

## **Study week 5.**

### **NOTE!**

Deadline for the **project report** (pdf-format) Thursday 28 November 2013, 17.00.

You can now sign-up for the **lab** on the home page

# Chapter 9

## An Introduction to Time-varying Multipath Channels

$$z(t) = \sum_n \alpha_n(t) s(t - \tau_n(t)) \quad (9.1)$$

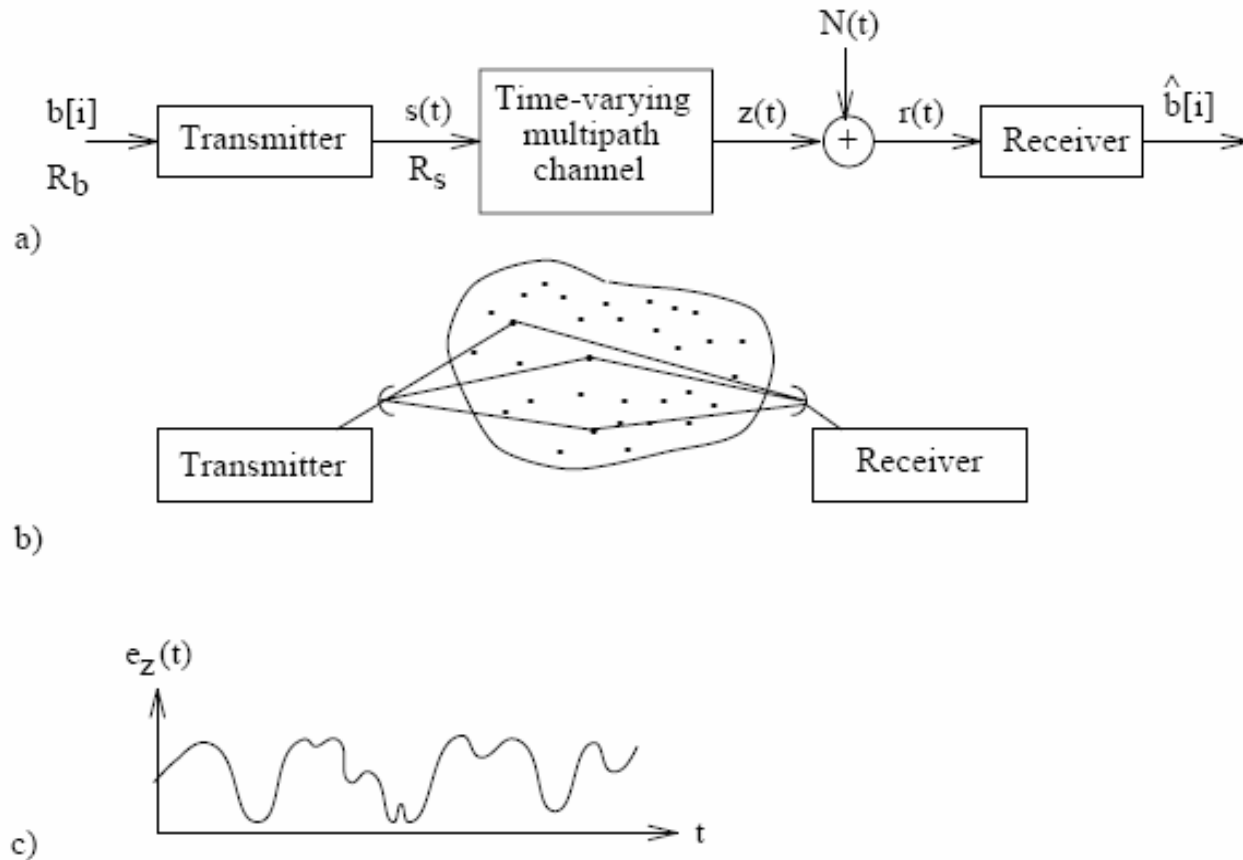


Figure 9.1: a) The digital communication system; b) A scattering medium; c) Illustrating the fading envelope  $e_z(t)$ .

$$s(t) = \cos((\omega_c + \omega_1)t) , \quad -\infty \leq t \leq \infty \quad (9.2)$$

