Study week 5.

NOTE!

All project groups should by now have a project group number (PG1 – PG16).

Deadline for the **project report** (pdf-format) Thursday 4 December 2014, 17.00.

You can now sign-up for the **lab** on the home page.

Chapter 9

An Introduction to Time-varying Multipath Channels

$$\boxed{z(t) = \sum_n \alpha_n(t) s(t - \tau_n(t))} \tag{9.1}$$

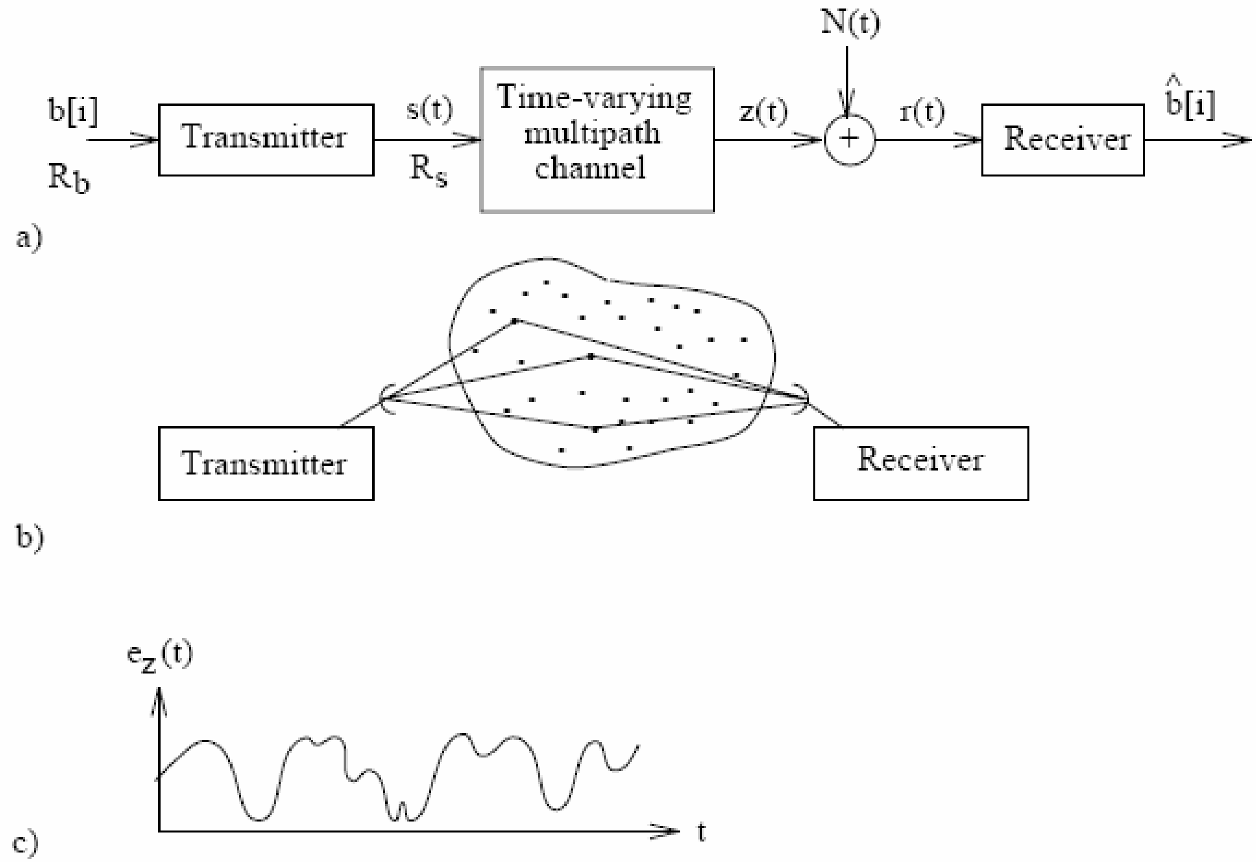


Figure 9.1: a) The digital communication system; b) A scattering medium; c) Illustrating the fading envelope $e_z(t)$.

$$s(t) = \cos((\omega_c + \omega_1)t) \ , \quad -\infty \leq t \leq \infty \tag{9.2}$$

