An introduction to OFDM – modeling and implementation

Lecture notes in the course Digital communications, advanced course (ETTN01)

Göran Lindell, Version 141124

1. Introduction

The modulation technique referred to as OFDM (Orthogonal Frequency Division Multiplexing) is of particular interest since it is used today in a number of important and high-performing communication systems. Some examples are Digital TV, LTE (4G), WLAN, WIMAX.

An OFDM signal extends over a T_s second long time-interval, which is referred to as the OFDM signal (or symbol) interval. In practice, a new OFDM signal is sent every T_s second, and the value of T_s depends on the application, but typically $T_s = 1$ ms, or smaller.

An OFDM signal can be described as the sum of K different QAM signals, where all QAM signals use the same T_s-long time-interval. Hence, K QAM signals are simultaneously transmitted within the OFDM signal interval. The value of K is typically quite large, several hundred or several thousand. A large value of K immediately leads to the question how to implement an OFDM signal in practice. Even if a single QAM-signal is easy to implement (as we will see below), it is not practical nor economical to implement say 1000 individual QAM-signals, and then add them together to create an OFDM signal (every T_s). As we will see there are very elegant engineering solutions to this problem.

The main purpose of these introductory lecture notes is to describe efficient, i.e. very fast and economical, implementations of the OFDM modulator (at the transmitter side) and the OFDM demodulator (at the receiver side). *The importance of efficient implementations should not be under-estimated, since the success of OFDM to a very large extent rests on the fact that efficient implementations exist!*

The transmitter typically consists of a digital part followed by an analog part, where discrete-time and continuous-time operations are performed, respectively. Digital-to-Analog (D/A) converters act as interface between the digital domain and the analog domain. In connection to the synthesis of an OFDM signal, the Inverse Discrete Fourier Transform (IDFT) plays a fundamental role in the digital domain while the analog domain typically includes filtering, mixing and power amplifying operations. An example of a transmitter structure is given in Figure 5 and Figure 6 on pages 20 and 24, respectively.

The received signal is a noisy and distorted version of the transmitted analog OFDM signal. The receiver typically consists of an analog part followed by a digital part, where continuous-time and discrete-time operations are performed, respectively. Analog-to-Digital (A/D) converters act as interface between the analog domain and the digital domain. The analog domain of the receiver typically includes filtering, amplifying and mixing operations, while the Discrete Fourier Transform (DFT) plays a fundamental role in the digital domain. An example of a receiver structure is given in Figure 8 and Figure 9 on pages 31 and 35, respectively.

The physical communication link (or channel) is analog. Hence, the complete communication chain transmitter-channel-receiver consists of a mixture of digital and analog operations. More details about these operations will be given as we continue in these lecture notes. It should also be noted (and will be illustrated in later sections) that the choice of sampling frequency in the transmitter (or the receiver) may significantly influence the implementation of the analog part.

The literature on different aspects of OFDM is extensive and the reader is strongly recommended to investigate, e.g., the important database IEEE Xplore [http://ieeexplore.ieee.org/Xplore/DynWel.jsp] to acquire information of recent advances related to OFDM, as well as tutorials. As examples of books that contain descriptions and/or applications of OFDM we have refs. [2]-[10].

Each of the K QAM signals has a different carrier frequency, which we refer to as a **sub-carrier frequency**. The choice of sub-carrier frequencies in OFDM is such that the frequency separation between neighboring sub-carriers is always equal to f_Δ Hz. How to choose f_Δ will be explained in detail in the next section but typically, depending on the application, f_Δ may be in the range 1-20 kHz. Let us number the K QAM signals from 0 up to K-1 according to increasing sub-carrier frequency. Then we have that the n:th QAM signal has the sub-carrier frequency f_n Hz,

$$f_n = f_0 + nf_\Delta, \;\; n = 0,1,\dots,K-1 \qquad (1.1)$$

The choice of *the overall carrier frequency* f_c is in principle arbitrary, but here we define f_c as the center frequency in the OFDM frequency band, i.e.,

$$f_c = f_0 + \frac{K-1}{2} f_\Delta \qquad (1.2)$$

It is seen in equation (1.2) that if K is an odd number then f_c coincide with the sub-carrier frequency $f_{(K-1)/2}$. On the other hand, if K is an even number then f_c is exactly in the middle between the two sub-carrier frequencies $f_{(K-2)/2}$ and $f_{K/2}$. In, e.g., LTE-applications the overall carrier frequency f_c typically is in the GHz range, and K is an odd number in the down-link while K is an even number in the up-link [9].

From equation (1.1) the ***bandwidth*** of the OFDM signal is found to be approximately $(K+1)f_\Delta$ Hz (here we have also taken into consideration the bandwidth consumption f_Δ at the two edges of the frequency band). In these lecture notes K is typically $K \gg 1$, and then the bandwidth of the OFDM signal is approximately Kf_Δ Hz.

Let us now take a closer look at the n:th QAM signal in the time-interval $0 \le t \le T_s$, where it can be expressed as,

$$g_{rec}(t)\big(a_{n,I}\cos(2\pi f_n t) - a_{n,Q}\sin(2\pi f_n t)\big) = g_{rec}(t)Re\{a_n e^{j2\pi f_n t}\} \qquad (1.3)$$

The left-hand side above is the conventional so-called I/Q description of a QAM signal. The pulse $g_{rec}(t)$ is equal to a constant value within the time-interval $0 \le t \le T_s$, and it equals zero outside this time-interval. The *information* is carried by the pair $(a_{n,I}, a_{n,Q})$ of values, and there are M_n unique pairs, each pair is usually referred to as a **signal point**. The size of the n:th QAM *signal constellation* is denoted M_n and it is typically a power of 4 (4, 16, 64, 256,…). Furthermore, different QAM signals can have different constellation sizes.

The right-hand side in equation (1.3) is also a conventional description of a QAM-signal, and it uses complex notation. *Observe that this description will be used almost exclusively in these lecture notes!* Here, the complex number a_n is defined as,

$$a_n = a_{n,I} + ja_{n,Q} \qquad (1.4)$$

where $a_{n,I}$ and $a_{n,Q}$ is the real part and the imaginary part of a_n, respectively. Hence, in this description the information is described with the complex number a_n (which also is referred to as a

signal point). *It is very important to understand that the two QAM-descriptions given in equation (1.3) are identical!*

Due to the rectangular pulse shape, the Fourier transform of the n:th QAM signal in equation (1.3) is **sinc-shaped** around the sub-carrier frequency f_n, with ***peak value*** $a_n T_s$, and it has **zero-crossings** at the frequencies $f = f_n + i/T_s$ (for any non-zero integer i). Hence, the width of the **main-lobe** is $2/T_s$ Hz.

We need a suitable description of the OFDM signal. By this is meant a description that is tailored to the modeling and implementation issues considered in these lecture notes. As will be more clear in section 2, it is convenient to introduce a so-called *reference* carrier frequency, denoted f_{rc}, chosen to be one of the sub-carrier frequencies closest to the overall carrier frequency f_c. Based on the discussion in connection to equation (1.2), the *reference carrier frequency* f_{rc} is defined by,

$$f_{rc} = f_c = f_{(K-1)/2} \qquad \text{if K is odd} \qquad (1.5)$$

$$f_{rc} = f_c - f_\Delta/2 = f_{(K-2)/2} \qquad \text{if K is even} \qquad (1.6)$$

Furthermore, now we can express the n:th sub-carrier frequency f_n by using f_{rc} as a reference in the following useful way,

$$f_n = f_{rc} + g_n f_\Delta, \quad n = 0,1,\dots,K-1 \qquad (1.7)$$

and this will be used a lot in section 2 where a *baseband* (low-frequency) version of the OFDM signal is investigated. The number g_n in equation (1.7) denotes an integer value defined by,

$$g_n = n - n_{rc} = n - (K-1)/2 \qquad \text{if K is odd} \qquad (1.8)$$

$$g_n = n - n_{rc} = n - (K-2)/2 \qquad \text{if K is even} \qquad (1.9)$$

Where n_{rc} denotes the value of n corresponding to f_{rc}, see equations (1.5)-(1.6). As will be seen in section 2, g_n is an alternative useful way of numbering the corresponding K *baseband* sub-carrier frequencies. The parameter n in equation (1.1) numbers the sub-carrier frequencies from 0 to K-1, but g_n in equation (1.7) instead numbers the sub-carrier frequencies ***relative*** to the reference carrier frequency f_{rc}. It is also seen in equations (1.7)-(1.9) that $g_n = 0$ corresponds to the sub-carrier number $n = n_{rc}$ of the reference carrier frequency f_{rc}, i.e. $g_{n_{rc}} = 0$.

The numbers g_n range from g_0 to g_{K-1},

$$g_n: -\frac{K-1}{2} = g_0, \dots, -1,0,1, \dots, \frac{K-1}{2} = g_{K-1} \qquad \text{if K is odd} \qquad (1.10)$$

$$g_n: -\frac{K-2}{2} = g_0, \dots, -1,0,1, \dots, \frac{K}{2} = g_{K-1} \qquad \text{if K is even} \qquad (1.11)$$

Examples: If K=8 then $g_0 = -3$, $g_3 = 0$ and $g_7 = 4$. If K=9 then $g_0 = -4$, $g_4 = 0$ and $g_8 = 4$.

By extending equation (1.3) to include the sum of K QAM signals, and also use the expression in equation (1.7), i.e., $f_n = f_{rc} + g_n f_\Delta$, $n = 0,1,\dots,(K-1)$ a compact expression of an OFDM signal in the time-interval $0 \le t \le T_s$, is obtained as,

$$\text{OFDM signal} = g_{rec}(t) Re\left\{\sum_{n=0}^{K-1} a_n e^{j2\pi(f_0 + n f_\Delta)t}\right\} = g_{rec}(t) Re\left\{\sum_{n=0}^{K-1} a_n e^{j2\pi g_n f_\Delta t} e^{j2\pi f_{rc} t}\right\} \quad (1.12)$$

This expression illustrates that an OFDM signal can be viewed as the sum of K QAM signals, and this is the most basic characteristics of an OFDM signal. Figure 1a on page 7 shows an example if $K = 8$ and this figure roughly indicates some frequency-domain properties of an OFDM signal. Figure 1a illustrates the main-lobes of the eight individual QAM-signals that constitute the OFDM signal. The side-lobes of each QAM-signal are, however, **not** shown in Figure 1.a.

For the moment we do not know how to efficiently create an OFDM signal in practice if K is large. As will be seen in section 2, the expression in the right-hand side of equation (1.12) is indeed very useful to find an efficient implementation. Note that both the numbering g_n and the reference carrier frequency f_{rc} is present in equation (1.12).

The number of *coded bits* carried by the OFDM signal above is denoted B_c, and $B_c = \sum_{n=0}^{K-1} log_2(M_n)$. So, if K=600 sub-carriers and if 64-QAM is used throughout, then 3600 coded bits are sent every T_s. Furthermore, if $T_s = 0.1$ ms then $36 \cdot 10^6$ coded bits are sent per second.

The communication channel is assumed to be a **multi-path** (or linear time-invariant filter) **channel**. Such a channel typically distorts the transmitted OFDM signal such that, e.g., in the beginning of the OFDM symbol interval *an initial relatively short transient* behavior of the signal occurs. This will be described in detail in section 5.

The first part of the transmitted OFDM signal is referred to as the **Cyclic Prefix** (CP) and its duration is denoted T_{CP}. In most cases $T_{CP} \ll T_s$. The main purpose of the CP is to allow for the initial transient behavior of the multi-path channel *output* signal to occur within the duration of the CP. In a sense, the CP acts as a "guard interval". The CP plays an important role at the receiver side since, as we will see in sections 5-6, it is possible to completely eliminate interference between OFDM signals provided that the CP is properly designed. Hence, as long as the CP is properly designed, such ***inter-symbol interference (ISI)*** between OFDM signals will ***not*** be present in the receiver, and that is a major advantage of OFDM and one of the main reasons why the ODFM technique is chosen in many applications. More details about the CP will be given in sections 3,5,6.

The remaining part of the OFDM signal is referred to as **the receiver's observation interval** and its duration is denoted T_{obs},

$$T_{obs} = T_s - T_{CP} \tag{1.13}$$

The receiver's observation interval is the time-interval of the received signal that the receiver uses for extraction of the K received noisy and distorted signal points. Hence, for efficient operation this time interval should be the major part of the OFDM symbol time T_s, i.e. $T_{CP} \ll T_s$, since otherwise too much signal power is spent on a signal (the CP) that actually will not be used in the detection process in the receiver.

Observe now, as will be seen in sections 2-3, that the strategy to create an OFDM signal with duration T_s, is to **first** create N time-domain samples corresponding to an OFDM signal with duration T_{obs}. When this is done, we can relatively easy (in section 3) create the cyclic prefix as a so-called *periodic extension* of the already created N samples.

For future reference we here also give the definition of ***orthogonal signals***. Two real or complex signals $s(t)$ and $z(t)$ are orthogonal over the time-interval $t_1 \leq t \leq t_2$ if and only if,

$$\int_{t_1}^{t_2} s(t) z^*(t)\, dt = 0 \qquad (1.14)$$

where the symbol * denotes conjugate.

The last part of this introduction is a list of the remaining topics in these lecture notes.

Section 2:

- How to obtain N **time-domain complex samples** of a complex baseband (low-frequency) version of an OFDM signal by using the size-N **IDFT** (Inverse Discrete Fourier Transform). A relatively low value of the sampling frequency is here assumed.

Section 3:

- Adding L **additional** time-domain complex samples corresponding to the **CP** (Cyclic Prefix), a new size-(L+N) sequence of complex samples is constructed
- Using **D/A converters**, applied to the size-(L+N) complex sequence, the continuous time analog **I- and Q-components** of the transmitted OFDM signal are obtained.

Section 4:

- I/Q frequency up-converting (mixing) to the carrier frequency, power amplifying and thereby obtaining **the transmitted OFDM signal**.

Section 5:

- How the OFDM signal is **changed** by the channel (H(f) and AWGN).

Section 6:

- Down-converting to baseband at the receiver side and **extracting** the information carrying I- and Q-components of the received distorted and noisy OFDM signal.
- **Sampling** the received I- and Q-components (**A/D conversion**).
- Using the size-N **DFT** (Discrete Fourier Transform) to obtain the received noisy and distorted signal point at each sub-carrier frequency. A relatively low value of the sampling frequency is here assumed.

Section 7:

- An **alternative modulator** implementation using a higher sampling frequency. The description given here is to a large extent influenced by the description in ref. [2].

Section 8:

- An **alternative demodulator** implementation using a higher sampling frequency.

2. A sampled version of a T_{obs}-long baseband OFDM signal

Our goal in this section is to use the size-N IDFT (Inverse Discrete Fourier transform) to efficiently create N time-domain complex samples of a complex baseband OFDM signal, within the time-interval $0 \leq t \leq T_{obs}$. *Observe however* that these N samples will in section 3 be time-shifted T_{CP} seconds to obtain the samples of the desired OFDM signal within the time-interval $T_{CP} \leq t \leq T_s$.

In section 3 we will also add the CP by adding L additional samples to the already created N samples, and the L samples correspond to the time-interval $0 \leq t \leq T_{CP}$. As will be seen, the CP is constructed by adding a so-called *periodic extension* of the N samples. In this way all $L + N$ samples of a complex baseband OFDM signal, within the time-interval $0 \leq t \leq T_s$, is obtained. The remaining steps are then D/A converters (also in section 3) and frequency up-converting (in section 4).

As was mentioned earlier the receiver´s observation interval constitutes the major part of the OFDM signal interval T_s. Hence the OFDM signal construction within the interval $T_{CP} \leq t \leq T_s$ is of course important.

The **sub-carrier frequency separation** f_Δ is a fundamental parameter in OFDM and it should be chosen such that,

$$f_\Delta = 1/T_{obs} \tag{2.1}$$

Among other advantages, this choice makes it possible for all K received QAM signals to be orthogonal over the receiver´s observation interval, and this is a fundamental desired property of an OFDM signal. Note however that the requirement in equation (2.1) assumes an overall *rectangular* pulse within T_{obs}, otherwise all K received QAM signals will not be orthogonal.

Step 1: The equivalent complex baseband signal of a T_{obs}-long OFDM signal .