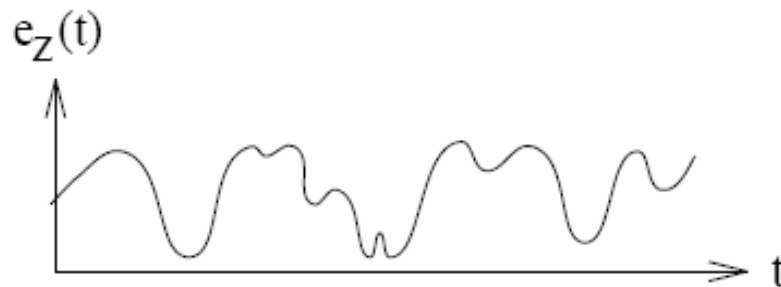
$$\begin{aligned}
 z(t) &= \sum_n \alpha_n(t) \cos((\omega_c + \omega_1)(t - \tau_n(t))) = \\
 &= \underbrace{\left[ \sum_n \alpha_n(t) \cos((\omega_c + \omega_1)\tau_n(t)) \right]}_{z_I(t) = \tilde{H}_{Re}(f_1, t)/2} \cos((\omega_c + \omega_1)t) - \\
 &\quad - \underbrace{\left[ \sum_n \alpha_n(t) \sin(-(\omega_c + \omega_1)\tau_n(t)) \right]}_{z_Q(t) = \tilde{H}_{Im}(f_1, t)/2} \sin((\omega_c + \omega_1)t) \\
 &= z_I(t) \cos((\omega_c + \omega_1)t) - z_Q(t) \sin((\omega_c + \omega_1)t) \\
 &= e_z(t) \cos((\omega_c + \omega_1)t + \theta_z(t)) \quad (9.3)
 \end{aligned}$$

**Compare with the time-invariant QAM-result:**

$$\begin{aligned}
 A_z + jB_z &= (A + jB)H(f_c) = \sqrt{A^2 + B^2}|H(f_c)|e^{j(\nu + \phi(f_c))} = \\
 &= (A + jB)(H_{Re}(f_c) + jH_{Im}(f_c)) \quad (3.110)
 \end{aligned}$$

$$s(t) = \cos((\omega_c + \omega_1)t) , \quad -\infty \leq t \leq \infty \quad (9.2)$$

$$\begin{aligned} z(t) &= \sum_n \alpha_n(t) \cos((\omega_c + \omega_1)(t - \tau_n(t))) = \\ &= e_z(t) \cos((\omega_c + \omega_1)t + \theta_z(t)) \end{aligned} \quad (9.3)$$



Observe that the quadrature components  $z_I(t)$  and  $z_Q(t)$  in (9.3) are *time-varying*. Hence, the output signal  $z(t)$  is *not* a pure sine wave with frequency  $f_c + f_1$ . *This is a significant difference compared with the linear time-invariant channel.* It is seen in (9.3) that the quadrature components depend

$$\begin{aligned}
z(t) &= \sum_n \alpha_n(t) \cos((\omega_c + \omega_1)(t - \tau_n(t))) = \\
&= z_I(t) \cos((\omega_c + \omega_1)t) - z_Q(t) \sin((\omega_c + \omega_1)t) \\
&= e_z(t) \cos((\omega_c + \omega_1)t + \theta_z(t))
\end{aligned}$$

Throughout this chapter it is assumed that  $z_I(t)$  and  $z_Q(t)$  may be modelled as baseband zero-mean wide-sense-stationary (WSS) *Gaussian random processes* (with variances  $\sigma_I^2 = \sigma_Q^2 = \sigma^2$ ). This is a commonly used assumption when the number of scatterers is large, implying that central limit theorem arguments can be used [43], [65], [68], [39]. For a fixed value of  $t$ , this assumption leads to a Rayleigh-distributed envelope  $e_z(t)$ ,

$$e_z(t) = \sqrt{z_I^2(t) + z_Q^2(t)} \quad (9.4)$$

$$p_{e_z}(x) = \frac{2x}{b} e^{-x^2/b}, \quad x \geq 0, \text{ Rayleigh distr.} \quad (9.5)$$

$$b = E\{e_z^2(t)\} = 2\sigma^2 = 2P_z \quad (9.6)$$

and a uniformly distributed phase  $\theta_z(t)$  (over a  $2\pi$  interval). The zero-mean assumption means that there is no deterministic signal path present in  $z(t)$ . If a