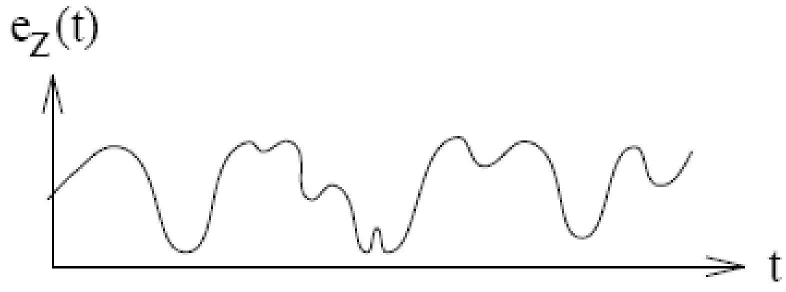
$$\begin{aligned}
z(t) &= \sum_n \alpha_n(t) \cos((\omega_c + \omega_1)(t - \tau_n(t))) = \\
&= \underbrace{\left[\sum_n \alpha_n(t) \cos((\omega_c + \omega_1)\tau_n(t))\right]}_{z_I(t) = \tilde{H}_{Re}(f_1,t)/2} \cos((\omega_c + \omega_1)t) - \\
&\quad - \underbrace{\left[\sum_n \alpha_n(t) \sin(-(\omega_c + \omega_1)\tau_n(t))\right]}_{z_Q(t) = \tilde{H}_{Im}(f_1,t)/2} \sin((\omega_c + \omega_1)t) \\
&= z_I(t) \cos((\omega_c + \omega_1)t) - z_Q(t) \sin((\omega_c + \omega_1)t) \\
&= e_z(t) \cos((\omega_c + \omega_1)t + \theta_z(t)) \qquad (9.3)
\end{aligned}$$

Compare with the time-invariant QAM-result:

$$\begin{aligned}
A_z + jB_z &= (A + jB)H(f_c) = \sqrt{A^2 + B^2}|H(f_c)|e^{j(\nu + \phi(f_c))} = \\
&= (A + jB)(H_{Re}(f_c) + jH_{Im}(f_c)) \qquad (3.110)
\end{aligned}$$

$$s(t) = \cos((\omega_c + \omega_1)t) \ , \quad -\infty \le t \le \infty \tag{9.2}$$

$$\begin{aligned} z(t) &= \sum_n \alpha_n(t) \cos((\omega_c + \omega_1)(t - \tau_n(t))) = \\ &= e_z(t) \cos((\omega_c + \omega_1)t + \theta_z(t)) \end{aligned} \tag{9.3}$$



Observe that the quadrature components $z_I(t)$ and $z_Q(t)$ in (9.3) are *time-varying*. Hence, the output signal $z(t)$ is *not* a pure sine wave with frequency $f_c + f_1$. ***This is a significant difference compared with the linear time-invariant channel.*** It is seen in (9.3) that the quadrature components depend

$$\begin{aligned} z(t) &= \sum_n \alpha_n(t)\cos((\omega_c+\omega_1)(t-\tau_n(t))) = \\ &= z_I(t)\cos((\omega_c+\omega_1)t) - z_Q(t)\sin((\omega_c+\omega_1)t) \\ &= e_z(t)\cos((\omega_c+\omega_1)t+\theta_z(t)) \end{aligned}$$

Throughout this chapter it is assumed that $z_I(t)$ and $z_Q(t)$ may be modelled as baseband zero-mean wide-sense-stationary (WSS) *Gaussian random processes* (with variances $\sigma_I^2 = \sigma_Q^2 = \sigma^2$). This is a commonly used assumption when the number of scatterers is large, implying that central limit theorem arguments can be used [43], [65], [68], [39]. For a fixed value of t, this assumption leads to a Rayleigh-distributed envelope $e_z(t)$,

$$e_z(t) = \sqrt{z_I^2(t) + z_Q^2(t)} \tag{9.4}$$

$$p_{e_z}(x) = \frac{2x}{b}\, e^{-x^2/b}, \quad x \geq 0\ , \text{ Rayleigh distr.} \tag{9.5}$$

$$b = E\{e_z^2(t)\} = 2\sigma^2 = 2P_z \tag{9.6}$$

and a uniformly distributed phase $\theta_z(t)$ (over a 2π interval). The zero-mean assumption means that there is no deterministic signal path present in $z(t)$. If a