We now use the same kind of description as in equation (1.12) to describe an OFDM signal within the time-interval $0 \leq t \leq T_{obs}$, here denoted $y(t)$,

$$y(t) = Re\{\sum_{n=0}^{K-1} a_n e^{j2\pi g_n f_\Delta t} e^{j2\pi f_{rc} t}\} = Re\{x(t) e^{j2\pi f_{rc} t}\} \tag{2.2}$$

where

$$x(t) = x_{Re}(t) + jx_{Im}(t) = \sum_{n=0}^{K-1} a_n e^{j2\pi g_n f_\Delta t}, \qquad 0 \leq t \leq T_{obs} \tag{2.3}$$

and $x(t) = 0$ outside this interval. Observe that the signal $x(t)$ does not contain any high-frequency components, it only contains frequency components around the so-called *equivalent baseband sub-carrier frequencies*:

$$g_0 f_\Delta, \ldots, -f_\Delta, 0, f_\Delta, \ldots\ldots, g_{K-1} f_\Delta \tag{2.4}$$

where g_n is defined in equations (1.8)-(1.9). Consequently, the signal $x(t)$ contains only *baseband frequencies* (low frequencies) and $x(t)$ is referred to as the ***equivalent complex baseband signal*** of the OFDM signal y(t).

Note that the n:th QAM symbol a_n is carried by *the baseband sub-carrier frequency* $g_n f_\Delta$ in the complex baseband OFDM signal $x(t)$!

The frequency contents of the signals y(t) and x(t), denoted $Y(f)$ and $X_a(f)$, respectively, are roughly indicated in Figure 1 below for an example where K=8. It is seen in Figure 1a that the high-frequency OFDM signal y(t) carries the QAM-symbols $a_0, a_1, \dots, a_7$ at the high-frequency sub-carrier frequencies $f_0, f_1, \dots, f_7$, respectively. For the specific example shown in Figure 1a it is concluded that $|a_7|$ is much larger than $|a_0|$, since the main-lobe around f_7 is much higher than the main-lobe around f_0. Figure 1b shows the corresponding complex baseband situation where the baseband OFDM signal $x(t)$ carries the QAM-symbols $a_0, a_1, \dots, a_7$ at the ***baseband*** sub-carrier frequencies $-3f_\Delta, -2f_\Delta, \dots, 4f_\Delta$, respectively (according to equation (2.4)).

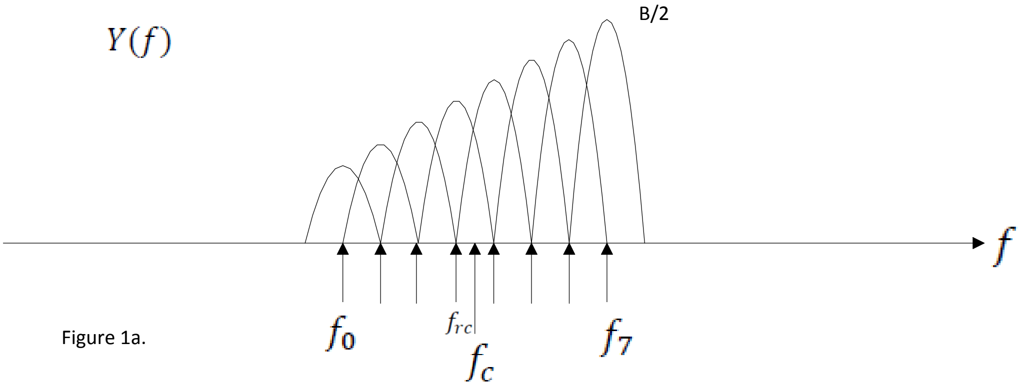

Figure 1a.

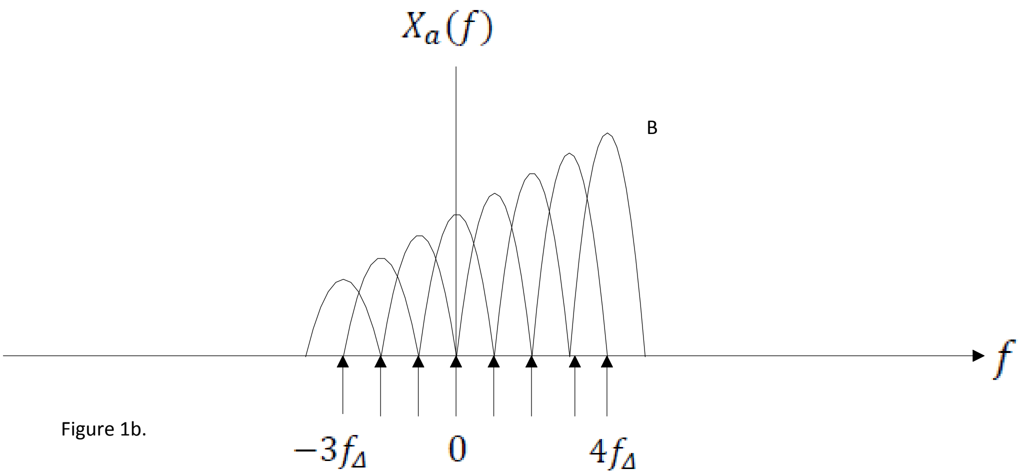

Figure 1b.

Figure 1a) A specific example where $K = 8$, illustrating the main-lobes of the eight individual QAM signals that constitute the OFDM signal $y(t)$ in equation (2.2). The side-lobes of each QAM-signal are, however, **not** shown in this figure. The Fourier transform $Y(f)$ of the OFDM signal $y(t)$ is only roughly indicated by this figure. The short arrows show the eight sub-carrier frequencies. Furthermore, $f_{rc} = f_3$ in this case. In this example it is also assumed that the specific set of K signal points to be transmitted are such that $|a_0| < |a_1| < |a_2| < \cdots < |a_6| < |a_7|$.
Figure 1b) The baseband version of Figure 1a is here considered. Illustrating the main-lobes for the eight individual *baseband* QAM signals that constitute the baseband OFDM signal $x(t)$ in equation (2.3). The Fourier transform $X_a(f)$ of the baseband OFDM signal $x(t)$ is only roughly indicated by this figure. The arrows show the 8 *baseband* sub-carrier frequencies.

Note that $y(t)$ in equation (2.2) can be written as,

$$y(t) = Re\{x(t)e^{j2\pi f_{rc}t}\} = x_{Re}(t)\cos(2\pi f_{rc}t) - x_{Im}(t)\sin(2\pi f_{rc}t) \qquad (2.5)$$

Observe in equation (2.5) that the OFDM-signal $y(t)$ is ***easily obtained*** as soon as we have created the real part $x_{Re}(t)$ and the imaginary part $x_{Im}(t)$ of $x(t)$.

We should therefore focus on creating $x(t)$, since $x_{Re}(t)$ and $x_{Im}(t)$ then are easy to find.

In connection to equation (2.5) we know that $X_a(f)$ denotes the Fourier transform of the baseband OFDM signal $x(t)$. The Fourier transform of the signal $x(t)e^{j2\pi f_{rc}t}$ then is,

$$\int_{-\infty}^{\infty} x(t)e^{j2\pi f_{rc}t}e^{-j2\pi ft}\,dt = X_a(f - f_{rc}) \qquad (2.6)$$

and this is a pure frequency shift of $X_a(f)$. In general, the signal $x(t)e^{j2\pi f_{rc}t}$ is a complex signal and for such signals the Fourier transform does not possess symmetry properties around the frequency $f = 0$. However, the OFDM signal $y(t)$ is real, and its Fourier transform $Y(f)$ can be shown to be,

$$Y(f) = (X_a(f - f_{rc}) + X_a^*(-(f + f_{rc})))/2 \qquad (2.7)$$

$Y(f)$ (at positive frequencies only) and $X_a(f)$ are roughly indicated in Figure 1. Symmetry exists in $Y(f)$ since, e.g., $|Y(f)| = |Y(-f)|$. The symbol * denotes conjugate.

It is of great importance to understand the frequency content in the baseband OFDM signal $x(t)$. As is seen in equation (2.3) the signal $x(t)$ is the sum of K complex baseband QAM signals, and let us denote these K individual baseband QAM signals by,

$$x_n(t) = a_n e^{j2\pi g_n f_\Delta t}, \quad n = 0,1,\dots,(K-1) \qquad (2.8)$$

where each signal is zero outside the time-interval $0 \le t \le T_{obs}$. The frequency content in the signal $x_n(t)$ is denoted $X_{a,n}(f)$ to stress that this is the Fourier transform of an *analog* signal, and

$$X_{a,n}(f) = a_n T_{obs}\frac{sin(\pi(f - f_{x,n})T_{obs})}{\pi(f - f_{x,n})T_{obs}}\,e^{-j\pi(f - f_{x,n})T_{obs}} \qquad (2.9)$$

where

$$f_{x,n} = g_n f_\Delta = \frac{g_n}{T_{obs}} \qquad (2.10)$$

is the baseband sub-carrier frequency of the signal $x_n(t)$. It is seen in equation (2.9) that the Fourier transform of the individual complex baseband QAM signal $x_n(t)$ is **sinc-shaped** around the baseband sub-carrier frequency $f_{x,n} = g_n f_\Delta$, with ***peak value*** $a_n T_{obs}$, and it has **zero-crossings** at the frequencies $f = f_{x,n} + if_\Delta$ (for any non-zero integer i). Hence, the width of the **main-lobe** is $2f_\Delta$. Observe, for future reference, that the signal point a_n appears in the peak value of the Fourier transform in equation (2.9), and the peak value is located at the frequency $f = f_{x,n} = g_n f_\Delta$.

The frequency content in the complex baseband OFDM signal $x(t)$, denoted $X_a(f)$, is now easily found as the sum of the frequency contents of the individual signals $x_n(t)$,

$$X_a(f) = \sum_{n=0}^{K-1} X_{a,n}(f) \qquad (2.11)$$

Figure 1b) roughly indicates $X_a(f)$ for an example where $K = 8$.

Step 2: Samples of the complex baseband OFDM signal $x(t)$, and the IDFT.

The strategy here is to construct $x(t)$ ***indirectly*** by first constructing time-domain complex samples of $x(t)$.

The *sampling theorem*, see ref. [1], states that if the highest frequency-component in a signal $s(t)$ is W Hz, then the signal $s(t)$ can be *completely reconstructed* from its samples, provided that the sampling frequency is *at least* 2W samples per second.

In case the signal $s(t)$ is complex, the theorem is still valid and both the real part of $s(t)$ and the imaginary part of $s(t)$ should then be considered.

As mentioned earlier, the K baseband sub-carriers in the baseband OFDM signal $x(t)$ are located from $g_0 f_\Delta$ Hz up to $g_{K-1} f_\Delta$ Hz. This means that the **baseband bandwidth** of $x(t)$ is approximately $(g_{K-1}+1) f_\Delta$ Hz (and the same bandwidth may therefore be assumed also for the real part and for the imaginary part).

Using equations (1.10) - (1.11), and assuming that $K \gg 1$, the baseband bandwidth of $x(t)$ is therefore approximately $K f_\Delta / 2$ Hz. We therefore conclude that the **sampling frequency** f_{samp} should be larger than $K f_\Delta$ samples per second (and large enough such that the sampling theorem can be considered to be sufficiently fulfilled, note that x(t) is **not** a band-limited signal).

Let us now sample the signal $x(t)$ in equation (2.3) every $\frac{T_{obs}}{N}$ second, i.e. with N samples within the time-interval $0 \leq t < T_{obs}$. This corresponds to a sampling frequency f_{samp} equal to,

$$f_{samp} = N/T_{obs} = N f_\Delta > K f_\Delta \qquad (2.12)$$

samples per second, and **N should be chosen larger than K**, and large enough such that the sampling theorem can be considered to be sufficiently fulfilled. In case a higher sampling frequency such that $f_{samp} = N f_\Delta > 2K f_\Delta$ is considered then the implementation treated in section 7 may be an option.

Let the column vector $\boldsymbol{x}$ contain the N time-domain complex samples $x_0, x_1, \dots, x_{N-1}$, of the signal $x(t)$ in equation (2.3). This means that the m:th sample x_m is,

$$x_m = x(mT_{obs}/N) = \sum_{n=0}^{K-1} a_n e^{j2\pi g_n m/N} \qquad m = 0,1 \dots, (N-1) \qquad (2.13)$$

The right hand side gives us a way to create the desired samples x_m. However, we need to do some additional work to get an expression that contains the desired size-N IDFT. Furthermore, we also need an understanding of both the size-N DFT and the size-N IDFT to better understand why and how the former is applied at the receiver side and the latter at the transmitter side.

The Fourier transform of the discrete-time signal $\boldsymbol{x}$ in equation (2.13) is defined by, see ref. [1],

$$X(\nu) = \sum_{n=0}^{N-1} x_n e^{-j2\pi\nu n} \qquad (2.14)$$

Note in equation (2.14) that $X(\nu)$ is *periodic in ν with period 1*. Furthermore, the variable ν can be viewed as a **normalized** frequency variable, $\nu = f/f_{samp}$. The periodicity in ν is illustrated in Figure 2.

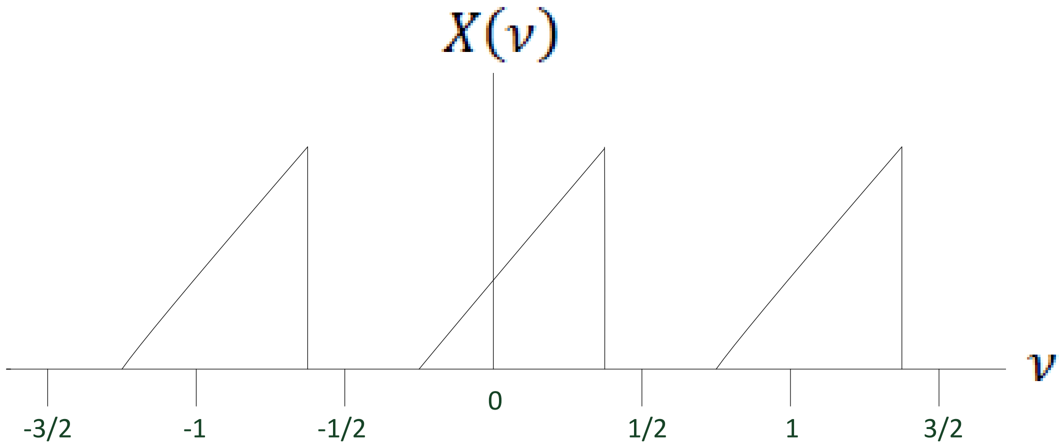

Figure 2. Illustrating that $X(\nu)$ is periodic in ν with period 1. The shape of $X(\nu)$ in this figure is an example of a Fourier transform of a discrete-time complex signal.

Due to the periodic structure of $X(\nu)$ it is clear that it is important to understand the behavior of $X(\nu)$ in the *fundamental interval* $-1/2 \leq \nu \leq 1/2$.

Furthermore, let X_k denote the k:th **frequency-domain sample** of $X(\nu)$ defined by,

$$X_k = X(\nu = k/N) = \sum_{n=0}^{N-1} x_n e^{-j2\pi kn/N}, k = 0,1, \dots, N-1 \qquad \textbf{(DFT)} \qquad (2.15)$$

This is the definition ([1]) of the size-N **DFT** (Discrete Fourier Transform) of the sequence $\boldsymbol{x}$.

However, for the moment we are particularly interested in the size-N **IDFT** (Inverse Discrete Fourier transform) which is defined by ([1]),

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}, n = 0,1, \dots, N-1 \qquad \textbf{(IDFT)} \qquad (2.16)$$

Hence, as soon as we have determined the samples in the frequency domain $X_0, X_1, \dots, X_{N-1}$ we should use them in the size-N IDFT in equation (2.16) to efficiently create the desired sequence $\boldsymbol{x}$. The values X_k will be determined in step 3.

In practice, **N is a power of 2** since fast Fourier transform (FFT) algorithms then can be used to significantly speed up the calculations in equation (2.16).

It is clear that since the discrete-time signal (or vector) $\boldsymbol{x}$ consists of samples of the analog signal $x(t)$, there should be a relationship between the frequency contents of these two signals.

The Fourier transform of the signal $x(t)$ is denoted by $X_a(f)$, and it can be shown that ([1]),

$$X(\nu) = f_{samp} \sum_{k=-\infty}^{\infty} X_a((\nu - k) f_{samp}) \qquad (2.17)$$

This relationship is very useful indeed, since it gives us complete knowledge about $X(\nu)$, since $X_a(f)$ is known (from equations (2.11) and (2.9)). Equation (2.17) tells us that $X(\nu)$ equals $f_{samp} X_a(\nu f_{samp})$ **plus** <u>*periodic repetitions*</u> of this function, spaced with normalized frequency 1 apart.