### 9.1.1 Doppler Power Spectrum and Coherence Time

$$\begin{aligned}
 R_{\mathcal{D}}(f) &= \mathcal{F}(\tilde{c}_z(\tau)) \\
 \tilde{c}_z(\tau) &= \frac{1}{2} E\{[z_I(t + \tau) + jz_Q(t + \tau)] [z_I(t) - jz_Q(t)]\} \\
 R_z(f) &= \frac{1}{2} (R_{\mathcal{D}}(f + f_c + f_1) + R_{\mathcal{D}}(f - f_c - f_1))
 \end{aligned}
 \tag{9.7}$$

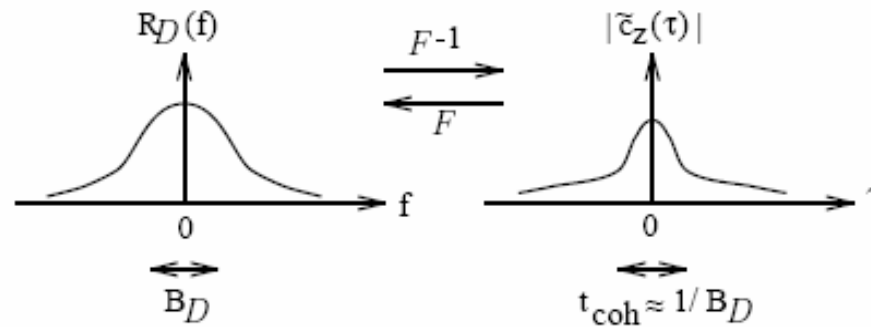


Figure 9.2: Illustrating the Fourier transform pair  $\tilde{c}_z(\tau) \longleftrightarrow R_{\mathcal{D}}(f)$ .

$$t_{\text{coh}} \approx 1/B_{\mathcal{D}}
 \tag{9.8}$$

If the channel is slowly changing, then the coherence time is large. Note that  $z_I(t + \tau)$  and  $z_I(t)$  (also  $z_Q(t + \tau)$  and  $z_Q(t)$ ) are correlated over time-intervals  $\tau$  (much) smaller than the coherence time  $t_{coh}$ . Hence, input signals within such intervals are therefore affected similarly by the fading channel. On the other hand, input signals that are separated in time by (much) more than  $t_{coh}$ , are affected differently by the channel, and at the output of the channel they become essentially independent of each other. If the former case apply (time flat fading), for a given time-interval, then we say that the channel is **time-nonselective**, and if the latter case apply, then the channel is said to be **time-selective**.



## 9.1.2 Coherence Bandwidth and Multipath Spread

$$z(t) = z(f_1, t) = \underbrace{\frac{1}{2} \tilde{H}_{Re}(f_1, t)}_{z_I(t)} \cos((\omega_c + \omega_1)t) - \underbrace{\frac{1}{2} \tilde{H}_{Im}(f_1, t)}_{z_Q(t)} \sin((\omega_c + \omega_1)t) \quad (9.9)$$

What can be said about the output signal  $z(t)$  if another frequency  $f_2 = f_1 + f_\Delta$  is used, instead of  $f_1$ ? Are different frequency-intervals, in the input signal spectrum, treated differently by the time-varying multipath channel? To answer these questions the correlation between  $z(f_1, t)$  and  $z(f_1 + f_\Delta, t)$  can be found by

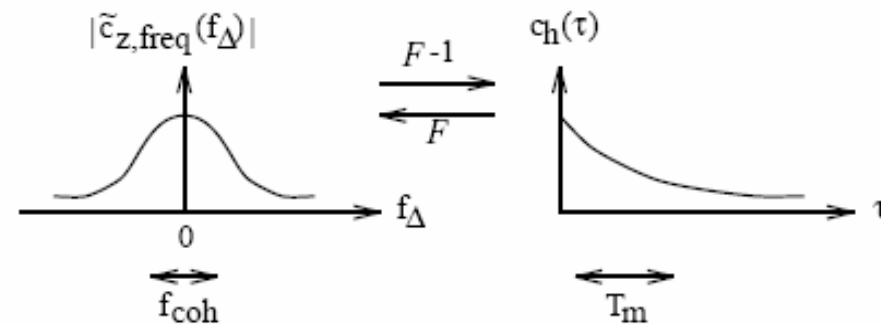


Figure 9.3: Illustrating the Fourier transform pair  $c_h(\tau) \longleftrightarrow \tilde{c}_{z, \text{freq}}(f_\Delta)$ .