9.1.1 Doppler Power Spectrum and Coherence Time

$$\boxed{\begin{aligned} &R_{\mathcal{D}}(f) = \mathcal{F}(\tilde{c}_z(\tau)) \\ &\tilde{c}_z(\tau) = \frac{1}{2}\, E\{[z_I(t+\tau) + jz_Q(t+\tau)]\ [z_I(t) - jz_Q(t)]\} \\ &R_z(f) = \frac{1}{2}\, (R_{\mathcal{D}}(f + f_c + f_1) + R_{\mathcal{D}}(f - f_c - f_1)) \end{aligned}} \qquad (9.7)$$

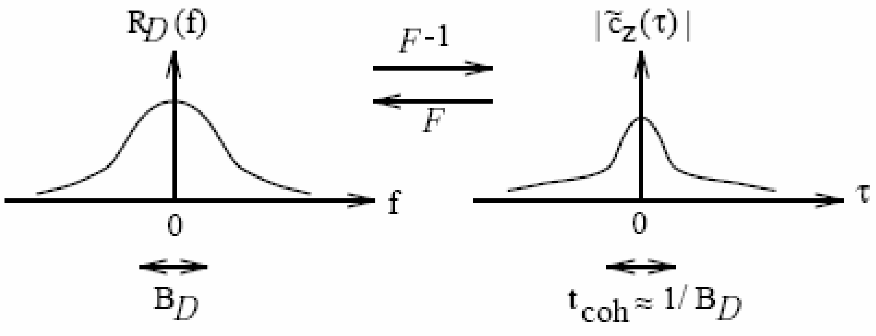


Figure 9.2: Illustrating the Fourier transform pair $\tilde{c}_z(\tau) \longleftrightarrow R_{\mathcal{D}}(f)$.

$$t_{coh} \approx 1/B_{\mathcal{D}} \qquad (9.8)$$

If the channel is slowly changing, then the coherence time is large. Note that $z_I(t+\tau)$ and $z_I(t)$ (also $z_Q(t+\tau)$ and $z_Q(t)$) are correlated over time-intervals τ (much) smaller than the coherence time t_{coh}. Hence, input signals within such intervals are therefore affected similarly by the fading channel. On the other hand, input signals that are separated in time by (much) more than t_{coh}, are affected differently by the channel, and at the output of the channel they become essentially independent of each other. If the former case apply (time flat fading), for a given time-interval, then we say that the channel is **time-nonselective**, and if the latter case apply, then the channel is said to be **time-selective**.

9.1.2 Coherence Bandwidth and Multipath Spread

$$z(t) = z(f_1, t) = \underbrace{\frac{1}{2}\, \tilde{H}_{Re}(f_1, t)}_{z_I(t)} \cos((\omega_c + \omega_1)t) - \underbrace{\frac{1}{2}\, \tilde{H}_{Im}(f_1, t)}_{z_Q(t)} \sin((\omega_c + \omega_1)t) \tag{9.9}$$

What can be said about the output signal $z(t)$ if another frequency $f_2 = f_1 + f_\Delta$ is used, instead of f_1? Are different frequency-intervals, in the input signal spectrum, treated differently by the time-varying multipath channel? To answer these questions the correlation between $z(f_1, t)$ and $z(f_1 + f_\Delta, t)$ can be found by

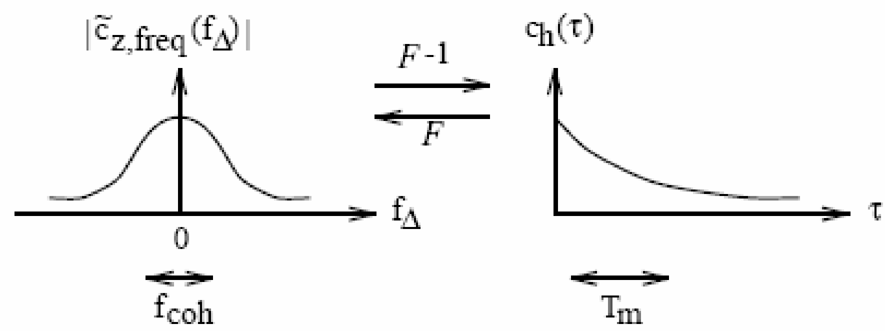


Figure 9.3: Illustrating the Fourier transform pair $c_h(\tau) \longleftrightarrow \tilde{c}_{z,freq}(f_\Delta)$.