Let us consider an example where $X_a(f)$ is given in Figure 3, and where the baseband bandwidth is denoted W. In figure 3, $X_a(f) = 0$ outside the frequency range $-2W/3 \leq f \leq W$. Furthermore assume that the sampling frequency $f_{samp} = \frac{8}{3}W$ samples per second is used. The reader is recommended to apply equation (2.17) to this example and show that $X(\nu)$ then will be identical to the Fourier transform given in Figure 2, and the peak value in Figure 2 then is equal to $f_{samp}B$.

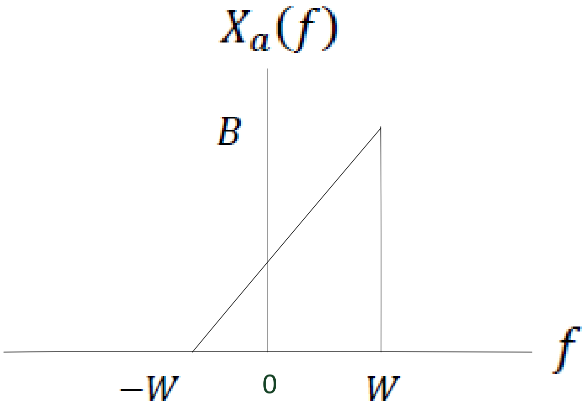

Figure 3. Illustrating $X_a(f)$. The shape of $X_a(f)$ in this figure is an example of a Fourier transform of an analog complex signal (and it does not follow the shape given by equation (2.11)).

Step 3: The sequences $a_0, a_1, \ldots, a_{K-1}$ and $X_0, X_1, \ldots, X_{N-1}$.

Let us take a closer look at the sample $X_l = X(\nu = l/N)$, where l is an arbitrary integer. $X(\nu)$ is roughly indicated in Figure 4 for the example given in Figure 1b where $K = 8$. It is also assumed in Figure 4 that $N = 12$, and this means that $X(\nu)$ is sampled at the normalized frequency $\nu = l/12$ to obtain X_l. The arrows in the figure indicate where the samples X_l are obtained in the normalized frequency domain (i.e. in the ν-domain). Observe that the Fourier Transform $X_a(f)$ of $x(t)$ (see Figure 1b) appears frequency-normalized and repeatedly in Figure 4. This means that the sequence of QAM symbols $a_0, a_1, \ldots, a_7$ also appears repeatedly in this figure. *The complex baseband QAM signal that carries a_n is located around the baseband sub-carrier frequency $g_n f_\Delta$ Hz (see Figure 1b) and equation (2.10)), and in Figure 4 this QAM signal appears in the ν-domain around $\nu = \frac{g_n}{N}$ (and periodically).* Since $X(\nu)$ is periodic in ν with period 1, the sample obtained at $\nu = \frac{l}{N}$ will give exactly the same result as the sample obtained at $\nu = \frac{l+N}{N}$. The samples X_l, X_{l-N} and X_{l+N} are therefore identical.

As an example in Figure 4: since $K = 8$ and $N = 12$, the QAM symbol a_3 appears at the sampling indices $l =$ ...-24,-12,0,12,24,.. (corresponding to $\nu = \cdots, -2, -1, 0, 1, 2, \ldots$).

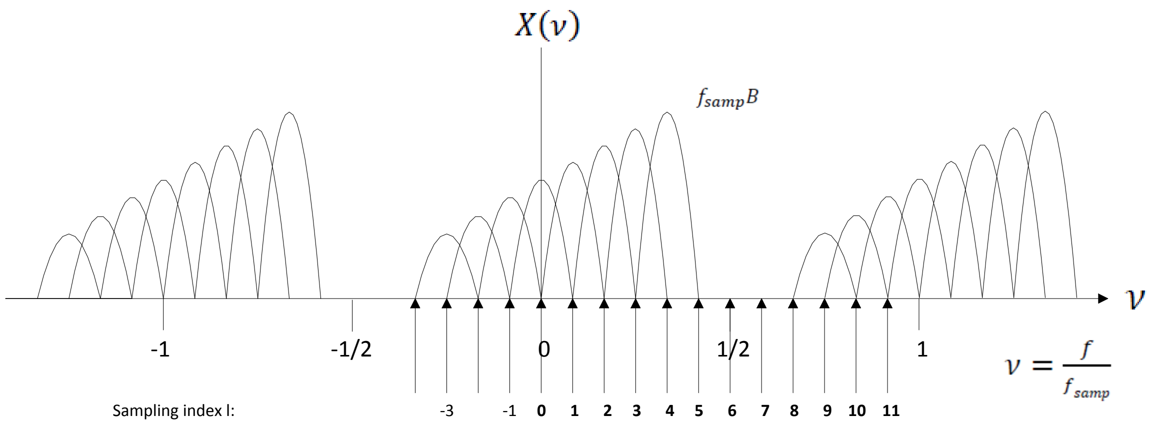

Figure 4. Roughly indicating $X(\nu)$ for the specific example given in Figure 1b where $K = 8$. It is in this figure also assumed that $N = 12$ and the arrows indicate where the frequency-samples X_l are obtained. The bold indices are those samples that are used by the size-12 IDFT. Since $K = 8$ and $N = 12$ the QAM symbol a_0 appears at the sampling indices $l =$...-15,-3,9,21,..

We know that the frequency interval between two successive sub-carrier frequencies is f_Δ Hz. Furthermore, the normalized frequency parameter ν is $\nu = \frac{f}{f_{samp}} = \frac{f}{N f_\Delta}$. This means that in the normalized frequency domain, i.e. in the ν-domain, the normalized frequency interval between two successive sub-carrier frequencies is $\frac{f_\Delta}{N f_\Delta} = \frac{1}{N}$.

By definition we also know that the sample X_l is obtained at the normalized frequency $\nu = l/N$. Hence, the spacing between two successive sampling instants is $1/N$ and, as was discussed above, this is ***identical*** with the spacing between two successive normalized baseband sub-carrier frequencies. This is of fundamental importance since this implies that the l:th sampling instance then occur at the **peak** of a particular normalized baseband QAM frequency spectrum, or at its **zero-crossings**! See equations (2.9)-(2.11), (2.17), and also Figure 4.

Examples in figure 4: The sample X_0 is obtained with $l = 0$, and $l = 0 = g_{n_{rc}} = g_3$. *The sample* X_0 *will therefore* ***only*** *carry information about the QAM symbol* $a_{n_{rc}}$, and in this example $a_{n_{rc}} = a_3$. Furthermore, the sample X_{10} is obtained with $l = 10$, and since the samples X_{10} and $X_{-2} = X_{g_1}$ are identical the sample X_{10} will **only** carry information about the QAM symbol a_1.

Note also in figure 4, that the baseband sub-carrier frequency $g_7 f_\Delta = 4f_\Delta$ appears at $\nu = 4/12$. If N would be doubled to N=24 in figure 4, then this baseband sub-carrier frequency would appear at $\nu = 4/24$ instead. In general, as N increases with K held fixed, the K normalized baseband sub-carrier frequencies will be more and more concentrated around the integer values of ν.

It is instructive to take a closer look at the impact of the individual signal $x_n(t)$ on X_l. Therefore, consider the Fourier transform of the discrete-time signal obtained if **only** samples from $x_n(t)$ is considered, i.e. the Fourier transform of the discrete-time signal,

$$a_n e^{j2\pi g_n m/N}, \qquad m = 0, 1, \ldots, (N-1) \tag{2.18}$$

The Fourier transform of the discrete-time signal in equation (2.18) can be expressed as,

$$\sum_{m=0}^{N-1} a_n e^{j2\pi g_n m/N} e^{-j2\pi \nu m} = a_n \frac{\sin(\pi N(\nu - g_n/N))}{\sin(\pi(\nu - g_n/N))} e^{-j\pi(\nu - g_n/N)(N-1)} \tag{2.19}$$

To get the result in equation (2.19) we have identified a geometric series in the left-hand side. Observe that the maximum value in equation (2.19) is $a_n N$, and it is obtained at $\nu = \frac{g_n}{N} + i$, where i denotes an arbitrary integer (periodicity).

It is found, see also Problem 2.6, that if the Fourier transform in equation (2.19) is sampled at the normalized frequency $\nu = l/N$, then a **non-zero** result is obtained **only** if $l = g_n + iN$, where i denotes an arbitrary integer, and the non-zero result equals the peak value $a_n N$.

Therefore, if the specific sample X_l is non-zero then only **one** of the K QAM signals will contribute to the value of X_l,

$$X_l = a_n N \tag{2.20}$$

and the particular value of n is defined by,

$$g_n = l - iN \tag{2.21}$$

This means that we now can determine the sample X_l.

For $0 \le l \le g_{K-1}$ we find from equation (2.21) that $g_n = l$, and the corresponding value of n is known from equations (1.8)-(1.9). Therefore,

$$n = n_{rc} + g_n = n_{rc} + l \tag{2.22}$$

$$X_l = N a_{n_{rc}+l} \qquad l = 0,1 \ldots, g_{K-1} \tag{2.23}$$

For $g_0 + N \leq l \leq (N-1)$ we find from equation (2.21) that $g_n = l - N$, and the corresponding values of n and X_l are,

$$n = n_{rc} + g_n = n_{rc} + l - N \qquad (2.24)$$

$$X_{-n_{rc}+N+n} = Na_n \qquad n = 0,1, \dots (n_{rc} - 1) \qquad (2.25)$$

For the <u>*intermediate interval*</u> $(g_{K-1} + 1) \leq l \leq (g_0 + N - 1)$, that covers the remaining (N-K) samples, the values of these samples are $X_l = 0$. Compare with Figure 4 where $(N-K) = 4$ and where $X_5 = X_6 = X_7 = X_8 = 0$.

Observe from equations (2.20)-(2.25) that the desired sequence $X_0, X_1, \dots, X_{N-1}$ is very easy to construct by using the K signal-points a_n, and $(N-K)$ zeroes! This is shown below.

If we first construct the size-N sequence $Na_0, Na_1, .., Na_{K-1}, 0,0, .., 0$, and then "left-rotate" this sequence n_{rc} positions (or "right-rotate" this sequence $(g_0 + N)$ positions), then the desired sequence $X_0, X_1, \dots, X_{N-1}$ in equations (2.20)-(2.25) is obtained!

Consider as an example the case K=8 and N=12. In this case $n_{rc} = 3$ and $g_{K-1} = 4$, and the desired sequence $X_0, X_1, \dots, X_{11}$ then equals: $Na_3, Na_4, Na_5, Na_6, Na_7, 0,0,0,0, Na_0, Na_1, Na_2$. See also figure 4.

The final step is to calculate the size-N **IDFT**,

$$x_n = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}, n = 0,1, \dots, N-1 \qquad (2.26)$$

In practice, **N is a power of 2** since fast Fourier transform (FFT) algorithms can then be used to significantly speed up the calculations in equation (2.26).

Equation (2.26) is the desired final expression to compute the discrete-time signal $\boldsymbol{x}$. *Equation (2.26), i.e. the size-N IDFT, is computationally very efficient when implemented using FFT algorithms (N is a power of 2). The sequence* $X_0, X_1, \dots, X_{N-1}$ *is given by equations (2.22)-(2.25), and also by the construction ("rotation") given above. See also Figure 5 on page 20.*

The $(N-K)$ zeroes in the sequence $\boldsymbol{X}$ can be interpreted as using zero-valued signal-points at baseband sub-carriers located at the edges but outside of the OFDM frequency band.

It should also be mentioned here that the "rotation" operation described above can alternatively be expressed as a matrix multiplication. Define the size-K column vector $\boldsymbol{a}$ by $\boldsymbol{a}^{tr} = (a_0\ a_1 \dots a_{K-1})$, where "tr" denotes transpose. Furthermore, define the size-N column vector $\boldsymbol{X}$ by $\boldsymbol{X}^{tr} = (X_0\ X_1 \dots X_{N-1})$. Then, by examining equations (2.23) and (2.25), the "rotation" operation can be described by the following expression,

$$\boldsymbol{X} = N\boldsymbol{Q}_t\boldsymbol{a} \qquad (2.27)$$

where the size NxK matrix $\boldsymbol{Q}_t$ has the value one in the (i,j):th elements given below (the rows are numbered from 0 to (N-1), and the columns are numbered from 0 to (K-1) :

$$(i,j): (0, n_{rc}), (1, n_{rc}+1), (2, n_{rc}+2), \dots, (g_{K-1}, K-1) \qquad (2.28)$$

$$(i,j): (N-n_{rc}, 0), (N-n_{rc}+1, 1), (N-n_{rc}+2, 2), \dots, (N-1, n_{rc}-1) \qquad (2.29)$$

The total number of matrix element positions given in equations (2.28)-(2.29) is K, and the value of each corresponding matrix element is one. For the remaining matrix elements in $\boldsymbol{Q}_t$ the value is zero.

Note that the operation in equation (2.27) automatically places N-K zeroes in the sequence $\boldsymbol{X}$ since the matrix $\boldsymbol{Q}_t$ has N-K rows that contain only zeroes. Furthermore, every column in the matrix $\boldsymbol{Q}_t$ has only one element that is equal to one.

Example: If K=8 and N=12 in equations (2.28)-(2.29), then $\boldsymbol{Q}_t$ equals ($n_{rc} = 3$ and $g_7 = 4$) :

$$\boldsymbol{Q}_t = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Example: If K=9 and N=12 in equations (2.28)-(2.29), then $\boldsymbol{Q}_t$ equals ($n_{rc} = 4$ and $g_8 = 4$) :

$$\boldsymbol{Q}_t = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For future reference let us also observe that,

$$\boldsymbol{a} = \frac{1}{N}\boldsymbol{Q}_r\boldsymbol{X} \tag{2.30}$$

where the size KxN matrix $\boldsymbol{Q}_r$ is defined by,

$$\boldsymbol{Q}_r = \boldsymbol{Q}_t{}^{tr} \tag{2.31}$$

Furthermore, the size KxK matrix $\boldsymbol{Q}_r\boldsymbol{Q}_t$ is the identity matrix, while the size NxN matrix $\boldsymbol{Q}_t\boldsymbol{Q}_r$ has K ones and N-K zeroes on the main diagonal (and the remaining elements are zero).

As an example of an application let us consider LTE-systems (Long-Term Evolution). In LTE (from [9]), OFDM is used and $f_\Delta = \frac{1}{T_{obs}} = 15$ kHz which means that $T_{obs} = 66.67$ µs. Furthermore, assume that, e.g., K=720 sub-carriers are used and that N=1024 is chosen. The OFDM bandwidth is then approximately $Kf_\Delta = 10.8$ MHz around the carrier frequency, and the chosen sampling frequency is $f_{samp} = Nf_\Delta = 15.36$ Msample per second. A typical OFDM interval in LTE is 71.36 µs, and 14 OFDM signals (i.e. 14 size-1024 IDFT calculations) are then generated every ms.

It is clear from the numbers in the example above that very fast and computationally efficient implementations are required to be able to build the state-of-the-art communication systems of today. It is recommended that the reader reflects over the implementation parameters given in the example above, including the size of the unit and its power-consumption (and cooling requirements).

Let us now introduce a compact and convenient description of the DFT and IDFT operations that is based on matrices. Therefore, consider the (symmetric) size NxN matrix $\boldsymbol{F}$ with matrix elements $F_{k,n}$,

$$F_{k,n} = e^{-j2\pi kn/N} \tag{2.32}$$

The sequence (column vector) of frequency-domain samples $\boldsymbol{X}$ obtained from the DFT operation in equation (2.15) can then be written as,

$$\boldsymbol{X} = \boldsymbol{F}\boldsymbol{x} \qquad \textbf{(DFT)} \tag{2.33}$$

and the column vector $\boldsymbol{x}$ contains the N time-domain samples $x_0, x_1, \dots, x_{N-1}$, of the signal $x(t)$ in equation (2.3).