The **coherence bandwidth**  $f_{coh}$  of the channel is defined as the width of the autocorrelation function  $\tilde{c}_{z,freq}(f\Delta)$ , see Figure 9.3. Note that frequencies within a frequency-interval (much) smaller than the coherence bandwidth  $f_{coh}$  are correlated, and they are affected similarly by the fading channel. On the other hand, two frequencies that are separated by (much) more than  $f_{coh}$ , are affected differently by the channel, and they are essentially independent of each other. If the former case apply (frequency flat fading), for a given frequency-interval, then we say that the channel is **frequency-nonselective**, and if the latter case apply, then the channel is said to be **frequency-selective**.

$$z(t) = \int_{-\infty}^{\infty} h(\tau, t) s(t - \tau) d\tau \quad (9.10)$$

**delay power spectrum**  $c_h(\tau)$  (also multipath intensity profile) of the time-varying impulse response  $h(\tau, t)$ ,

$$c_h(\tau) = E \left\{ \frac{h^2(\tau, t)}{2} \right\} = \frac{1}{2} E \{ h_I^2(\tau, t) + h_Q^2(\tau, t) \} = \frac{1}{2} E \{ \tilde{h}(\tau, t) \tilde{h}^*(\tau, t) \} \quad (9.15)$$

An example of the delay power spectrum  $c_h(\tau)$  is illustrated in Figure 9.3. The width of the delay power spectrum is referred to as the **multipath spread** of the channel and it is denoted by  $T_m$ . This is an important parameter since if  $T_m$  is too large, compared with e.g. the symbol time, then intersymbol interference can occur.

$$T_m \approx 1/f_{coh} \quad (9.16)$$

## 9.2 Frequency-Nonselective, Slowly Fading Channel

$$T_s \ll t_{coh} \quad (9.27)$$

or equivalently,

$$B_{\mathcal{D}} \ll R_s \quad (9.28)$$

This means that the channel is **slowly fading**, which imply that it can be treated as a time-invariant channel within the coherence time.

In this subsection a frequency-nonselective channel is investigated. To obtain this situation it is required that the bandwidth of the transmitted signal, denoted  $W$ , is much smaller than the coherence bandwidth  $f_{coh}$  of the channel,

$$W \ll f_{coh} \quad (9.29)$$

or equivalently,

$$T_m \ll 1/W \quad (9.30)$$

$$\tilde{z}(t) = \frac{1}{2} \int_{-\infty}^{\infty} \tilde{S}(f) \tilde{H}(f, t) e^{j2\pi ft} df \quad (9.26)$$

$$z_I(t) + jz_Q(t) = \frac{1}{2} \int_{-\infty}^{\infty} [S_I(f) + jS_Q(f)] [H_I(f, t) + jH_Q(f, t)] e^{j2\pi ft} df \quad (9.33)$$

$$z_I(t) + jz_Q(t) = \frac{1}{2} \int_{-\infty}^{\infty} [S_I(f) + jS_Q(f)] \cdot (H_I + jH_Q) e^{j2\pi ft} df \quad (9.36)$$

$$\begin{aligned} z_I(t) + jz_Q(t) &= \frac{1}{2} (s_I(t) + js_Q(t))(H_I + jH_Q) = \\ &= e_s(t) e^{j\theta_s(t)} \cdot a e^{j\phi} = e_z(t) e^{j\theta_z(t)} \end{aligned} \quad (9.37)$$

$$\begin{aligned}
z_I(t) + jz_Q(t) &= \frac{1}{2} (s_I(t) + js_Q(t))(H_I + jH_Q) = \\
&= e_s(t)e^{j\theta_s(t)} \cdot ae^{j\phi} = e_z(t)e^{j\theta_z(t)}
\end{aligned} \tag{9.37}$$

$$\boxed{z(t) = ae_s(t) \cos(\omega_c t + \theta_s(t) + \phi)} \tag{9.38}$$

$$p_a(x) = \frac{2x}{b} e^{-x^2/b}, \quad x \geq 0 \quad (\text{Rayleigh distribution}) \tag{9.39}$$

where,

$$E\{a\} = \frac{1}{2} \sqrt{\pi b} \tag{9.40}$$

$$E\{a^2\} = b \tag{9.41}$$

and,

$$p_\phi(y) = \begin{cases} 1/2\pi & , \quad -\pi \leq y \leq \pi \\ 0 & , \quad \text{otherwise} \end{cases} \tag{9.42}$$