The **coherence bandwidth f_{coh}** of the channel is defined as the width of the autocorrelation function $\tilde{c}_{z,freq}(f_\Delta)$, see Figure 9.3. Note that frequencies within a frequency-interval (much) smaller than the coherence bandwidth f_{coh} are correlated, and they are affected similarly by the fading channel. On the other hand, two frequencies that are separated by (much) more than f_{coh}, are affected differently by the channel, and they are essentially independent of each other. If the former case apply (frequency flat fading), for a given frequency-interval, then we say that the channel is **frequency-nonselective**, and if the latter case apply, then the channel is said to be **frequency-selective**.

$$\boxed{z(t) = \int_{-\infty}^{\infty} h(\tau, t)s(t-\tau)d\tau} \qquad (9.10)$$

delay power spectrum $c_h(\tau)$ (also multipath intensity profile) of the time-varying impulse response $h(\tau, t)$,

$$c_h(\tau) = E\left\{\frac{h^2(\tau,t)}{2}\right\} = \frac{1}{2}\,E\{h_I^2(\tau,t) + h_Q^2(\tau,t)\} = \frac{1}{2}\,E\{\tilde{h}(\tau,t)\tilde{h}^*(\tau,t)\} \quad (9.15)$$

An example of the delay power spectrum $c_h(\tau)$ is illustrated in Figure 9.3. The width of the delay power spectrum is referred to as the **multipath spread** of the channel and it is denoted by $\boldsymbol{T_m}$. This is an important parameter since if T_m is too large, compared with e.g. the symbol time, then intersymbol interference can occur.

$$T_m \approx 1/f_{coh} \qquad (9.16)$$

9.2 Frequency-Nonselective, Slowly Fading Channel

$$T_s \ll t_{coh} \tag{9.27}$$

or equivalently,

$$B_{\mathcal{D}} \ll R_s \tag{9.28}$$

This means that the channel is **slowly fading**, which imply that it can be treated as a time-invariant channel within the coherence time.

In this subsection a frequency-nonselective channel is investigated. To obtain this situation it is required that the bandwidth of the transmitted signal, denoted W, is much smaller than the coherence bandwidth f_{coh} of the channel,

$$W \ll f_{coh} \tag{9.29}$$

or equivalently,

$$T_m \ll 1/W \tag{9.30}$$

$$\boxed{\boxed{\tilde{z}(t) = \frac{1}{2}\int_{-\infty}^{\infty} \tilde{S}(f)\tilde{H}(f,t)e^{j2\pi ft}df}} \qquad (9.26)$$

$$\boxed{z_I(t) + jz_Q(t) = \frac{1}{2}\int_{-\infty}^{\infty}[S_I(f) + jS_Q(f)]\ [H_I(f,t) + jH_Q(f,t)]e^{j2\pi ft}df} \qquad (9.33)$$

$$z_I(t) + jz_Q(t) = \frac{1}{2}\int_{-\infty}^{\infty}[S_I(f) + jS_Q(f)] \cdot (H_I + jH_Q)e^{j2\pi ft}df \qquad (9.36)$$

$$\begin{aligned} z_I(t) + jz_Q(t) &= \frac{1}{2}\ (s_I(t) + js_Q(t))(H_I + jH_Q) = \\ &= e_s(t)e^{j\theta_s(t)} \cdot ae^{j\phi} = e_z(t)e^{j\theta_z(t)} \end{aligned} \qquad (9.37)$$

$$\begin{aligned} z_I(t) + jz_Q(t) &= \frac{1}{2}\,(s_I(t) + js_Q(t))(H_I + jH_Q) = \\ &= e_s(t)e^{j\theta_s(t)} \cdot ae^{j\phi} = e_z(t)e^{j\theta_z(t)} \end{aligned} \tag{9.37}$$

$$\boxed{\boxed{z(t) = ae_s(t)\cos(\omega_c t + \theta_s(t) + \phi)}} \tag{9.38}$$

$$p_a(x) = \frac{2x}{b}\,e^{-x^2/b}, \quad x \geq 0 \quad \text{(Rayleigh distribution)} \tag{9.39}$$

where,

$$E\{a\} = \frac{1}{2}\,\sqrt{\pi b} \tag{9.40}$$

$$E\{a^2\} = b \tag{9.41}$$

and,

$$p_\phi(y) = \begin{cases} 1/2\pi & , \quad -\pi \leq y \leq \pi \\ 0 & , \quad \text{otherwise} \end{cases} \tag{9.42}$$