In the same way consider the (symmetric) size NxN matrix $\boldsymbol{G}$ with matrix elements $G_{k,n}$,

$$G_{k,n} = e^{j2\pi kn/N} \tag{2.34}$$

The sequence of time-domain samples $\boldsymbol{x}$ obtained from the IDFT operation in equation (2.16) can then be written as,

$$\boldsymbol{x} = \frac{1}{N}\boldsymbol{G}\boldsymbol{X} \qquad \textbf{(IDFT)} \tag{2.35}$$

By combining equation (2.27) and equation (2.35) we obtain the compact description,

$$\boldsymbol{x} = \frac{\mathbf{1}}{\boldsymbol{N}}\boldsymbol{G}N\boldsymbol{Q}_t\boldsymbol{a} \tag{2.36}$$

The matrices $\boldsymbol{F}$ and $\boldsymbol{G}$ are such that the size NxN matrix $\frac{1}{N}\boldsymbol{F}\boldsymbol{G}$ is the identity matrix. Furthermore, for future reference (in section 6), by combining equation (2.30), equation (2.33) and equation (2.36) we obtain the equalities,

$$\boldsymbol{a} = \frac{1}{N}\boldsymbol{Q}_r\boldsymbol{X} = \frac{1}{N}\boldsymbol{Q}_r\boldsymbol{F}\boldsymbol{x} = \frac{1}{N}\boldsymbol{Q}_r\boldsymbol{F}\frac{\mathbf{1}}{\boldsymbol{N}}\boldsymbol{G}N\boldsymbol{Q}_t\boldsymbol{a} \tag{2.37}$$

An alternative and equally important way to determine the impact of the individual signal $x_n(t)$ on X_l is to first use equation (2.17) to determine the Fourier transform of the discrete-time signal in equation (2.18). The second step is then to sample this Fourier transform at $\nu = l/N$ and investigate its value.

We have already found the Fourier transform of the analog signal $x_n(t)$, see equation (2.9), and it was found that its spectrum is **sinc-shaped** around the baseband sub-carrier frequency $\frac{g_n}{T_{obs}}$ Hz , with **peak value** $a_n T_{obs}$, and with **zero-crossings** at the frequencies $f = \frac{g_n}{T_{obs}} + i/T_{obs}$ (for any non-zero integer i). Hence, the width of the **main-lobe** is $2f_\Delta$ Hz.

From equation (2.17) it is also known how to obtain the Fourier transform of the discrete-time signal if the Fourier transform of the corresponding analog signal is known.

Therefore, the Fourier transform of the discrete-time signal in equation (2.18) equals,

$$f_{samp} X_{a,n}\left(\nu f_{samp}\right) = f_{samp} a_n T_{obs} \frac{sin(\pi\left(\nu f_{samp} - \frac{g_n}{T_{obs}}\right)T_{obs})}{\pi\left(\nu f_{samp} - \frac{g_n}{T_{obs}}\right)T_{obs}} \; e^{-j\pi\left(\nu f_{samp} - \frac{g_n}{T_{obs}}\right)T_{obs}} \qquad (2.38)$$

plus *periodic repetitions of this function*, spaced with normalized frequency 1 apart.

Let us now use that $f_{samp} = N/T_{obs}$. Then equation (2.38) is simplified to,

$$f_{samp} X_{a,n}\left(\nu f_{samp}\right) = N a_n \frac{sin(\pi N(\nu - g_n/N))}{\pi N(\nu - g_n/N)} \; e^{-j\pi N(\nu - g_n/N)} \qquad (2.39)$$

An important observation now is that sampling the frequency-function in equation (2.39), at the normalized frequency $\nu = l/N$, results in a **non-zero** value **only** if $l = g_n$, and this non-zero value equals $a_n N$.

So, the impact on X_l from the individual signal $x_n(t)$ is therefore found to be equal to $a_n N$ **only** for the indices $l = g_n + iN$, where i denotes an arbitrary integer (due to the periodicity in ν with period 1), and zero for any other l. *Observe that this is exactly the same result as was obtained earlier in connection to equation (2.20) following an alternative path.*

It is also very instructive to compare equation (2.39) with equation (2.19), especially in the fundamental interval $-1/2 \leq \nu \leq 1/2$. The reason why these two expressions are different is that the Fourier transform in equation (2.19) includes the total effect of *aliasing* (aliasing is a consequence when the chosen sampling frequency does not satisfy the sampling theorem, and the effect of aliasing therefore decreases as N increases), while equation (2.39) only represents the contribution to the Fourier transform given by the term corresponding to the index k=0 in equation (2.17).

In the next section we will add the so-called cyclic prefix (CP) to the vector $\boldsymbol{x}$, and also use digital-to-analog (D/A) converters to create the analog real signals that are needed in the construction of the OFDM signal.

3. The Cyclic Prefix (CP) and Digital-to-Analog (D/A) conversion

Adding the Cyclic prefix (CP)

In the previous section we found how the size-N IDFT could be used to efficiently create N time-domain complex samples of the equivalent complex baseband OFDM signal $x(t)$ in equations (2.2) and (2.3). These N samples will in this section be time-shifted T_{CP} seconds and they then become the samples of the OFDM signal $x(t - T_{CP})$ within the time-interval $T_{CP} \leq t \leq T_s$, i.e. within the observation interval of the receiver. We will here also specify the remaining L samples corresponding to the cyclic prefix (CP) within the time-interval $0 \leq t \leq T_{CP}$.

The OFDM signal that we are in progress to synthesize has a CP of duration T_{CP} in the beginning of the OFDM interval T_s, and the remaining part of the OFDM signal has duration T_{obs}=$T_s - T_{CP}$. As was mentioned in the first section, the main purpose of the CP is to allow for the initial transient behavior of the multi-path channel *output* signal to occur within the duration of the CP. In a sense, the CP acts as a “guard interval”.

It is also very important that the duration of the cyclic prefix is at least as large as the duration of the impulse response of the channel. This will be explained in detail in section 5. An additional requirement is that $T_{CP} \ll T_s$.

As an example, in LTE (from [9]) a typical choice of T_{CP} is $T_{CP} = 4.69$ µs which is 6.57 % of the OFDM symbol interval T_s.

In this section we will construct the CP by adding L additional samples before the already created N samples. As will be seen, the CP is constructed by adding a so-called size-L *periodic extension* of the N samples. In this way all $L + N$ samples of a complex baseband OFDM signal within the time-interval $0 \leq t \leq T_s$ is obtained, and $T_{CP} = LT_{obs}/N$. The remaining steps is then D/A converters (also in this section) and frequency up-converting (in section 4).

Now observe that the signal $x(t)$ in equation (2.3) has duration T_{obs}. However, the expression that is used to define $x(t)$ equals $\sum_{n=0}^{K-1} a_n e^{j2\pi g_n f_\Delta t}$, and this expression is **periodic in t with period T_{obs}**. Therefore, this expression is *identical* within the two time-intervals $-T_{CP} \leq t \leq 0$ and $(T_{obs} - T_{CP}) \leq t \leq T_{obs}$. Alternatively, the values of this expression in the former time-interval above is a “copy” of the values in the latter time-interval. Observe that the L last samples in the size-N vector **x** are the samples that correspond to the latter time-interval $(T_{obs} - T_{CP}) \leq t \leq T_{obs}$. Hence, the values of these L last samples are identical to values of the L samples corresponding to the time-interval $-T_{CP} \leq t \leq 0$, i.e. these L samples constitute the CP.

So, if we consider the expression $\sum_{n=0}^{K-1} a_n e^{j2\pi g_n f_\Delta t}$ *only* within the time-interval $-T_{CP} \leq t \leq T_{obs}$ it defines an OFDM signal, and the OFDM signal interval is $T_s = T_{CP} + T_{obs}$. Furthermore, all L+N time-domain samples are known since the size-N vector **x** has already been created, and, as was described above, the L samples corresponding to the CP are identical to the last L samples in **x**.

Since we should have a *causal* OFDM signal we need to introduce a time-delayed version of the expression studied above. With a time delay equal to T_{CP} seconds we then get the expression $\sum_{n=0}^{K-1} a_n e^{j2\pi g_n f_\Delta (t - T_{CP})}$ within the OFDM signal interval $0 \leq t \leq T_S$. This is the final desired expression for the synthesized OFDM signal.

What remains to be done in the digital domain is to construct a new size-(L+N) vector $\boldsymbol{u}$ that contains all (L+N) time-domain samples of the expression $\sum_{n=0}^{K-1} a_n e^{j2\pi g_n f_\Delta (t-T_{CP})}$ within the OFDM signal interval $0 \leq t \leq T_S$.

Based on the discussion about periodicity above let us therefore construct a new size-(L+N) vector $\boldsymbol{u}$ as a so-called <u>*periodic extension*</u> of the size-N vector $\mathbf{x}$. This means that *the L last samples in $\boldsymbol{x}$ are copied and placed as the first L samples in $\boldsymbol{u}$*. The remaining N samples in $\boldsymbol{u}$ are identical to $\boldsymbol{x}$. This means that,

$$u_0 = x_{N-L}, \ldots, u_{L-1} = x_{N-1}, u_L = x_0, \ldots, u_{L+N-1} = x_{N-1}. \tag{3.1}$$

The construction of the vector $\boldsymbol{u}$ above implies that the first L samples in $\boldsymbol{u}$ are identical with the last L samples in $\boldsymbol{u}$, and this reflects the periodicity discussed above.

The duration of the OFDM signal interval is T_s, and it can be expressed as,

$$T_s = \frac{(L+N)T_{obs}}{N} = T_{CP} + T_{obs} \tag{3.2}$$

The vector $\boldsymbol{u}$ in equation (3.1) contains (L+N) time-domain complex samples of a complex baseband OFDM signal defined over the entire interval $0 \leq t \leq T_s$. This baseband OFDM signal is here denoted $u(t)$, and based on the previous discussion in this section, the OFDM signal $u(t)$ is,

$$u(t) = u_{Re}(t) + ju_{Im}(t) = \sum_{n=0}^{K-1} a_n e^{j2\pi g_n f_\Delta (t-T_{CP})}, \quad 0 \leq t \leq T_s \tag{3.3}$$

and $u(t)$ equals zero outside this interval. The OFDM signal $u(t)$ in equation (3.3) includes the CP of duration T_{CP} and also the observation interval of duration T_{obs}. Furthermore, the m:th sample in the vector $\boldsymbol{u}$ is,

$$u_m = u(mT_{obs}/N), \;\; m = 0,1, \ldots, (L+N-1) \tag{3.4}$$

As we will see in section 4, and in analogy with equations (2.2) and (2.5), the desired real high-frequency OFDM signal, here denoted $s(t)$, is obtained as,

$$s(t) = Re\{u(t)e^{j2\pi f_{rc} t}\} = u_{Re}(t)\cos(2\pi f_{rc} t) - u_{Im}(t)\sin(2\pi f_{rc} t) \tag{3.5}$$

Our work in the discrete-time (digital) domain is now finished and it is time to enter the continuous-time (analog) domain. The practical way to do this is to use D/A converters.

D/A converting

In analogy with equation (3.5) the construction of the desired real high-frequency OFDM signal $s(t)$ is straight-forward ($s_I(t) = u_{Re}(t)$, and $s_Q(t) = u_{Im}(t)$),

$$s(t) = s_I(t)\cos(2\pi f_{rc}t) - s_Q(t)\sin(2\pi f_{rc}t) \qquad (3.6)$$

Hence, we first need to construct the two analog signals $s_I(t)$ and $s_Q(t)$ that correspond to the real part $\boldsymbol{u}_{Re}$ and to the imaginary part $\boldsymbol{u}_{Im}$ of the vector $\boldsymbol{u}$ in equations (3.1) and (3.4), respectively, where

$$\boldsymbol{u} = \boldsymbol{u}_{Re} + j\boldsymbol{u}_{Im} \qquad (3.7)$$

These two real-valued sequences each feeds a separate D/A converter as is illustrated in Figure 5, and the samples in each sequence arrive with the rate $f_{samp} = Nf_\Delta$ samples per second.

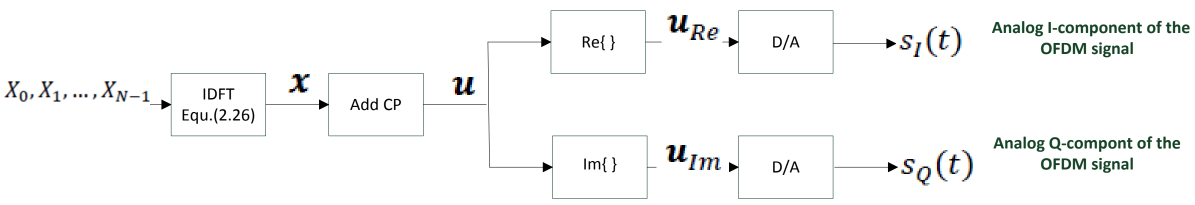

Figure 5. Block diagram illustrating the operations in the digital domain, and the transition to the analog domain.

Let $s_I(t)$ denote the analog output signal (i.e. the I-component of the OFDM signal) that results from the sequence (or vector) $\boldsymbol{u}_{Re}$. In the same way, let $s_Q(t)$ denote the analog output signal (i.e. the Q-component of the OFDM signal) that results from the sequence (or vector) $\boldsymbol{u}_{Im}$.

We will now go through the D/A conversion of the digital sequence $\boldsymbol{u}_{Re}$ to the analog signal $s_I(t)$.

The ideal goal of the D/A converter is to create as output signal an analog (time-continuous) voltage waveform that preserves the samples in $\boldsymbol{u}_{Re}$ in the time-domain, and **also** preserves the frequency content of $\boldsymbol{u}_{Re}$ corresponding to the fundamental interval $-1/2 \leq \nu \leq 1/2$ (compare with Figure 4, but assume in this figure a spectrum that is symmetric around $\nu = 0$, since $\boldsymbol{u}_{Re}$ is a *real* sequence). Ideally, given the samples in $\boldsymbol{u}_{Re}$, and given the sampling frequency f_{samp}, the output analog waveform can be determined and it is *unique*.

The overall operation of the D/A converter may be interpreted as an *interpolation,* and typically this can be achieved by constructing the signal $s_I(t)$ as,

$$s_I(t) = \sum_{m=0}^{L+N-1} u_{Re,m}\, g_i(t - \frac{mT_{obs}}{N}) \qquad (3.8)$$

where $g_i(t)$ is referred to as an *interpolation filter* or reconstruction filter. Note in equation (3.8) that $s_I(t)$ is created by weighting the m:th sample $u_{Re,m}$ with a time-dependent and m-dependent weighting-factor $g_i(t - \frac{mT_{obs}}{N})$, and then add all L+N contributions. Hence, in general the value $s_I(t)$

is created as a weighted sum of all elements in the vector $\boldsymbol{u}_{Re}$. However, as is also seen in equation (3.8), the total contribution to $s_I(t)$ from the individual sample $u_{Re,m}$ equals,

$$u_{Re,m} g_i(t - \frac{mT_{obs}}{N}) \quad (3.9)$$

and equation (3.8) is the superposition of L+N such contributions.

However, at the m:th sampling-time we know that $s_I\left(\frac{mT_{obs}}{N}\right) = u_{Re,m}$ which means that $g_i(t = 0) = 1$, and $g_i\left(t = \frac{lT_{obs}}{N}\right) = 0$ where l is any non-zero integer.

Furthermore, it is reasonable to assume that the contribution in equation (3.9) is large at, and around, $t = \frac{mT_{obs}}{N}$ and then gets "smaller" as t moves away from this value. In summary; the filter $g_i(t)$ should have its peak value 1 at $t = 0$, should be equal to zero at $t = \frac{lT_{obs}}{N}$ where l is any non-zero integer, and in general "decay" as $|t|$ increases.

Let $U_{Re}(\nu)$ denote the Fourier transform of the input discrete-time signal $\boldsymbol{u}_{Re}$, and let $S_I(f)$ denote the Fourier transform of the output continuous-time signal $s_I(t)$. Equation (3.8) can be interpreted as a filtering operation, where $g_i(t)$ is the impulse response of the filter and $G_i(f)$ denotes the transfer function of the filter. It is left as an exercise to the reader to show that, with $f_{samp} = N/T_{obs}$,

$$S_I(f) = U_{Re}(\nu = f/f_{samp})G_i(f) \quad (3.10)$$

Note that since $U_{Re}(\nu)$ is periodic in ν with period 1, $U_{Re}(\nu = f/f_{samp})$ is periodic in f with period f_{samp}.

Hence, to extract *only* the desired frequency content in $\boldsymbol{u}_{Re}$ corresponding to only the fundamental interval $-f_{samp}/2 \le \nu \le f_{samp}/2$ the transfer function $G_i(f)$ of the filter should have low-pass characteristics with an ideal cut-off frequency equal to $f_{samp}/2$ Hz.