If we assume uncoded equally likely binary signals over a Rayleigh fading channel ( $z_1(t) = as_1(t), z_0(t) = as_0(t)$ ), then the bit error probability of the ideal coherent ML receiver is ( $0 < d^2 = \frac{D_{s_1, s_0}^2}{2E_{b, sent}} \leq 2$ )

$$P_b = \int_0^\infty \Pr\{\text{error}|a\} p_a(x) dx = E\{\Pr\{\text{error}|a\}\} \quad (9.43)$$

$$\begin{aligned} P_b &= \int_0^\infty Q(\sqrt{d^2 x^2 E_{b, sent} / N_0}) \frac{2x}{b} e^{-x^2/b} dx = \\ &= -e^{-x^2/b} Q(x\sqrt{d^2 E_{b, sent} / N_0}) \Big|_0^\infty - \int_0^\infty (-e^{-x^2/b}) \\ &\quad \left( \frac{-\sqrt{d^2 E_{b, sent} / N_0}}{\sqrt{2\pi}} e^{-\frac{x^2 d^2 E_{b, sent} / N_0}{2}} \right) dx = \\ &= \frac{1}{2} - \sqrt{d^2 E_{b, sent} / N_0} \cdot \underbrace{\beta \int_0^\infty \frac{e^{-x^2/2\beta^2}}{\beta\sqrt{2\pi}} dx}_{1/2} \end{aligned} \quad (9.44)$$

$$\mathcal{E}_b = E\{a^2\}E_{b,sent} = bE_{b,sent} \quad (9.45)$$

$$P_b = \frac{1}{2} \left( 1 - \sqrt{\frac{d^2 \mathcal{E}_b / N_0}{2 + d^2 \mathcal{E}_b / N_0}} \right) = \frac{1}{2 + d^2 \mathcal{E}_b / N_0 + \sqrt{2 + d^2 \mathcal{E}_b / N_0} \sqrt{d^2 \mathcal{E}_b / N_0}}$$

$\mathcal{E}_b / N_0$  “large”  
 $\downarrow$   
 $\approx \frac{1}{2d^2 \mathcal{E}_b / N_0}$ 
(9.46)

where  $d^2 = 2$  for antipodal signals and  $d^2 = 1$  for orthogonal signals.

*Observe the dramatic increase in  $P_b$  due to the Rayleigh fading channel.  $P_b$  is no longer exponentially decaying in  $\mathcal{E}_b / N_0$ , it now decays essentially as  $(\mathcal{E}_b / N_0)^{-1}$ !*



## EXAMPLE 9.1

Assume that equally likely, binary orthogonal FSK signals, with equal energy, are sent from the transmitter. Hence,  $s_i(t) = \sqrt{2E_{b, \text{sent}}/T_b} \cos(2\pi f_i t)$  in  $0 \leq t \leq T_b$ ,  $i = 0, 1$ .

These signals are communicated over a Rayleigh fading channel, i.e. the received signal is (see (9.38)),

$$r(t) = a\sqrt{2E_{b, \text{sent}}/T_b} \cos(2\pi f_i t + \phi) + N(t)$$

Assume that the incoherent receiver in Figure 5.28 on page 397 is used. From (5.109) it is known that for a given value of  $a$ ,

$$P_b = \frac{1}{2} e^{-a^2 E_{b, \text{sent}}/2N_0}$$

since  $a^2 E_{b, \text{sent}}$  then is the average received energy per bit.

