. From Equations (2.19) – (2.21) it is then concluded that the sub-sequence $X_0, X_1, X_2 X_3, X_4$ contains the QAM signal points a_3, a_4, a_5, a_6, a_7, the sub-sequence X_5, X_6, X_7, X_8 contains only zero values, and the sub-sequence X_9, X_{10}, X_{11} contains the QAM signal points a_0, a_1, a_2.