It is here instructive to investigate some properties of the **ideal** reconstruction filter. This filter can be derived from the sampling theorem and the result is a filter with infinite duration, see ref. [1],

$$g_{i,ideal}(t) = \frac{\sin(\pi f_{samp} t)}{\pi f_{samp} t} \quad (3.11)$$

If this ideal filter is used in equation (3.8) then we find that the particular sample $s_I(t = kT_{obs}/N) = s_I(t = k/f_{samp}) = u_{Re,k}$, and this is due to the zero-crossings in $g_{i,ideal}(t)$. Furthermore, this ideal filter is actually an ideal low-pass filter i.e. having a non-zero (and symmetric) transfer function **only** within the frequency interval $-f_{samp}/2 \le f \le f_{samp}/2$,

$$G_{i,ideal}(f) = 1/f_{samp}, \quad -f_{samp}/2 \le f \le f_{samp}/2 \quad (3.12)$$

Hence, using this ideal filter in equation (3.8) indeed extracts only the fundamental frequency interval.

However, there are some practical difficulties with the ideal filter such as its non-causal property and its ideal transfer function.

Instead of trying to implement a single filter $g_i(t)$ with sufficiently good properties, it is common to split $g_i(t)$ into two filters denoted $g_{i,1}(t)$ and $g_{i,2}(t)$. First the simpler filter $g_{i,1}(t)$ is used in equation (3.8). The resulting analog signal is then filtered with the second filter $g_{i,2}(t)$. The final output signal from the second filter then has the Fourier transform,

$$U_{Re}\left(\nu = f/f_{samp}\right)G_{i,1}(f)G_{i,2}(f) = U_{Re}(\nu = f/f_{samp})G_i(f) \qquad (3.13)$$

Hence, the convolution between $g_{i,1}(t)$ and $g_{i,2}(t)$ equals the overall filter $g_i(t)$.

A typical choice of the first filter is $g_{i,1}(t) = 1$ in the time-interval $0 \leq t \leq T_{obs}/N$, and $g_{i,1}(t) = 0$ outside this interval. The analog output signal from this filter is then a **staircase**-function, i.e. the output signal consists of a sequence of T_{obs}/N-long rectangular pulses where the amplitude of the m:th pulse equals the value of the sample $u_{Re,m}$. Therefore, this choice of filter is referred to as S/H filter (**Sample-and-Hold**).

The Fourier transform of the S/H filter $g_{i,1}(t)$ above is,

$$G_{i,1}(f) = \frac{1}{f_{samp}}\frac{sin(\pi f/f_{samp})}{\pi f/f_{samp}}\ e^{-j\pi f/f_{samp}} \qquad (3.14)$$

which is significantly different then $G_{i,ideal}(f)$ in equation (3.12).

To improve the D/A conversion let us, as an example, choose the second filter as $g_{i,2}(t) = f_{samp}g_{i,1}(t)$ where $G_{i,1}(f)$ is given by equation (3.14). The overall filter $g_i(t)$ is then **triangular**, with peak value equal to 1 at $t = T_{obs}/N$ and with duration $2T_{obs}/N$ (symmetric around $t = T_{obs}/N$). *Observe* that the analog output signal from this second filter then consists of **straight lines** connecting the sample values. *Observe also* that a **time-delay** equal to T_{obs}/N (corresponding to 1 sample) here is obtained. See also Problem 3.5b.

The Fourier transform of the overall triangular filter $g_i(t)$ is,

$$G_i(f) = \frac{1}{f_{samp}}\left(\frac{\sin\left(\frac{\pi f}{f_{samp}}\right)}{\frac{\pi f}{f_{samp}}}\right)^2\ e^{-j2\pi f/f_{samp}} \qquad (3.15)$$

and with this filter a more efficient overall low-pass filtering is obtained. Furthermore, the phase function is a linear function of frequency over the fundamental frequency interval of interest, and it is seen in the exponential in equation (3.15) that the linear phase corresponds to the one-sample time-delay T_{obs}/N (=$1/f_{samp}$).

In case additional improvements are needed additional low-pass filtering can be implemented. One possibility may be to try to synthesize an overall interpolation filter that is a symmetrically time-truncated and time-delayed version of the ideal filter. There is also the possibility of digital filtering of the sequence $\boldsymbol{u}_{Re}$ before D/A-conversion, which may relax the requirements on the analog interpolation filter. It should however be observed that efficient high precision (e.g. 16 bits per sample or more) D/A and A/D converters operating at high sampling frequencies are challenging to implement in practice, and different implementation strategies may be used.

This concludes our introductory treatment on D/A-converters. To simplify the presentations in the next sub-sections we will assume that the two signals $s_I(t)$ and $s_Q(t)$ are almost perfectly reconstructed using almost ideal D/A-converters in Figure 5. This means that we will assume throughout that both $S_I(f)$ and $S_Q(f)$ can be closely approximated to be ideal. Hence, it is assumed that both these frequency functions are equal to zero outside the frequency interval $-f_{samp}/2 \leq f \leq f_{samp}/2$, and within this interval they are equal to,

$$S_I(f) = \frac{1}{f_{samp}} U_{Re}\left(\nu = \frac{f}{f_{samp}}\right), \quad in \quad -f_{samp}/2 \leq f \leq f_{samp}/2 \tag{3.16}$$

$$S_Q(f) = \frac{1}{f_{samp}} U_{Im}\left(\nu = \frac{f}{f_{samp}}\right), \quad in \quad -f_{samp}/2 \leq f \leq f_{samp}/2 \tag{3.17}$$

In the time domain equations (3.16)-(3.17) means that we assume that (from equation (3.3)),

$$s_I(t) + js_Q(t) = \sum_{n=0}^{K-1} a_n e^{j2\pi g_n f_\Delta (t-T_{CP})}, \quad 0 \leq t \leq T_s \tag{3.18}$$

We are now ready for the final stages in the transmitter, i.e. frequency up-converting, power amplification and the antenna coupling unit.

4. Frequency up-converting and power amplification

Similar to equation (3.6) the construction of the OFDM signal $s(t)$ is straight-forward,

$$s(t) = s_I(t)\cos(2\pi f_{rc}t + \phi) - s_Q(t)\sin(2\pi f_{rc}t + \phi) \quad (4.1)$$

where $s_I(t)$ and $s_Q(t)$ are given in equation (3.18). Note however that in equation (4.1) we have also taken into account the *actual phase* ϕ of the high-frequency signal oscillators. This is illustrated in Figure 6 for the case when K is odd for which $f_{rc} = f_c$.

To get an OFDM signal with sufficient signal power, a power amplifier (PA) is needed. This means that if the amplifier amplifies the input signal amplitude with a factor A then the signal power is changed with a factor A^2.

It should be remembered that an OFDM signal basically equals the summation of K signals, see e.g., equation (1.12). Therefore, we can expect that there will be time-intervals where the OFDM signal is small due to *destructive addition* of the K signals. In the same way we can expect that there will be time-intervals where the OFDM signal is strong due to *constructive addition* of the K signals. Hence, there will be amplitude (envelope) variations in the OFDM signal, and if these variations are too large they will cause some problems concerning the implementation of the power amplifier. The so-called Peak-to-Average-Power-Ratio (PAPR, see, e.g., ref. [9]) is a common parameter to quantify the instantaneous power variation in a signal. For signals with high PAPR it is a challenge to implement power-efficient and low-cost power amplifiers, and this is especially important in, e.g., the uplink in mobile communication systems.

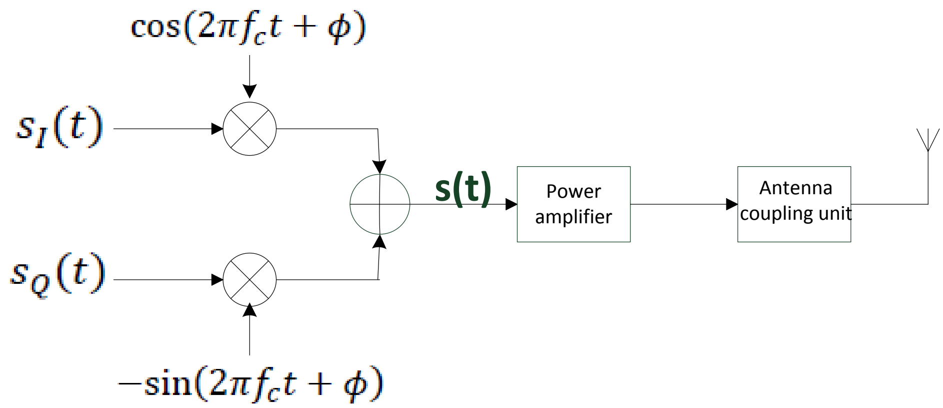

Figure 6. Block diagram illustrating frequency up-converting to the carrier frequency (K is odd), the power amplifier, and the antenna coupling unit. The OFDM signal $s(t)$ is given in equation (4.1).

The last block in Figure 6 is the antenna coupling unit which is the interface to the antenna. Finally, the OFDM signal is radiated out from the antenna as an electro-magnetic wave propagating in space.

We have here only briefly mentioned the power amplifier and the antenna. The reader is recommended to study these important parts in more advanced literature.

This concludes the transmitter part of these lecture notes, and we will in the next section investigate how the multi-path channel changes the transmitted OFDM signal.

5. The multi-path (linear time-invariant filter) channel, and the additive white Gaussian noise (AWGN)

The OFDM signal $s(t)$ in equation (4.1) is amplified and transmitted through a multi-path channel (or a linear time-invariant filter channel), and in this section we will determine how the transmitted signal is changed by this channel. It is here assumed that the filter can be considered to be time-invariant over a few OFDM symbol intervals.

Since the channel is linear it is convenient to first determine how the n:th QAM signal in the OFDM signal is changed by the channel. The result obtained will then also tell us how the remaining K-1 QAM signals are changed by the channel.

Using equations (3.18) and (4.1) we conclude that the n:th transmitted QAM signal can be expressed as,

$$ARe\left\{a_n e^{j2\pi g_n f_\Delta (t-T_{CP})} e^{j(2\pi f_{rc} t+\phi)}\right\} = ARe\left\{a_n e^{j(2\pi f_n t+\theta_n)}\right\}, \quad 0 \le t \le T_s \qquad (5.1)$$

and this signal is zero outside the OFDM symbol interval. A denotes the amplitude amplification in the power-amplifier. The sub-carrier frequency is (see also equation (1.7)) $f_n = f_{rc} + g_n f_\Delta$, and the phase θ_n in equation (5.1) is,

$$\theta_n = -\frac{2\pi g_n T_{CP}}{T_{obs}} + \phi \qquad (5.2)$$

The linear filter channel has the (real) impulse response $h(t)$, and its input signal is denoted $q(t)$, see also Figure 7 below.

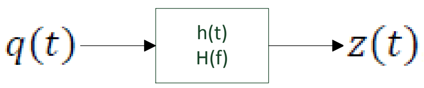

Figure 7. Illustrating the linear time-invariant filter channel.

In case the linear filter is a multi-path channel with P signal paths its impulse response is,

$$h(t) = \sum_{n=1}^{P} \alpha_n \delta(t-\tau_n) \qquad (5.3)$$

If we assume (for simplicity) that the delays in the multi-path channel are numbered in increasing order then we have that $\tau_1 < \tau_2 < \cdots < \tau_P$. Furthermore, in this case it is easy to identify the *duration of the impulse response*, here denoted T_{ch}, to be equal to the largest delay, i.e. $T_{ch} = \tau_P$.

It is also straight-forward to obtain the transfer function $H(f)$ for the multi-path channel in equation (5.3),

$$H(f) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi f t} dt = \sum_{n=1}^{P} \alpha_n \, e^{-j2\pi f \tau_n} \qquad (5.4)$$

Let us now investigate the output signal, here denoted $z(t)$, from a channel that can be represented by the impulse response $h(t)$.

The input signal $q(t)$ is given by equation (5.1).

Observe! Here it is assumed that $h(t) = 0$ outside the time-interval $0 \leq t \leq T_{ch}$.

Observe! Here it is also assumed that $T_{CP} \geq T_{ch}$ (in practice we also have that $T_s \gg T_{CP}$).

The convolutional integral gives us a way to calculate the output signal $z(t)$,

$$z(t) = \int_{-\infty}^{\infty} h(x)q(t-x)dx = \int_{0}^{T_{ch}} h(x)q(t-x)dx \qquad (5.5)$$

Since the function $q(t)$ equals zero outside the interval $0 \leq t \leq T_s$, the function $q(t-x)$ in equation (5.5) equals zero outside the interval $(t - T_s) \leq x \leq t$.

The output signal $z(t)$ is therefore zero outside the time-interval $0 \leq t \leq T_s + T_{ch}$. However, to calculate the output signal within this interval it is convenient to study the three sub-intervals below.

$\mathbf{0 \leq t \leq T_{ch}}$**:**

In this time-interval the output signal $z(t)$ in equation (5.5) is,

$$z(t) = \int_{0}^{t} h(x)ARe\{a_n e^{j(2\pi f_n(t-x)+\theta_n)}\}dx \qquad (5.6)$$

and this is the *initial transient* behavior of the output signal.

$\mathbf{T_{ch} \leq t \leq T_s}$**:**

In this time-interval the output signal $z(t)$ in equation (5.5) is,

$$z(t) = \int_{0}^{T_{ch}} h(x)ARe\{a_n e^{j(2\pi f_n(t-x)+\theta_n)}\}dx = ARe\{a_n e^{j(2\pi f_n t+\theta_n)} \int_{0}^{T_{ch}} h(x)e^{-j2\pi f_n x}dx\} \qquad (5.7)$$

Observe that the integral on the right-hand side equals the ***transfer function*** of the filter, evaluated at the sub-carrier frequency f_n, i.e. $H(f_n)$!

Therefore, we can write,

$$z(t) = ARe\{a_n H(f_n)e^{j(2\pi f_n t+\theta_n)}\} \qquad (5.8)$$

This result is ***indeed very important*** and it should be compared with the input signal in equation (5.1). Equation (5.8) shows that within the time-interval $T_{ch} \leq t \leq T_s$ *the output signal is also a QAM signal*, **but the signal point is modified** to be $a_n H(f_n)$. Multiplication with $H(f_n) = |H(f_n)|e^{jarg(H(f_n))}$ means that the original signal point a_n is *attenuated* with $|H(f_n)|$ and *phase rotated* with $arg(H(f_n))$.

$\mathbf{T_s \leq t \leq T_s + T_{ch}}$**:**

In this time-interval the output signal $z(t)$ in equation (5.5) is,

$$z(t) = \int_{t-T_s}^{T_{ch}} h(x)ARe\{a_n e^{j(2\pi f_n(t-x)+\theta_n)}\}dx \qquad (5.9)$$

and this is the *ending transient* behavior of the output signal.

Now, if the input signal to the filter equals the OFDM signal $As(t)$ where $s(t)$ is given by equations (3.18) and (4.1) we conclude that the input OFDM signal can be expressed as,

INPUT OFDM: $As(t) = ARe\left\{\sum_{n=0}^{K-1} a_n e^{j2\pi g_n f_\Delta (t-T_{CP})} e^{j(2\pi f_{rc} t+\phi)}\right\}, \quad 0 \le t \le T_s$ (5.10)

or alternatively as,

INPUT OFDM: $As(t) = ARe\left\{\sum_{n=0}^{K-1} a_n \, e^{j(2\pi f_n t+\theta_n)}\right\}, \quad 0 \le t \le T_s$ (5.11)

and this input OFDM signal is zero outside the OFDM symbol interval. The sub-carrier frequency f_n, the phase θ_n and the reference carrier frequency f_{rc} are given in connection to equation (5.1).

Applying the results in equations (5.6)-(5.9) to each of the K input QAM signals, and using that $T_{CP} \ge T_{ch}$ we find that the output signal $z(t)$ also is an OFDM signal in the time-interval $T_{CP} \le t \le T_s$, where it can be expressed as,

OUTPUT OFDM: $z(t) = ARe\left\{\sum_{n=0}^{K-1} a_n H(f_n) e^{j2\pi g_n f_\Delta (t-T_{CP})} e^{j(2\pi f_{rc} t+\phi)}\right\}, \quad T_{CP} \le t \le T_s$ (5.12)

or alternatively as,

OUTPUT OFDM: $z(t) = ARe\left\{\sum_{n=0}^{K-1} a_n H(f_n) \, e^{j(2\pi f_n t+\theta_n)}\right\}, \quad T_{CP} \le t \le T_s$ (5.13)

Furthermore, in the relatively short time-intervals $0 \le t \le T_{CP}$ and $T_s \le t \le T_s + T_{CP}$ the output signal $z(t)$ contains a transient behavior. Note that if the *next* input OFDM signal starts at $t = T_s$ then the corresponding initial output transient will coincide (i.e. overlap) with the ending output transient of the current OFDM signal. This overlap of OFDM signals may cause inter-symbol interference in the receiver unless counter-measures are made at the receiver side. As we will see in section 6, a simple solution can be used to eliminate the possibility of this inter-symbol interference.