For the Rayleigh fading channel, and the same receiver,  $P_b$  can be calculated by using (9.43),

$$P_b = \int_0^\infty \Pr\{\text{error}|a = x\} p_a(x) dx = E\{\Pr\{\text{error}|a\}\}$$

$$E\{\Pr\{error|a\}\} = E\left\{\frac{1}{2} e^{-a^2 E_{b, sent}/2N_0}\right\} =$$

$$E\left\{\frac{1}{2} e^{-a_1^2 E_{b, sent}/2N_0}\right\} \cdot E\left\{e^{-a_2^2 E_{b, sent}/2N_0}\right\}$$

$$P_b = \frac{1/2}{1 + \frac{E_{b, sent}}{N_0} \cdot \frac{E\{a^2\}}{2}} = \frac{1}{2 + \mathcal{E}_b/N_0}$$

*Observe the dramatic increase in  $P_b$  due to the Rayleigh fading channel.  $P_b$  is no longer exponentially decaying in  $\mathcal{E}_b/N_0$ , it now decays essentially as  $(\mathcal{E}_b/N_0)^{-1}$ ! As an example, assuming  $\mathcal{E}_b/N_0 = 1000$  (30 dB), we obtain*

$$P_b = \begin{cases} 0.5e^{-500} \approx 3.6 \cdot 10^{-218} & , \text{ AWGN} \\ (1002)^{-1} \approx 10^{-3} & , \text{ Rayleigh+AWGN} \end{cases}$$

**DIVERSITY IS NEEDED!**

## 7.3 Reception and Detection

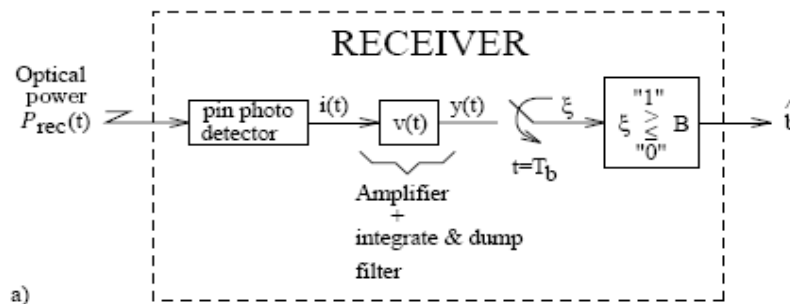
**Within a bit interval:** A received random number of photons generates a random number of photo-electrons after the photo-detector.

The Poisson Process:

In (7.27), the arrival times  $\dots, t_{i-1}, t_i, t_{i+1}$ , are modeled as a **Poisson process** with an intensity  $\mathcal{I}(t)$ . This means that the number of arrivals  $\mathcal{N}_{\mathcal{T}}$ , within a time interval of length  $\mathcal{T}$ , is a random variable having the properties

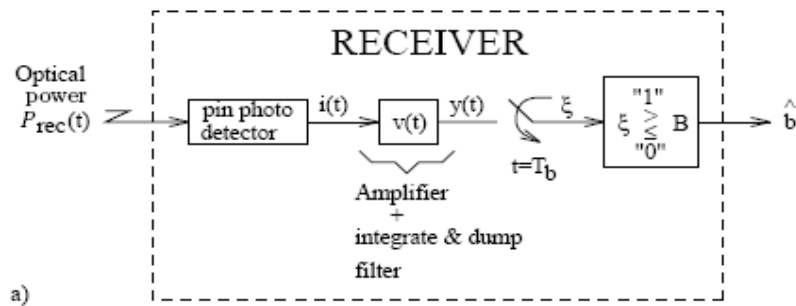
$$\begin{aligned} \text{Prob}\{\mathcal{N}_{\mathcal{T}} = n\} &= \frac{\mu^n e^{-\mu}}{n!} \\ \mu &= E\{\mathcal{N}_{\mathcal{T}}\} = \int_{t_0}^{t_0+\mathcal{T}} I(t) dt \\ \sigma^2 &= E\{(\mathcal{N}_{\mathcal{T}} - \mu)^2\} = \mu \end{aligned} \quad (7.29)$$

Note that the mean and the variance are identical.

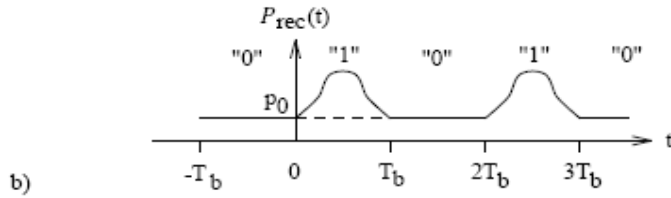


Compare with Chapter 4!