If we assume uncoded equally likely binary signals over a Rayleigh fading channel ($z_1(t) = as_1(t), z_0(t) = as_0(t)$), then the bit error probability of the ideal coherent ML receiver is ($0 < d^2 = \frac{D^2_{s_1,s_0}}{2E_{b,sent}} \leq 2$)

$$\boxed{\boxed{\boldsymbol{P_b = \int_0^\infty \mathrm{Pr}\{\mathrm{error}|a\} p_a(x) dx = E\{\mathrm{Pr}\{\mathrm{error}|a\}\}}}} \tag{9.43}$$

$$\begin{aligned}
P_b &= \int_0^\infty Q(\sqrt{d^2 x^2 E_{b,sent}/N_0})\ \frac{2x}{b}\ e^{-x^2/b} dx = \\
&= -e^{-x^2/b}\ Q(x\sqrt{d^2 E_{b,sent}/N_0})\Big]_0^\infty - \int_0^\infty (-e^{-x^2/b}) \\
&\quad \left(\frac{-\sqrt{d^2 E_{b,sent}/N_0}}{\sqrt{2\pi}}\ e^{-\frac{x^2 d^2 E_{b,sent}/N_0}{2}} \right) dx = \\
&= \frac{1}{2} - \sqrt{d^2 E_{b,sent}/N_0} \cdot \beta \underbrace{\int_0^\infty \frac{e^{-x^2/2\beta^2}}{\beta\sqrt{2\pi}} dx}_{1/2}
\end{aligned} \tag{9.44}$$

$$\mathcal{E}_b = E\{a^2\}E_{b,sent} = bE_{b,sent} \tag{9.45}$$

$$P_b = \frac{1}{2}\left(1 - \sqrt{\frac{d^2\mathcal{E}_b/N_0}{2 + d^2\mathcal{E}_b/N_0}}\right) = \frac{1}{2 + d^2\mathcal{E}_b/N_0 + \sqrt{2 + d^2\mathcal{E}_b/N_0}\sqrt{d^2\mathcal{E}_b/N_0}}$$

$$\underset{\approx}{\overset{\mathcal{E}_b/N_0 \text{ "large"}}{\downarrow}} \quad \frac{1}{2d^2\mathcal{E}_b/N_0} \tag{9.46}$$

where $d^2 = 2$ for antipodal signals and $d^2 = 1$ for orthogonal signals.
Observe the dramatic increase in P_b due to the Rayleigh fading channel. P_b is no longer exponentially decaying in $\mathcal{E}_b/N_0$, it now decays essentially as $(\mathcal{E}_b/N_0)^{-1}$!

EXAMPLE 9.1

Assume that equally likely, binary orthogonal FSK signals, with equal energy, are sent from the transmitter. Hence, $s_i(t) = \sqrt{2E_{b,sent}/T_b}\cos(2\pi f_i t)$ *in* $0 \le t \le T_b$, $i = 0, 1$.

These signals are communicated over a Rayleigh fading channel, i.e. the received signal is (see (9.38)),

$$r(t) = a\sqrt{2E_{b,sent}/T_b}\cos(2\pi f_i t + \phi) + N(t)$$

Assume that the incoherent receiver in Figure 5.28 on page 397 is used. From (5.109) it is known that for a given value of a,

$$P_b = \frac{1}{2}\ e^{-a^2 E_{b,sent}/2N_0}$$

since $a^2 E_{b,sent}$ *then is the average received energy per bit.*

For the Rayleigh fading channel, and the same receiver, P_b *can be calculated by using (9.43),*

$$\boxed{P_b = \int_0^\infty \Pr\{error|a = x\}p_a(x) = E\{\Pr\{error|a\}\}}$$

$$E\{\Pr\{error|a\}\} = E\left\{\frac{1}{2}\ e^{-a^2 E_{b,sent}/2N_0}\right\} =$$
$$E\left\{\frac{1}{2}\ e^{-a_1^2 E_{b,sent}/2N_0}\right\} \cdot E\left\{e^{-a_2^2 E_{b,sent}/2N_0}\right\}$$

$$P_b = \frac{1/2}{1 + \frac{E_{b,sent}}{N_0} \cdot \frac{E\{a^2\}}{2}} = \frac{1}{2 + \mathcal{E}_b/N_0}$$

Observe the dramatic increase in P_b due to the Rayleigh fading channel. P_b is no longer exponentially decaying in $\mathcal{E}_b/N_0$, it now decays essentially as $(\mathcal{E}_b/N_0)^{-1}$! *As an example, assuming* $\mathcal{E}_b/N_0 = 1000$ *(30 dB), we obtain*

$$P_b = \begin{cases} 0.5e^{-500} \approx 3.6 \cdot 10^{-218} & , \quad AWGN \\ (1002)^{-1} \approx 10^{-3} & , \quad Rayleigh{+}AWGN \end{cases}$$

DIVERSITY IS NEEDED!