The signal appearing at the receiving antenna can now be expressed as,

$$z(t) + n(t) + \gamma(t) \quad (5.14)$$

where $z(t)$ is the desired signal described in equations (5.12)-(5.13) and in the text just below these equations. The term $n(t)$ in equation (5.14) is assumed to be *additive white Gaussian noise (AWGN)*. The power spectral density of the random process $n(t)$ equals a constant over the entire frequency axis ("white"), and this constant is denoted $N_0/2$. The signal $\gamma(t)$ in equation (5.14) represents all other signals that may be present at the receiving antenna. The frequency content of $\gamma(t)$, within the bandwidth occupied by $z(t)$, is assumed to be *virtually zero*. Hence, the only *in-band disturbance* assumed in these lecture notes are in-band noise originating from $n(t)$.

This completes the description of the communication channel studied in these lecture notes. In the next sections we will investigate different steps in the receiver and also show how the K received noisy signal-points efficiently can be extracted by using the DFT.

6. The Receiver: Frequency down-converting, sampling (A/D) and the DFT

The ultimate goal of the receiver is to, as good as possible, recover the sequence of information bits that corresponds to the original sequence of K QAM signal points $a_0, a_1, \dots, a_{K-1}$. To obtain this goal the first sub-goal of the receiver typically is to extract the corresponding K *received distorted and noisy signal points,* and the purpose of this section is to show how the DFT can be used to accomplish this.

The second, and last, sub-goal is to use the sequence of K received distorted and noisy signal points as input to a suitable *decoding algorithm*. The output sequence from the decoding algorithm is the receiver´s decision of the original sequence of information bits that was sent from the transmitter. In these lecture notes decoding algorithms will not be investigated, so the interested reader is recommended to study more advanced literature on this topic.

Before going into details concerning frequency down-converting to baseband let us start with a fundamental example of how a received noisy QAM signal point may be extracted from a received signal. This example is very important since it gives a good illustration to why the DFT can be used in OFDM to extract the K received noisy signal points.

In this example we assume that the received signal $r(t)$ equals (compare with equation (1.4)),

$$r(t) = b_I \cos(2\pi f_B t) - b_Q \sin(2\pi f_B t) + n(t)\,, \quad 0 \le t \le T \qquad (6.1)$$

Hence, the received signal is a single QAM signal disturbed by AWGN, and our goal in this example is to show how to extract the received noisy signal point.

Basic decision theory tells us that in this case the receiver should use two orthonormal basis functions, here denoted $\psi_1(t)$ and $\psi_2(t)$. These (real) functions should satisfy the two conditions,

$$\int_0^T \psi_1(t)\psi_2(t)\,dt = 0 \qquad \text{(orthogonal)} \qquad (6.2)$$

$$\int_0^T \psi_1(t)^2\,dt = \int_0^T \psi_2(t)^2\,dt = 1 \qquad \text{(normalization)} \qquad (6.3)$$

In this example the choice of orthogonal basis functions are,

$$\psi_1(t) = \cos(2\pi f_B t)/C\,, \qquad 0 \le t \le T \qquad (6.4)$$

$$\psi_2(t) = -\sin(2\pi f_B t)/C\,, \qquad 0 \le t \le T \qquad (6.5)$$

where C is a normalization constant chosen such that equation (6.3) is satisfied. It is also assumed that f_B and T are such that equation (6.2) is satisfied.

The received noisy signal point consists of two coordinates since a QAM signal is assumed in this example. The first coordinate, denoted r_1, is the correlation between the received signal and the first basis function,

$$r_1 = \int_0^T r(t)\psi_1(t)\,dt = Cb_I + n_1 \qquad (6.6)$$

and the second coordinate is calculated in a similar way,

$$r_2 = \int_0^T r(t)\psi_2(t)\,dt = Cb_Q + n_2 \qquad (6.7)$$

In equations (6.6)-(6.7) n_1 and n_2 denotes non-desired noise contributions from the AWGN $n(t)$. It can be shown that these two noise contributions each have a Gaussian probability density function, zero mean and variance $N_o/2$. Furthermore, they are also independent. Note that equations (6.6)-(6.7) may be interpreted as frequency down-conversion to baseband.

The two values (r_1, r_2) in equations (6.6)-(6.7) ***define the received noisy signal point*** and they are of *fundamental importance* in the decision process.

Let us now express the complex value r in the following way,

$$r = r_1 + jr_2 = \int_0^T r(t)e^{-j2\pi f_B t}\, dt/C = R(f_B)/C = Cb + n \qquad (6.8)$$

where $b = b_I + jb_Q$ and $n = n_1 + jn_2$.

It is now very important to observe in equation (6.8) that **the received noisy signal point r can be found by calculating the Fourier transform $R(f)$ of the received signal $r(t)$ over the time interval $0 \leq t \leq T$, and then sample $R(f)$ at $f = f_B$ to obtain $R(f_B)$.** As will be seen later on, *using the DFT in an OFDM receiver can be viewed as a natural extension of this result*. This concludes the example, and it is time to focus on frequency down-converting to baseband.

Frequency down-converting to baseband:

The input signal to the receiving antenna is given in equation (5.14), and the first part of the receiver is illustrated in Figure 8. It is seen in Figure 8 that the antenna coupling unit is followed by a band-pass (BP) filter. This filter typically has a relatively large pass-band to allow for several frequency bands that might be of interest to the receiver, e.g. the OFDM frequency band that contains the signal $z(t)$ in equation (5.13). Signals located outside the pass-band of the band-pass filter are rejected (more attenuated) and are of no interest to the receiver.

The received signal is normally a very weak signal and it needs to be amplified. However, with weak signals additional noise can be very hurtful to the *signal-to-noise-ratio*, and therefore a *low-noise amplifier* (LNA) is used in the receiver. LNA:s are designed with the purpose to keep the internal generated noise to a minimum, and only a very small loss in signal-to-noise ratio due to the LNA is acceptable.

It is *in principle* possible to extract the K received noisy signal points directly from the output signal of the LNA, here denoted $y_r(t)$. We learned from the previous example in this section, see equation (6.8), that the n:th received noisy signal point is found if we calculate the Fourier transform,

$$\int_{T_C}^{T_S} y_r(t)e^{-j2\pi f_n t}\,dt, \qquad n = 0,1,\dots,K-1 \tag{6.9}$$

Since the sub-carrier frequency separation is $f_\Delta = 1/T_{obs}$ and since the sub-carrier frequency f_n is at a high frequency, different QAM-signals are virtually orthogonal, see equation (1.14), and this implies that there will be no “leakage” from the other (K-1) QAM-signals in the value given by equation (6.9).

However, instead of doing the calculations at high frequencies as in equation (6.9) it is, as we will see, much more practical and efficient to do the calculations in baseband, and especially in the digital domain by using the DFT. This means that we need to create a *suitable* baseband spectrum from the high frequency OFDM spectrum that is contained in the signal $y_r(t)$.

Let us use the reference frequency f_{rc} of the received OFDM signal in the demodulation process, and we know from section 1 that $f_{rc} = f_{n_{rc}}$.

Frequency down-conversion is illustrated in Figure 8 for the case when K is odd for which $f_{rc} = f_c$. It is seen in this figure that the LNA is followed by multipliers and low pass filters, and this part is referred to as *homodyne* reception. The goal of homodyne reception is frequency down-conversion to baseband and thereby create the desired baseband components, denoted $r_I(t)$ and $r_Q(t)$, such that *ideally*,

$$R_I(f) + jR_Q(f) = Y_r(f + f_{rc}), \qquad |f| \leq W_{lp} \tag{6.10}$$

where W_{lp} denotes the baseband cut-off frequency (bandwidth) of each low-pass filter in Figure 8.

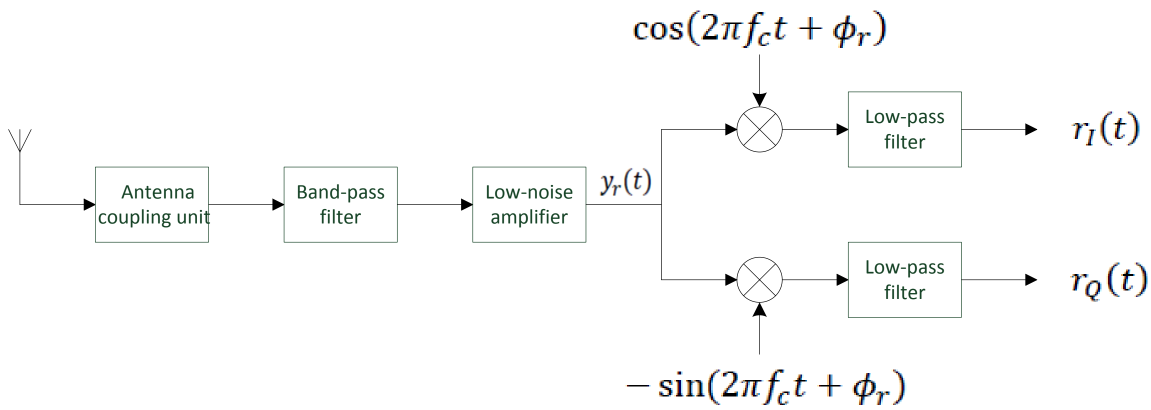

Figure 8. Illustrating the first part of the receiver: the antenna coupling unit, band-pass filter (wide), low-noise amplifier (LNA) and a homodyne unit for frequency down-converting and extracting the baseband signals $r_I(t)$ and $r_Q(t)$. It is here assumed that K is odd for which $f_{rc} = f_c$.

The output signal $y_r(t)$ from the LNA in Figure 8 contains several applications located in different frequency bands, but the receiver is here only interested in the frequency band that contains the desired OFDM signal $z(t)$.

We know the reference carrier frequency f_{rc} and the approximate bandwidth Kf_Δ of the desired signal $z(t)$ in equation (5.13). Therefore, the frequency of the oscillator signals in Figure 8 should be equal to f_{rc} Hz and the bandwidth W_{lp} of the low pass filters should be approximately $W_{lp} \approx Kf_\Delta/2$ Hz. In Figure 8, the actual phase of the high-frequency oscillator signals is denoted ϕ_r.

Now let $r_1(t)$ denote **only** the part of the signal $y_r(t)$ that corresponds to the desired OFDM spectrum. This means that the signal $(y_r(t) - r_1(t))$ has no frequency components within the approximate frequency range $f_{rc} - \frac{Kf_\Delta}{2} \leq f \leq f_{rc} + \frac{Kf_\Delta}{2}$ and the signal $(y_r(t) - r_1(t))$ will therefore (ideally) not contribute to the signals $r_I(t)$ and $r_Q(t)$ in Figure 8, only the signal $r_1(t)$ will.

So, the high-frequency signal $r_1(t)$ is important and it can be expressed as,

$$r_1(t) = Re\{(r_{1,I}(t) + jr_{1,Q}(t))e^{j2\pi f_{rc}t}\} \tag{6.11}$$

where, using equation (5.12), the corresponding complex baseband signal is found to be,

$$r_{1,I}(t) + jr_{1,Q}(t) = A\sum_{n=0}^{K-1} a_n H(f_n) e^{j2\pi g_n f_\Delta (t-T_{CP})} e^{j\phi}\, G_1(f_n) + n_1(t), \quad T_{CP} \leq t \leq T_s \tag{6.12}$$

The frequency function $G_1(f_n)$ represents the combined effect of the band-pass filter and the LNA on the n:th QAM signal, and the complex signal $n_1(t)$ represents “white” baseband noise (assuming that $|G_1(f)|$ is constant within the OFDM spectrum).

Let us now, for convenience, assume that the two low-pass filters in Figure 8 are almost ideal, i.e. it is here assumed that the transfer function of each filter is constant within the approximate frequency range $-Kf_\Delta/2 \leq f \leq Kf_\Delta/2$, and has a high attenuation at all other frequencies.

By studying the homodyne part in Figure 8, and also using equation (6.11), we conclude that the two outputs from the low-pass filters can be determined from the relationship below,

$$r_I(t) + jr_Q(t) = [Re\{(r_{1,I}(t) + jr_{1,Q}(t))e^{j2\pi f_{rc}t}\}e^{-j(2\pi f_{rc}t+\phi_r)}]_{LP} \tag{6.13}$$

where the notation $[c(t)]_{LP}$ means "baseband (or low-frequency) component of $c(t)$".

The reader is recommended to start with the right-hand side of equation (6.13) and show that,

$$r_I(t) + jr_Q(t) = \frac{\left(r_{1,I}(t)+jr_{1,Q}(t)\right)e^{-j\phi_r}}{2} \circledast\ g_{lp}(t) \tag{6.14}$$

where the symbol $\circledast$ in equation (6.14) denotes convolution, and $g_{lp}(t)$ denotes the impulse response of each low-pass filter (the transfer function is denoted $G_{lp}(f)$). Combining equations (6.12) and (6.14), we obtain the ***important result***,

$$r_I(t) + jr_Q(t) = \sum_{n=0}^{K-1} a_n H_{eq}(f_n) e^{j2\pi g_n f_\Delta (t-T_{CP})} + w(t), \quad T_{CP} \le t \le T_s \tag{6.15}$$

Observe that a so-called "***equivalent channel***" $H_{eq}(f_n)$ is introduced in equation (6.15), and the noise is represented by the complex signal $w(t)$ (assumed to be "white" baseband noise). The "equivalent channel" is a convenient concept since it represents the combined effect of all sub-blocks in the complete communication chain. Mathematically, we have from equations (6.12) and (6.14) that,

$$H_{eq}(f_n) = H_{eq,n} = AH(f_n)e^{j\phi}G_1(f_n)e^{-j\phi_r}G_{lp}(f_n - f_{rc} = g_n f_\Delta)/2 \tag{6.16}$$

It is seen in equations (6.15)-(6.16) that $H_{eq}(f_n)$, alternatively $H_{eq,n}$, includes, e.g., the physical channel filter model, the band-pass filter, and the low-pass filters. Since $H_{eq,n} = |H_{eq,n}|e^{jarg(H_{eq,n})}$ the phase $arg(H_{eq,n})$ may also include (absorb) a phase-component that is a consequence of the particular description in equation (6.15).

Note that for the result in equation (6.15) to be valid, the duration of the *overall impulse response* should not exceed T_{CP} (compare with equation (5.7)). Another way to rephrase this is to say that if virtually all of the energy in the overall impulse response is contained within the time-interval $0 \le t \le T_{CP}$, then equations (6.15)-(6.16) are sufficiently accurate, and this is typically the case in well-designed communication systems.

Observe also that the description in equation (6.15) is independent of the particular implementation of the OFDM signal at the transmitter side.

It should also be observed in equation (6.15) that $H_{eq,n}$ represents the overall attenuation and rotation of the n:th original signal point a_n. Furthermore, as we will see, the value $a_n H_{eq,n}$ is a very important component in the n:th received noisy signal point. In most cases the physical channel filter $H(f)$ causes *random* and *frequency dependent* attenuation and rotation, so called *fading*. Hence, the sent K QAM signals may then be treated differently by the channel (concerning attenuation and rotation). If the remaining blocks in the communication chain overall acts as a frequency independent complex constant, denoted B, then we have $H_{eq,n} = H(f_n)B$.

Instead of calculating equation (6.9) to obtain the n:th received noisy signal point we can calculate the Fourier transform of the complex signal $(r_I(t) + jr_Q(t))$ in equation (6.15), and evaluate this Fourier transform at the baseband sub-carrier frequency $f = f_n - f_{rc} = g_n f_\Delta$. The reason for this is the frequency shift down to baseband that is performed by the homodyne unit, which implies that the spectral characteristics of the high-frequency signal $r_1(t)$ at the high frequency f_n Hz will, except for the factor $e^{-j\phi_r}$ in equation (6.14) and the influence of $G_{lp}(g_n f_\Delta)$, be the same as the spectral characteristics of the low-frequency signal $(r_I(t) + jr_Q(t))$ at the baseband frequency $(f_n - f_{rc}) = g_n f_\Delta$ Hz. However, as we will see, the DFT will instead be used to recover the K received noisy signal points.