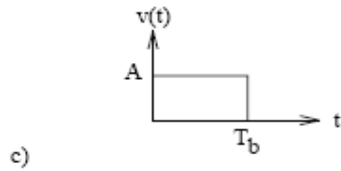
Fig. 7.8:



”0”:  $p_0$   
 ”1”:  $p_0 + p(t)$



Received optical power.



$$P_{rec}(t) = p_0 + \sum_{i=-\infty}^{\infty} m[i]p(t - iT_b), \quad m[i] \in \{0, 1\}, \quad -\infty \leq t \leq \infty \quad (7.31)$$

$$\begin{aligned} \xi &= y(T_b) = \int_{-\infty}^{\infty} i(\tau)v(T_b - \tau)d\tau = A \int_0^{T_b} i(\tau)d\tau = \\ &= A \int_0^{T_b} (i_r(t) + i_d(t))dt = AqN_{T_b} \end{aligned} \quad (7.32)$$

q=charge of an electron.  
 id(t)=”dark current”.

## Bit error probability:

$$\begin{aligned} P_b &= P_0 \underbrace{Prob\{\text{error}|m_0 \text{ sent}\}}_{P_F} + P_1 \underbrace{Prob\{\text{error}|m_1 \text{ sent}\}}_{P_M} \\ &= P_0 Prob\{\xi > B|m_0 \text{ sent}\} + P_1 Prob\{\xi \leq B|m_1 \text{ sent}\} = \\ &= P_0 Prob\{\mathcal{N}_{T_b} > (B/Aq)|m_0 \text{ sent}\} + \\ &\quad + P_1 Prob\{\mathcal{N}_{T_b} \leq (B/Aq)|m_1 \text{ sent}\} \end{aligned} \tag{7.33}$$

$$\begin{aligned} P_F &= Prob\{\mathcal{N}_{T_b} > \alpha|m_0 \text{ sent}\} = \sum_{n=\alpha+1}^{\infty} \frac{\mu_0^n e^{-\mu_0}}{n!} \\ P_M &= Prob\{\mathcal{N}_{T_b} \leq \alpha|m_1 \text{ sent}\} = \sum_{n=0}^{\alpha} \frac{\mu_1^n e^{-\mu_1}}{n!} \\ \alpha &= B/Aq \end{aligned} \tag{7.35}$$

**Exact expressions!**

**We need the averages!**

$$\begin{aligned}
 \text{Prob}\{\mathcal{N}_T = n\} &= \frac{\mu^n e^{-\mu}}{n!} \\
 \mu &= E\{\mathcal{N}_T\} = \int_{t_0}^{t_0+T} I(t) dt \\
 \sigma^2 &= E\{(\mathcal{N}_T - \mu)^2\} = \mu
 \end{aligned}
 \tag{7.29}$$

$$\mathcal{I}_e(t) = \eta \cdot \mathcal{M} \cdot \mathcal{I}_{ph}(t) + \mathcal{I}_d = \eta \cdot \mathcal{M} \cdot \frac{\mathcal{P}_{rec}(t)}{hf} + \mathcal{I}_d \text{ [electrons/s]}
 \tag{7.8}$$

Id=id/q  
Page 476.

Combining (7.29), (7.8) and (7.31) it is found that

$$\begin{aligned}
 \mu_0 &= E\{\mathcal{N}_{T_b} | m_0 \text{ sent}\} = \int_0^{T_b} \left( \frac{\eta}{hf} p_0 + \mathcal{I}_d \right) dt = \mathcal{I}_d T_b + \frac{\eta\lambda}{hc} p_0 T_b \\
 \mu_1 &= E\{\mathcal{N}_{T_b} | m_1 \text{ sent}\} = \mu_0 + \frac{\eta\lambda}{hc} \int_0^{T_b} p(t) dt = \mu_0 + \frac{\eta\lambda}{hc} \cdot \mathcal{E}_p
 \end{aligned}
 \tag{7.34}$$