7.3 Reception and Detection

<u>Within a bit interval</u>: A received random number of photons generates a random number of photo-electrons after the photo-detector.

The Poisson Process:

In (7.27), the arrival times ..., t_{i-1}, t_i, t_{i+1}, are modeled as a **Poisson process** with an intensity $\mathcal{I}(t)$. This means that the number of arrivals $\mathcal{N}_\mathcal{T}$, within a time interval of length $\mathcal{T}$, is a random variable having the properties

$$\boxed{\begin{aligned} &Prob\{\mathcal{N}_\mathcal{T} = n\} = \frac{\mu^n e^{-\mu}}{n!} \\ &\mu = E\{\mathcal{N}_\mathcal{T}\} = \int_{t_0}^{t_0+\mathcal{T}} I(t)dt \\ &\sigma^2 = E\{(\mathcal{N}_\mathcal{T} - \mu)^2\} = \mu \end{aligned}} \qquad (7.29)$$

Note that the mean and the variance are identical.

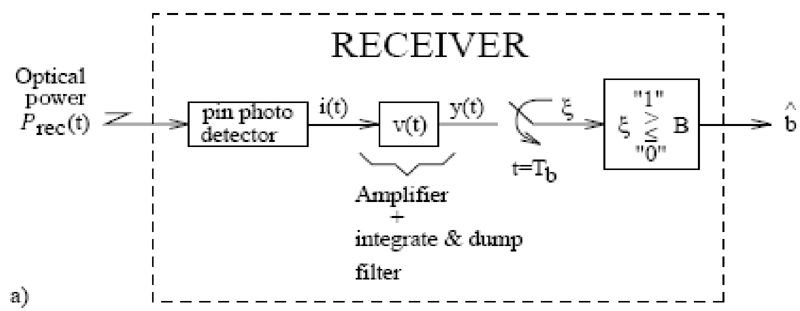
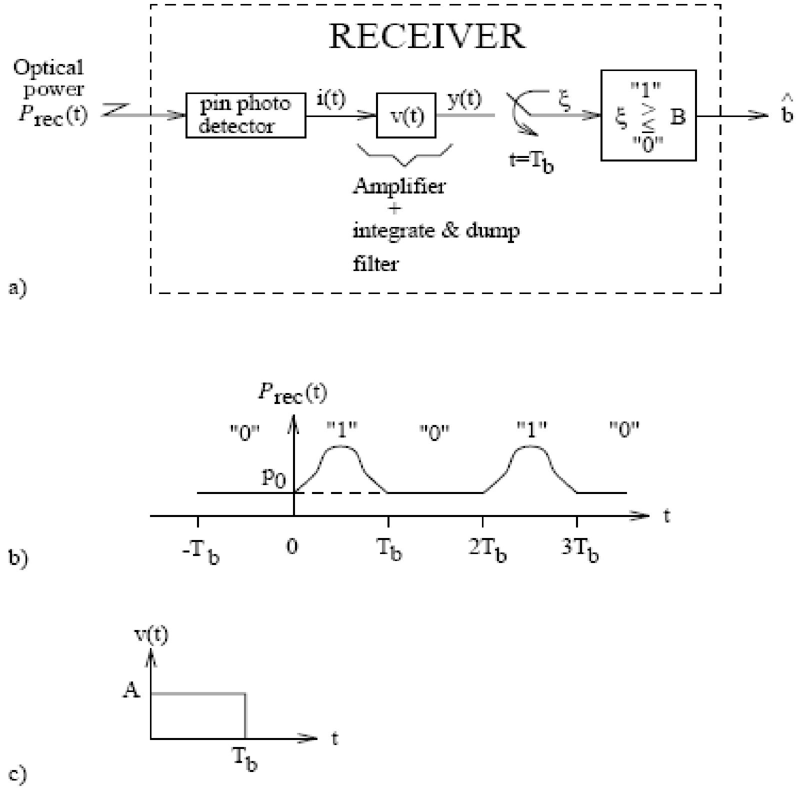


Fig. 7.8a

Compare with Chapter 4!