It should also be mentioned here that most decoding algorithms need knowledge about the overall channel parameters $H_{eq,n}$, $n = 0,1, \ldots (K-1)$. Therefore, a *channel estimation algorithm* is needed in the receiver. The result of this algorithm will be channel parameter estimates. Normally, completely known so-called pilot-symbols or channel-reference symbols are transmitted at specific sub-carriers and in specific OFDM symbol intervals (i.e. in a two-dimensional time-frequency grid). Based on the received signal at the corresponding sub-carrier and OFDM interval, the corresponding channel parameter can be estimated. From all these specific estimates (two-dimensional “samples”) it is possible to reconstruct an estimate of the two-dimensional channel filter (similar to a “two-dimensional D/A converter”), and thereby obtain estimates of $H_{eq,n}, n = 0,1, \ldots (K-1)$ over *several* OFDM intervals. In these lecture notes we will not analyze any channel estimation algorithms, so the interested reader is recommended to study more advanced literature on this topic.

This completes the description of the analog part of the receiver (Figure 8). In the next sub-section we will study the transition to the digital domain by assuming ideal sampling (A/D).

Sampling (A/D):

In this sub-section only ideal A/D converters are considered and by this we mean *instantaneous sampling* and *infinite precision* in the samples. See also Figure 9. As an example, consider ideal sampling of the signal $r_I(t)$ at $t = t_0$. With infinite precision the value obtained is then $r_I(t_0)$, in contrast to if finite precision is used where a *quantized* value of $r_I(t_0)$ instead is the result (and it is represented by a finite number of bits). In practice A/D converters are non-ideal, and efficient "close to ideal" A/D converters are non-trivial to implement. However, in these lecture notes we assume that the non-ideal A/D converters are fast and accurate enough such that we may approximate them with ideal A/D converters.

The goal in this sub-section is to obtain sampled versions of the two signals $r_I(t)$ and $r_Q(t)$ in Figures 8-9. Using the same arguments that was used in step 2 in section 2, we conclude that the *sampling frequency* f_{samp} may be chosen as $f_{samp} = Nf_\Delta$ samples per second, and N should be chosen larger than K (see equation (2.12)), and large enough such that the sampling theorem can be considered to be sufficiently fulfilled. Observe that the choice of N (i.e. the sampling frequency) to use in the receiver is a decision of the *engineers of the receiver*, and it does ***not*** need to be the same value as was chosen at the transmitter side (which typically is unknown in many cases). In general, how the transmitted OFDM signal was created may not be known at the receiver side.

Corresponding to the current OFDM interval $0 \leq t \leq T_s$, the two sequences of samples generated from the A/D converters in Figure 9 are $r_I(mT_{obs}/N)$ and $r_Q(mT_{obs}/N)$, respectively, $m = 0,1,\dots,(L+N-1)$.

Note however that the first L samples (the CP) will normally not be used in the detection process and these samples are therefore ***ignored*** in the future, see Figure 9! This means that the signal interval where an overlap between OFDM signals may occur is ignored by the receiver, and thereby possible interference between OFDM signals is eliminated in the receiver (provided that $T_{ch} \leq T_{CP}$), see also section 5. Since the CP is assumed to be much smaller than T_s the loss of ignoring the CP is acceptable.

The remaining N samples represent the receiver's observation interval and they are on the other hand extremely important. These samples are collected in the two vectors $\boldsymbol{r}_I$ and $\boldsymbol{r}_Q$ defined below (see also Figure 9),

$$r_{I,m} = r_I((L+m)T_{obs}/N), \quad m = 0,1,\dots,N-1 \qquad (6.17)$$

$$r_{Q,m} = r_Q((L+m)T_{obs}/N), \quad m = 0,1,\dots,N-1 \qquad (6.18)$$

We now create the complex size-N vector $\boldsymbol{r}$ as,

$$\boldsymbol{r} = \boldsymbol{r}_I + j\boldsymbol{r}_Q \qquad (6.19)$$

From equation (6.15) it is seen that the discrete-time signal $\boldsymbol{r}$ is a sampled version of the complex signal $x_r(t)$, where $x_r(t)$ is defined as,

$$x_r(t) = r_I(t+T_{CP}) + jr_Q(t+T_{CP}) = \sum_{n=0}^{K-1} a_n H_{eq,n} e^{j2\pi g_n f_\Delta t} + w'(t), \ \ 0 \leq t \leq T_{obs} \qquad (6.20)$$

The signal $x_r(t)$ in equation (6.20) should be compared with the signal $x(t)$ in equation (2.3) on page 6!

Observe that the effects of the complete communication chain are that the n:th original signal point a_n is *changed* to $a_n H_{eq,n}$, and that Gaussian baseband *noise* is present in the received complex baseband OFDM signal $x_r(t)$.

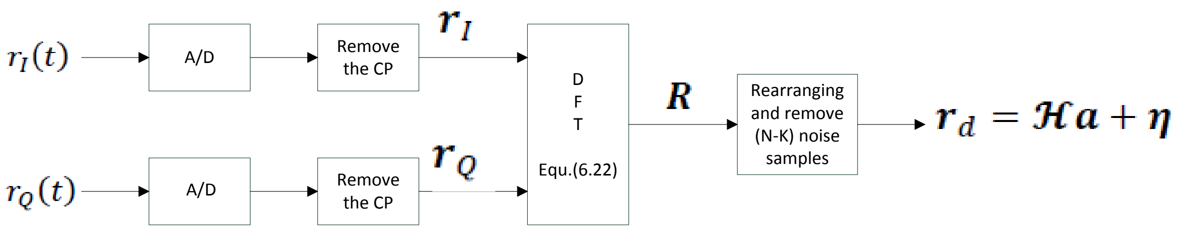

Figure 9. Illustrating sampling and the size-N DFT in the receiver to extract the K received noisy signal points collected in the size-K vector $\boldsymbol{r}_d$.

Using the DFT:

The remaining investigation aims to find the K received distorted and noisy signal points. Note that the received signal point obtained at subcarrier frequency f_n can be found by calculating the Fourier transform of the signal $x_r(t)$ in equation (6.20) at the corresponding baseband sub-carrier frequency $g_n f_\Delta$ Hz. However, to efficiently obtain these values with the size-N DFT, let us continue as follows.

Consider the Fourier transform of the discrete-time signal $\boldsymbol{r}$ in equation (6.19),

$$R(\nu) = \sum_{n=0}^{N-1} r_n e^{-j2\pi\nu n} \quad (6.21)$$

Remember from , e.g., Figure 4 on page 12 that $R(\nu)$ is periodic in ν with period 1. Furthermore, the Fourier transform of the signal $x_r(t)$ in equation (6.20) appears as a very significant part of $R(\nu)$ within the fundamental interval $-1/2 \leq \nu \leq 1/2$, see equation (2.17), and the example in connection to Figure 3 on page 11.

As has been pointed out several times the K received distorted and noisy signal points can be obtained by calculating the Fourier transform at K specific frequencies, i.e. by ***sampling the Fourier transform*** at K specific frequencies. This is precisely what the DFT does!

Let us therefore calculate the size-N DFT of the discrete-time signal $\boldsymbol{r}$,

$$R_k = R(\nu = k/N) = \sum_{n=0}^{N-1} r_n e^{-j2\pi kn/N}, k = 0,1, \dots, N-1 \quad (6.22)$$

In practice, **N is a power of 2** since fast Fourier transform (FFT) algorithms can then be used to significantly speed up the calculations in equation (6.22).

In Figure 9 on page 35 the size-N column vector $\boldsymbol{R}$ contains the samples R_k.

Observe that the sample R_k is obtained at the normalized frequency (compare with Figure 4),

$$\nu = \frac{k}{N} = \frac{f}{f_{samp}} = \frac{f}{N f_\Delta} = \frac{k f_\Delta}{N f_\Delta} \quad (6.23)$$

It is important to see that the only differences between the signal $x_r(t)$ in equation (6.20) and the signal $x(t)$ in equation (2.3), are that the original signal point a_n is changed to $a_n H_{eq,n}$, and that Gaussian baseband noise is present in $x_r(t)$.

We can now write $\boldsymbol{R}$ as,

$$\boldsymbol{R} = \boldsymbol{X}_r + \boldsymbol{w}_r \quad (6.24)$$

where $\boldsymbol{X}_r$ is the noise-free part of $\boldsymbol{R}$, and $\boldsymbol{w}_r$ is the noise vector.

Let us also introduce the size-K column vector $\boldsymbol{\mathcal{H}a}$, where $\boldsymbol{a}$ is a size-K column vector containing the original QAM symbols a_n. $\boldsymbol{\mathcal{H}}$ is a size KxK matrix containing the equivalent channel parameters $H_{eq,n}$ on the main diagonal, and the off-diagonal elements are equal to zero. This means that,

$$(\boldsymbol{\mathcal{H}a})^{tr} = (a_0 H_{eq,0} \;\; a_1 H_{eq,1} \dots . a_{K-1} H_{eq,K-1}) \quad (6.25)$$

Hence, the size-K column vector $\boldsymbol{\mathcal{H}a}$ contains the noise-free received distorted signal points.

From the general results obtained in section 2 we know that $\boldsymbol{X}_r$ is a "rotated" version of $\boldsymbol{\mathcal{H}a}$, or more precisely from equation (2.27).

$$\boldsymbol{X}_r = N\boldsymbol{Q}_t\boldsymbol{\mathcal{H}a} \tag{6.26}$$

As an example, the first value $X_{r,0}$, which is contained in the frequency sample $R_0 = R(\nu = 0)$, equals $X_{r,0} = a_{n_{rc}}H_{eq,n_{rc}}$.

Equation (6.24) can then be expressed as,

$$\boldsymbol{R} = N\boldsymbol{Q}_t\boldsymbol{\mathcal{H}a} + \boldsymbol{w}_r \tag{6.27}$$

To recover the K received noisy signal points we "re-rotate" the vector $\boldsymbol{R}$ according to equation (2.30),

$$\boldsymbol{r}_d = \frac{1}{N}\boldsymbol{Q}_r\boldsymbol{R} = \boldsymbol{Q}_r\boldsymbol{Q}_t\boldsymbol{\mathcal{H}a} + \frac{1}{N}\boldsymbol{Q}_r\boldsymbol{w}_r = \boldsymbol{\mathcal{H}a} + \boldsymbol{\eta} \tag{6.28}$$

Observe that the elements in the size-K column vector $\boldsymbol{r}_d$ *are **the desired received distorted and noisy signal points***,

$$r_{d,n} = a_n H_{eq,n} + \eta_n, \qquad n = 0,1,\dots,(K-1) \tag{6.29}$$

where η_n denotes additive complex Gaussian noise. Note also that the operation in equation (6.28) automatically ignores the (N-K) positions in $\boldsymbol{R}$ that only contains out-of-band noise.

The result in equation (6.28) is very important! See also Figure 9 on page 35.The next step in the receiver is to feed the vector $\boldsymbol{r}_d$ to the decoding unit, but this is not covered in these lecture notes.

It should also be pointed out here that the equivalent channel parameter $H_{eq,n}$ in equation (6.29) includes a multiplying component $e^{-j2\pi g_n f_\Delta t_\varepsilon}$, compared with the equivalent channel parameter in equation (6.20). The reason for this is that, in equations (6.17)-(6.18), the first (m=0) sampling instant occurs at the time t_0,

$$t_0 = \frac{LT_{obs}}{N} = T_{CP} + t_\varepsilon \tag{6.30}$$

where t_ε is a (small) non-negative "offset" in time.

In the next section we will take a look at some alternative implementations that are possible in case the sampling frequency is approximately at least twice as high as the sampling frequency used so far.

7. An alternative transmitter implementation using a higher sampling frequency

In this section we will study an alternative implementation at the transmitter side that is possible if the sampling frequency is significantly higher than the sampling frequency used in the sections 2-3 (where $f_{samp} = Nf_{\Delta}$ and $N > K$) . A higher sampling frequency normally implies a higher cost, e.g., for D/A converters. However, as we will see, interesting and useful alternative implementations can then be obtained. The description given here is to a large extent influenced by the description in ref. [2].

Let us start with the description of an OFDM signal in the time-interval $0 \le t \le T_s$ according to equation (1.12),

$$s(t) = g_{rec}(t)Re\left\{\sum_{n=0}^{K-1} a_n e^{j2\pi(f_0+nf_{\Delta})t}\right\} \tag{7.1}$$

The frequency content, for positive frequencies, is roughly indicated in Figure 1.1a on page 7. A significant difference however, compared to section 1 is that here we assume that the K sub-carriers in the OFDM signal $s(t)$ have relatively **low frequencies**. More specifically it is here assumed that the sub-carrier f_0 equals,

$$f_0 = N_g f_{\Delta} \tag{7.2}$$

where the parameter N_g is a design parameter and it is a non-zero positive integer. A purpose of the parameter N_g may be to have a suitable guard band in the frequency domain around $f = 0$, e.g., to simplify filtering (this will be more clear later in this section).

The OFDM signal $s(t)$ in equation (7.1) can be expressed in the following way within the time-interval $0 \le t \le T_{obs}$,

$$s(t) = g_{rec}(t)Re\left\{\sum_{n=0}^{K-1} a_n e^{j2\pi(f_0+nf_{\Delta})t}\right\} = \frac{g_{rec}(t)}{2}\left(\sum_{n=0}^{K-1} a_n e^{j2\pi(f_0+nf_{\Delta})t} + \sum_{n=0}^{K-1} a_n^* e^{-j2\pi(f_0+nf_{\Delta})t}\right) \tag{7.3}$$

and the bandwidth of this signal, including the N_g unused sub-carriers, is approximately equal to $(N_g + K)f_{\Delta}$ Hz.

Lets us now determine N time-domain samples of the signal $s(t)$ within the time interval $0 \le t \le T_{obs}$ (compare with section 2). From the description in equations (7.1)-(7.2), we find that the sampling frequency can be chosen as $f_{samp} = Nf_{\Delta}$ where,

$$N > 2(N_g + K) \tag{7.4}$$

Hence, the sampling frequency in this section is approximately ***at least twice as high*** as in sections 2-3 (for fixed value of K).

Let the vector $\boldsymbol{s}$ contain the N <u>*real*</u> samples $s_0, s_1, \dots, s_{N-1}$, of the signal $s(t)$, where

$$s_m = s(mT_{obs}/N) = \frac{1}{2}\left(\sum_{n=0}^{K-1} a_n e^{j2\pi(f_0+nf_{\Delta})mT_{obs}/N} + \sum_{n=0}^{K-1} a_n^* e^{-j2\pi(f_0+nf_{\Delta})mT_{obs}/N}\right) \tag{7.5}$$

This can be simplified to,

$$s_m = s(mT_{obs}/N) = \frac{1}{2}\left(\sum_{n=0}^{K-1} a_n e^{\frac{j2\pi(N_g+n)m}{N}} + \sum_{n=0}^{K-1} a_n^* e^{-\frac{j2\pi(N_g+n)m}{N}}\right), m = 0,1,\dots,(N-1) \tag{7.6}$$

Observe that the samples s_m in equation (7.6) are ***real***.

Continuing as in section 2, we need the Fourier transform of the discrete-time signal $\boldsymbol{s}$ in equation (7.6) defined by,

$$S(\nu) = \sum_{n=0}^{N-1} s_n e^{-j2\pi\nu n} \qquad (7.7)$$

and we are especially interested in the samples S_k of $S(\nu)$, given by the size-N DFT,

$$S_k = S(\nu = k/N) = \sum_{n=0}^{N-1} s_n e^{-j2\pi kn/N}, k = 0,1,\dots,N-1 \qquad \textbf{(DFT)} \qquad (7.8)$$

It should be observed that since the discrete-time signal $\boldsymbol{s}$ is real there are ***symmetries*** in its Fourier transform $S(\nu)$.

As in section 2, we need to specify the sequence of frequency samples $S_0, S_1, \dots, S_{N-1}$ since this sequence will be used in the size-N IDFT to create the desired sequence of samples $\boldsymbol{s}$. Actually, to be precise, the multiplication (or scaling) factor ½ that appears in equation (7.6) will be *ignored* below. It is assumed that this can be adjusted for later in the transmitter chain, e.g., in the power amplifier (PA).

Note also that,

$$N = 2N_g + 2K - 1 + N_x \qquad (7.9)$$

where N_x is a positive number of zero-valued samples S_k determined by the choice of N, and these zero-valued samples are located symmetrically around $\nu = 1/2$. A relatively large value of N_x may simplify the filtering operation in the D/A in order to extract the fundamental frequency interval $-f_{samp}/2 \le f \le f_{samp}/2$, see also section 3.