## *A very useful approximate expression of the bit error probability:*

The key to the Gaussian approximation is to approximate the conditional random variable  $\mathcal{N}_{T_b}$  in (7.35), with a Gaussian random variable having the same mean and variance. Doing this,  $P_F$  and  $P_M$  are approximated by

$$P_F = \text{Prob} \left\{ \frac{\mathcal{N}_{T_b} - \mu_0}{\sqrt{\mu_0}} > \frac{\alpha - \mu_0}{\sqrt{\mu_0}} \mid m_0 \text{ sent} \right\} \approx Q \left( \frac{\alpha - \mu_0}{\sqrt{\mu_0}} \right) \quad (7.37)$$

$$P_M = \text{Prob} \left\{ \frac{\mathcal{N}_{T_b} - \mu_1}{\sqrt{\mu_1}} \leq \frac{\alpha - \mu_1}{\sqrt{\mu_1}} \mid m_1 \text{ sent} \right\} \approx Q \left( \frac{\mu_1 - \alpha}{\sqrt{\mu_1}} \right)$$

A very useful approximation on the bit error probability is obtained by also approximating the threshold  $\alpha$  in (7.37) by

$$\alpha \approx \sqrt{\mu_0 \mu_1} \quad (7.38)$$

which makes the approximations of  $P_F$  and  $P_M$  in (7.37) identical. The resulting approximate expression of the bit error probability then becomes

OBS!

$$\boxed{P_b \approx Q(\varrho)} \quad (7.39)$$
$$\varrho = \sqrt{\mu_1} - \sqrt{\mu_0}$$



$$\begin{aligned}
 P_b &\approx Q(\varrho) \\
 \varrho &= \sqrt{\mu_1} - \sqrt{\mu_0}
 \end{aligned}
 \tag{7.39}$$

$$\begin{aligned}
 \mu_0 &= E\{\mathcal{N}_{T_b} | m_0 \text{ sent}\} = \int_0^{T_b} \left( \frac{\eta}{hf} p_0 + \mathcal{I}_d \right) dt = \mathcal{I}_d T_b + \frac{\eta\lambda}{hc} p_0 T_b \\
 \mu_1 &= E\{\mathcal{N}_{T_b} | m_1 \text{ sent}\} = \mu_0 + \frac{\eta\lambda}{hc} \int_0^{T_b} p(t) dt = \mu_0 + \frac{\eta\lambda}{hc} \cdot \mathcal{E}_p
 \end{aligned}
 \tag{7.34}$$

$$\mathcal{I}_d = i_d/q$$

## 7.3.2 Additive Noise

Consider the receiver in Figure 7.8a, and assume now that noise is introduced by the amplifier. This means that the decision variable  $\xi$  will contain a noisy component, here denoted by  $U$ ,

$$\xi = y(T_b) = Aq\mathcal{N}_{T_b} + U \quad (7.40)$$

$$\begin{aligned} P_F &= \text{Prob}\{\mathcal{N}_{T_b} + w > \alpha | m_0 \text{ sent}\} = & (7.43) \\ &= \text{Prob}\left\{\frac{\mathcal{N}_{T_b} + w - \mu_0}{\sqrt{\mu_0 + \sigma_w^2}} > \frac{\alpha - \mu_0}{\sqrt{\mu_0 + \sigma_w^2}} | m_0 \text{ sent}\right\} \approx Q\left(\frac{\alpha - \mu_0}{\sqrt{\mu_0 + \sigma_w^2}}\right) \end{aligned}$$

$$P_b \approx Q(\varrho)$$

$$\varrho = \sqrt{\mu_1 + \sigma_w^2} - \sqrt{\mu_0 + \sigma_w^2} = \frac{\mu_1 - \mu_0}{\sqrt{\mu_0 + \sigma_w^2} + \sqrt{\mu_1 + \sigma_w^2}}$$

(7.46)

$$\varrho = \frac{\frac{\eta\lambda}{hc} \mathcal{P}_p T_b}{\sqrt{\mathcal{I}_d T_b + \frac{\eta\lambda}{hc} p_0 T_b + k_\sigma T_b} + \sqrt{\mathcal{I}_d T_b + \frac{\eta\lambda}{hc} (p_0 T_b + \mathcal{P}_p T_b) + k_\sigma T_b}} \quad (7.47)$$

$$\mathcal{P}_p = \mathcal{E}_p / T_b$$

$$\frac{\mathcal{P}_{p,1}}{\sqrt{R_{b,1}}} = \frac{\mathcal{P}_{p,2}}{\sqrt{R_{b,2}}} \quad (7.48)$$