Fig. 7.8:



"0": po
"1": po+p(t)

Received optical power.

$$\mathcal{P}_{rec}(t) = p_0 + \sum_{i=-\infty}^{\infty} m[i]p(t - iT_b), \quad m[i] \in \{0,1\}, \quad -\infty \le t \le \infty \qquad (7.31)$$

$$\begin{aligned} \xi &= y(T_b) = \int_{-\infty}^{\infty} i(\tau)v(T_b - \tau)d\tau = A\int_0^{T_b} i(\tau)d\tau = \\ &= A\int_0^{T_b} (i_r(t) + i_d(t))dt = Aq\mathcal{N}_{T_b} \end{aligned} \qquad (7.32)$$

q=charge of an electron.
id(t)="dark current".

Bit error probability:

$$
\begin{aligned}
P_b &= P_0 \underbrace{Prob\{\text{error}|m_0 \text{ sent}\}}_{P_F} + P_1 \underbrace{Prob\{\text{error}|m_1 \text{ sent}\}}_{P_M} \\
&= P_0 Prob\{\xi > B|m_0 \text{ sent}\} + P_1 Prob\{\xi \leq B|m_1 \text{ sent}\} = \\
&= P_0 Prob\{\mathcal{N}_{T_b} > (B/Aq)|m_0 \text{ sent}\} + \\
&\quad + P_1 Prob\{\mathcal{N}_{T_b} \leq (B/Aq)|m_1 \text{ sent}\} \qquad (7.33)
\end{aligned}
$$

$$
\begin{aligned}
P_F &= Prob\{\mathcal{N}_{T_b} > \alpha|m_0 \text{ sent}\} = \sum_{n=\alpha+1}^{\infty} \frac{\mu_0^n e^{-\mu_0}}{n!} \\
P_M &= Prob\{\mathcal{N}_{T_b} \leq \alpha|m_1 \text{ sent}\} = \sum_{n=0}^{\alpha} \frac{\mu_1^n e^{-\mu_1}}{n!} \\
\alpha &= B/Aq \qquad (7.35)
\end{aligned}
$$

Exact expressions!

We need the averages!

$$\boxed{\begin{aligned} &Prob\{\mathcal{N}_{\mathcal{T}} = n\} = \frac{\mu^n e^{-\mu}}{n!} \\ &\mu = E\{\mathcal{N}_{\mathcal{T}}\} = \int_{t_0}^{t_0+\mathcal{T}} I(t)dt \\ &\sigma^2 = E\{(\mathcal{N}_{\mathcal{T}} - \mu)^2\} = \mu \end{aligned}} \tag{7.29}$$

$$\mathcal{I}_e(t) = \eta \cdot \mathcal{M} \cdot \mathcal{I}_{ph}(t) + \mathcal{I}_d = \eta \cdot \mathcal{M} \cdot \frac{\mathcal{P}_{rec}(t)}{hf} + \mathcal{I}_d \text{ [electrons/s]} \tag{7.8}$$

Id=id/q
Page 476.

Combining (7.29), (7.8) and (7.31) it is found that

$$\boxed{\boxed{\begin{aligned} &\mu_0 = E\{\mathcal{N}_{T_b}|m_0 \text{ sent}\} = \int_0^{T_b} \left(\frac{\eta}{hf}\, p_0 + \mathcal{I}_d\right) dt = \mathcal{I}_d T_b + \frac{\eta\lambda}{hc}\, p_0 T_b \\ &\mu_1 = E\{\mathcal{N}_{T_b}|m_1 \text{ sent}\} = \mu_0 + \frac{\eta\lambda}{hc} \int_0^{T_b} p(t)dt = \mu_0 + \frac{\eta\lambda}{hc} \cdot \mathcal{E}_p \end{aligned}}} \tag{7.34}$$

A very useful approximate expression of the bit error probability:

The key to the Gaussian approximation is to approximate the conditional random variable $\mathcal{N}_{T_b}$ in (7.35), with a Gaussian random variable having the same mean and variance. Doing this, P_F and P_M are approximated by

$$\begin{aligned} P_F &= Prob\left\{\frac{\mathcal{N}_{T_b}-\mu_0}{\sqrt{\mu_0}} > \frac{\alpha-\mu_0}{\sqrt{\mu_0}}|m_0 \text{ sent}\right\} \approx Q\left(\frac{\alpha-\mu_0}{\sqrt{\mu_0}}\right) \\ P_M &= Prob\left\{\frac{\mathcal{N}_{T_b}-\mu_1}{\sqrt{\mu_1}} \leq \frac{\alpha-\mu_1}{\sqrt{\mu_1}}|m_1 \text{ sent}\right\} \approx Q\left(\frac{\mu_1-\alpha}{\sqrt{\mu_1}}\right) \end{aligned} \tag{7.37}$$

A very useful approximation on the bit error probability is obtained by **also** approximating the threshold α in (7.37) by

$$\alpha \approx \sqrt{\mu_0\mu_1} \tag{7.38}$$

which makes the approximations of P_F and P_M in (7.37) identical. The resulting approximate expression of the bit error probability then becomes

OBS!

$$\boxed{\begin{aligned} &P_b \approx Q(\varrho) \\ &\varrho = \sqrt{\mu_1} - \sqrt{\mu_0} \end{aligned}} \tag{7.39}$$

$$\boxed{\begin{aligned} &P_b \approx Q(\varrho) \\ &\varrho = \sqrt{\mu_1} - \sqrt{\mu_0} \end{aligned}} \qquad (7.39)$$

$$\boxed{\begin{aligned} \mu_0 &= E\{\mathcal{N}_{T_b} | m_0 \text{ sent}\} = \int_0^{T_b} \left(\frac{\eta}{hf}\, p_0 + \mathcal{I}_d \right) dt = \mathcal{I}_d T_b + \frac{\eta\lambda}{hc}\, p_0 T_b \\ \mu_1 &= E\{\mathcal{N}_{T_b} | m_1 \text{ sent}\} = \mu_0 + \frac{\eta\lambda}{hc} \int_0^{T_b} p(t) dt = \mu_0 + \frac{\eta\lambda}{hc} \cdot \mathcal{E}_p \end{aligned}} \qquad (7.34)$$

$$\mathcal{I}_d = i_d / q$$

7.3.2 Additive Noise

Consider the receiver in Figure 7.8a, and assume now that noise is introduced by the amplifier. This means that the decision variable ξ will contain a noisy component, here denoted by U,

$$\xi = y(T_b) = Aq\mathcal{N}_{T_b} + U \tag{7.40}$$

$$\begin{aligned} P_F &= Prob\{\mathcal{N}_{T_b} + w > \alpha | m_0 \text{ sent}\} = \qquad (7.43) \\ &= Prob\left\{\frac{\mathcal{N}_{T_b} + w - \mu_0}{\sqrt{\mu_0 + \sigma_w^2}} > \frac{\alpha - \mu_0}{\sqrt{\mu_0 + \sigma_w^2}} | m_0 \text{ sent}\right\} \approx Q\left(\frac{\alpha - \mu_0}{\sqrt{\mu_0 + \sigma_w^2}}\right) \end{aligned}$$

$$\boxed{\begin{aligned}&P_b \approx Q(\varrho)\\ &\varrho = \sqrt{\mu_1 + \sigma_w^2} - \sqrt{\mu_0 + \sigma_w^2} = \frac{\mu_1 - \mu_0}{\sqrt{\mu_0 + \sigma_w^2} + \sqrt{\mu_1 + \sigma_w^2}}\end{aligned}} \qquad (7.46)$$

$$\varrho = \frac{\frac{\eta\lambda}{hc}\,\mathcal{P}_p T_b}{\sqrt{\mathcal{I}_d T_b + \frac{\eta\lambda}{hc}\,p_0 T_b + k_\sigma T_b} + \sqrt{\mathcal{I}_d T_b + \frac{\eta\lambda}{hc}\,(p_0 T_b + \mathcal{P}_p T_b) + k_\sigma T_b}} \qquad (7.47)$$

$$\mathcal{P}_p = \mathcal{E}_p / T_b$$

$$\frac{\mathcal{P}_{p,1}}{\sqrt{R_{b,1}}} = \frac{\mathcal{P}_{p,2}}{\sqrt{R_{b,2}}} \qquad (7.48)$$