If we compare with the situation treated in section 2 it is concluded from equation (7.6) that:

For $N_g \le l \le N_g + K - 1$ we find that S_l is,

$$S_{N_g+k} = Na_k, \; 0 \le k \le K-1 \qquad (7.10)$$

For $N_g + K + N_x \le l \le N_g + 2K + N_x - 1$ we find that S_l is,

$$S_{N_g+K+N_x+k} = Na_{K-1-k}^*, \quad 0 \le k \le K-1 \qquad (7.11)$$

For the remaining $(N - 2K)$ samples in the sequence $S_0, S_1, \dots, S_{N-1}$ the value equals zero.

Consider as an example the case K=3, $N_g = 2$ and N=12. In this case the desired sequence $S_0, S_1, \dots, S_{11}$ then equals: $0,0, Na_0, Na_1, Na_2, 0,0,0, Na_2^*, Na_1^*, Na_0^*, 0$.

Consider a similar example where K=3, $N_g = 2$ and N=11. In this case the desired sequence $S_0, S_1, \dots, S_{10}$ then equals: $0,0, Na_0, Na_1, Na_2, 0,0, Na_2^*, Na_1^*, Na_0^*, 0$.

Hence, the sequence of samples $S_0, S_1, \dots, S_{N-1}$ is completely determined and the desired real sequence $\boldsymbol{s}$ is found from the size-N IDFT,

$$s_n = \frac{1}{N}\sum_{k=0}^{N-1} S_k e^{j2\pi kn/N}, n = 0,1,\dots,N-1 \qquad \textbf{(IDFT)} \qquad (7.12)$$

In practice, **N is a power of 2** since fast Fourier transform (FFT) algorithms can then be used to significantly speed up the calculations in equation (7.12). See also Figure 10 on page 41.

To create the CP we proceed in the same way as in section 3. Therefore, if we for a moment allow the definition of the signal $s(t)$ in equation (7.3) to be valid for all t, then it is seen that the signal construction of $s(t)$ is such that $s(t)$ is **periodic** in t with period T_{obs}, i.e. $s(t) = s(t + T_{obs})$. This means, e.g., that $s(-t_1) = s(T_{obs} - t_1)$.

We now want to preserve the OFDM signal properties in the N samples that represent $s(t)$, over an **extended** period of time. This can conveniently be done by using the inherent periodicity in $s(t)$ discussed above. Hence, let us therefore construct a new size-(L+N) vector $\boldsymbol{u}$ as a *periodic extension* of the size-N vector $\boldsymbol{s}$. This means that *the L last samples in* $\boldsymbol{s}$ *are copied and placed as the first L samples in* $\boldsymbol{u}$. The remaining N samples in $\boldsymbol{u}$ are identical to $\boldsymbol{s}$. This means that,

$$u_0 = s_{N-L},\ldots, u_{L-1} = s_{N-1}, u_L = s_0,\ldots, u_{L+N-1} = s_{N-1}. \tag{7.13}$$

The construction of the vector $\boldsymbol{u}$ above implies that the first L samples in $\boldsymbol{u}$ are identical with the last L samples in $\boldsymbol{u}$, and this reflects the periodicity discussed above.

The duration of the OFDM signal interval is T_s, and it can be expressed as,

$$T_s = \frac{(L+N)T_{obs}}{N} = T_{CP} + T_{obs} \tag{7.14}$$

The vector $\boldsymbol{u}$ in equation (7.13) contains (L+N) samples of a real OFDM signal defined over the entire interval $0 \leq t \leq T_s$. This OFDM signal is here denoted $u(t)$, where $u(t)$ is a delayed version of the signal $s(t)$, or more precisely,

$$u(t) = s(t - T_{CP}) = g_{rec}(t)Re\{\sum_{n=0}^{K-1} a_n e^{j2\pi(f_0+nf_\Delta)(t-T_{CP})}\}, \quad 0 \leq t \leq T_s \tag{7.15}$$

and $u(t)$ equals zero outside this interval. Furthermore, the m:th sample in the vector $\boldsymbol{u}$ is,

$$u_m = u(mT_{obs}/N), \ \ m = 0,1,\ldots,(L + N - 1) \tag{7.16}$$

The final operation is to send the real sequence $\boldsymbol{u}$ to a D/A converter (compare with section 3), and the ideal output from the D/A will be the desired OFDM signal $u(t)$, see Figure 10. Note that only a single D/A converter is needed here, but on the other hand it operates with approximately at least twice as high sampling frequency compared to the sampling frequency used in section 3 (for fixed K).

The overall carrier frequency of the created OFDM signal $u(t)$ is,

$$f_{c,u} = f_0 + \frac{K-1}{2} f_\Delta = (N_g + \frac{K-1}{2}) f_\Delta \tag{7.17}$$

and typically $f_{c,u}$ is a relatively low frequency.

In case the OFDM signal $u(t)$ needs to be up-converted to a higher overall carrier frequency f_c, mixing and filtering according to Figure 10 is a possibility. The local oscillator frequency f_l used in the mixing operation in Figure 10 should then be chosen as,

$$f_l = f_c - f_{c,u} = f_c - (N_g + \frac{K-1}{2})f_\Delta \tag{7.18}$$

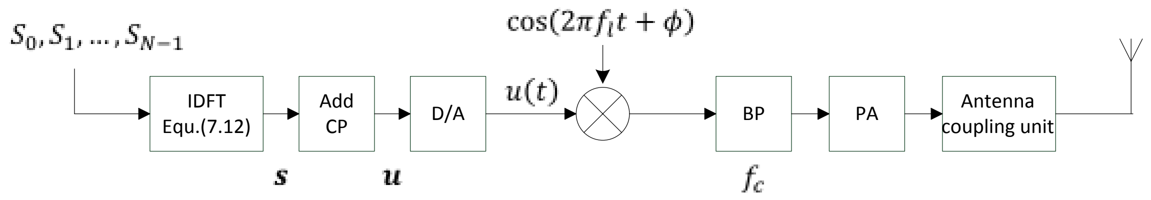

Figure 10. Illustrating how to create the OFDM signal $u(t)$ in equation (7.15). The size-N IDFT is used and N is given by equation (7.4). The construction of the sequence $S_0, S_1, \ldots, S_{N-1}$ is given by equations (7.10)-(7.11). This figure also includes a possible frequency up-converting to a higher carrier frequency f_c. The band-pass (BP) filter is centered around f_c, and PA means the power amplifier.

The requirements on the band-pass filter in Figure 10 can be somewhat relaxed due to the parameter N_g, since this parameter gives a guard band in the frequency domain equal to $2N_g f_\Delta$ Hz that separates the lower sideband from the desired upper sideband (which is the desired up-modulated OFDM signal).

As an example consider an OFDM signal $u(t)$ with the parameters: K=420, $N_g = 20$, and $f_\Delta = 4$ kHz. The values of the sub-carrier frequencies f_0 and f_{K-1} then equals $f_0 = 80$ kHz and $f_{K-1} = 1.76$ Mhz. In this case N should be chosen larger than 880. If N is chosen to be N=1024 then the sampling frequency 4.096 MHz is used, and $N_x = 145$. In case it is desired to up-convert this OFDM signal to higher frequencies, the width of the guard band that separates the lower and the upper (desired) sideband equals 160 kHz which simplifies the implementation of the band-pass filter.

8. An alternative receiver implementation using a higher sampling frequency

In this section we will study an alternative implementation at the receiver side that is possible if the sampling frequency is significantly higher than the sampling frequency used in sections 1-6 (where $f_{samp} = N/T_{obs}$ and $N > K$) . A higher sampling frequency normally implies a higher cost, e.g., for A/D converters. However, as we will see, interesting and useful alternative implementations can then be obtained.

In this section it is assumed that *at a certain stage in the receiver* the real noisy OFDM signal $r(t)$ is available where,

$$r(t) = u_r(t) + w(t) \tag{8.1}$$

In equation (8.1) the signal $u_r(t)$ denotes the noise-free part of $r(t)$, and it is assumed that the received OFDM signal $u_r(t)$ here can be expressed as,

$$u_r(t) = Re\{\sum_{n=0}^{K-1} a_n H_{eq,n} e^{j2\pi f_n(t-T_{CP})}\}, \qquad T_{CP} \leq t \leq T_s \tag{8.2}$$

where the different sub-carrier frequencies are,

$$f_n = N_g f_\Delta + n f_\Delta, \qquad\qquad n = 0,1,\ldots,K-1 \tag{8.3}$$

Hence, the sub-carrier frequencies here have relatively low frequencies, and N_g is a design parameter of the receiver.

As in section 6 the "equivalent channel" parameter $H_{eq,n}$ represents the combined effect on the n:th sub-carrier frequency of all sub-blocks in the complete communication chain. Compare with equation (6.16). Since $H_{eq,n} = |H_{eq,n}|e^{j arg(H_{eq,n})}$ the phase $arg(H_{eq,n})$ may also include (absorb) a phase-component that is a consequence of the particular description in equation (8.2).

The signal $w(t)$ in equation (8.1) represents band-limited additive "white" Gaussian noise within the bandwidth of $u_r(t)$, i.e. within the approximate frequency range $|f| \leq (N_g + K)f_\Delta$.

Observe that the description of $u_r(t)$ in equation (8.2) should be independent of the particular implementation of the OFDM signal at the transmitter side.

We now want to obtain a sampled version of the signal $r(t)$. Using the same arguments that was used in step 2 in section 2, we conclude that the *sampling frequency* f_{samp} may be chosen as $f_{samp} = N/T_{obs}$ samples per second, and N should be chosen as,

$$N > 2(N_g + K) \tag{8.4}$$

and large enough such that the sampling theorem ([1]) is sufficiently fulfilled. Hence, the sampling frequency is in this section approximately ***at least twice as high*** as in section 6 (for fixed value of K).

Note that the choice of N (i.e. the sampling frequency) to use in the receiver is a decision of the *engineers of the receiver*, and it does ***not*** need to be the same value as was chosen at the transmitter side (which typically also is unknown in many cases).

Corresponding to the current OFDM interval $0 \leq t \leq T_s$, the sequence of samples generated from the A/D converter in Figure 11 on page 43 is $r(mT_{obs}/N)$, $m = 0,1,\dots,(L+N-1)$.

Note however that the first L samples (the CP) will not be used in the detection process since these samples are ***ignored*** in the future! This means that the signal interval where an overlap between OFDM signals may occur is ignored by the receiver, and thereby possible interference between OFDM signals is eliminated in the receiver (provided that $T_{ch} \leq T_{CP}$), see also section 5. Since the CP is assumed to be much smaller than T_s the loss of ignoring the CP is small.

The remaining N samples are on the other hand extremely important and they are collected in the *real* vector $\boldsymbol{r}$ and defined below,

$$r_m = r((L+m)T_{obs}/N), \quad m = 0,1,\dots,N-1 \tag{8.5}$$

From equations (8.1)-(8.2) it is seen that the discrete-time signal $\boldsymbol{r}$ is a sampled version of the signal $x_r(t)$, where $x_r(t)$ is,

$$x_r(t) = r(t+T_{CP}) = Re\{\sum_{n=0}^{K-1} a_n H_{eq,n} e^{j2\pi f_n t}\} + w'(t), \qquad 0 \leq t \leq T_{obs} \tag{8.6}$$

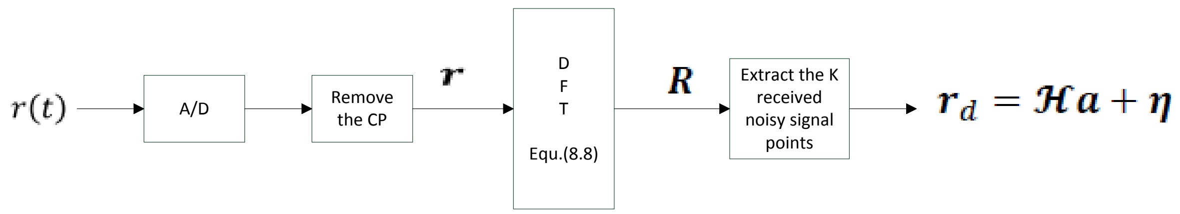

Figure 11. Illustrating a possible way to extract the K received noisy signal points if a high sampling frequency is used. The real noisy OFDM signal $r(t)$ is given in equation (8.1), and it is assumed to be available at a certain stage in the receiver. The size-N DFT is used and N is given by equation (8.4). The final result $\boldsymbol{r}_d$ is given in equations (8.11)-(8.13).

Using the DFT:

The remaining investigation aims to find the K received distorted and noisy signal points. Note that the received noisy signal point obtained at subcarrier frequency f_n can be found by calculating the Fourier transform of the signal $x_r(t)$ in equation (8.6) at the corresponding sub-carrier frequency f_n Hz. However, to efficiently obtain these values with the size-N DFT, let us continue as follows.

Consider the Fourier transform of the real discrete-time signal $\boldsymbol{r}$ in equation (8.5),

$$R(\nu) = \sum_{n=0}^{N-1} r_n e^{-j2\pi\nu n} \tag{8.7}$$

Remember from, e.g., Figure 2 on page 10 that $R(\nu)$ is periodic in ν with period 1. Furthermore, the Fourier transform of the signal $x_r(t)$ in equation (8.6) appears as a very significant part of $R(\nu)$ within the fundamental interval $-1/2 \leq \nu \leq 1/2$, see equation (2.17) and the example in connection to Figure 3 on page 11.

As has been pointed out several times the K received noisy signal points can be obtained by calculating the Fourier transform at K specific frequencies, i.e. by ***sampling the Fourier transform*** at K specific frequencies. This is precisely what the DFT does!

Let us therefore calculate the size-N **DFT** of the discrete-time signal $\boldsymbol{r}$,

$$R_k = R(\nu = k/N) = \sum_{n=0}^{N-1} r_n e^{-j2\pi kn/N}, k = 0,1, \dots, N-1 \tag{8.8}$$

In practice, **N is a power of 2** since fast Fourier transform (FFT) algorithms can then be used to significantly speed up the calculations in equation (8.8).

In Figure 11 the size-N column vector $\boldsymbol{R}$ contains the samples R_k.

Observe that the sample R_k is obtained at the normalized frequency,

$$\nu = \frac{k}{N} = \frac{f}{f_{samp}} = \frac{f}{Nf_\Delta} = \frac{kf_\Delta}{Nf_\Delta} \tag{8.9}$$

We can now write $\boldsymbol{R}$ as,

$$\boldsymbol{R} = \boldsymbol{X}_r + \boldsymbol{w}_r \tag{8.10}$$

where $\boldsymbol{X}_r$ is the noise-free part of $\boldsymbol{R}$, and $\boldsymbol{w}_r$ is the noise vector.

By comparing with the situation treated in section 7, it is concluded from equation (7.10) that the sequence of values $X_{r,N_g}, X_{r,N_g+1}, \dots, X_{r,N_g+K-1}$ is identical with the sequence of noise-free signal points $(Na_0H_{eq,0}, Na_1H_{eq,1}, \dots., Na_{K-1}H_{eq,K-1})$.

The desired received noisy signal points, denoted $r_{d,n}$ are thereby found since,

$$r_{d,n} = \frac{1}{N} R_{N_g+n} = a_n H_{eq,n} + \frac{1}{N} w_{r,N_g+n} \qquad ,n = 0,1, \dots, K-1 \tag{8.11}$$

This fundamental result can be written in the same way as in section 6,

$$\boldsymbol{r}_d = \boldsymbol{\mathcal{H}} \boldsymbol{a} + \boldsymbol{\eta} \tag{8.12}$$

where the elements in the size-K column vector $\boldsymbol{r}_d$ *are* ***the desired received noisy signal points***,

$$r_{d,n} = a_n H_{eq,n} + \eta_n, \quad n = 0,1, \dots, (K-1) \tag{8.13}$$

where η_n denotes additive complex Gaussian noise.

The result in equation (8.12) is very important! See also Figure 11.The next step in the receiver is to feed the vector $\boldsymbol{r}_d$ to the decoding unit, but this is not covered in these lecture notes.

It should also be pointed out here that the equivalent channel parameter $H_{eq,n}$ in equation (8.11) includes a multiplying component $e^{-j2\pi g_n f_\Delta t_\varepsilon}$, compared with the equivalent channel parameter in equation (8.6). The reason for this is that, in equation (8.5), the first (m=0) sampling instant occurs at the time t_0,

$$t_0 = \frac{LT_{obs}}{N} = T_{CP} + t_\varepsilon \tag{8.14}$$

where t_ε is a (small) non-negative "offset" in time. This is investigated in more detail in Problem 6.8.

This concludes these lecture notes where an introduction to OFDM has been given. Typical IDFT-based implementations at the transmitter side, and DFT-based implementations at the receiver side has been the focus of these lecture notes.

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