1 GRUNDLÄGGANDE SANNOLIKHETSTEORI

Definition: Fördelningsfunktion

$$F_{\tilde{x}}(x) = P(\tilde{x} \leq x) \qquad -\infty < x < \infty$$

Sats:

$$\lim_{x \to -\infty} F_{\tilde{x}}(x) = 0$$

$$\lim_{x \to \infty} F_{\tilde{x}}(x) = 1$$

$$P(a < \tilde{x} \leq b) = F_{\tilde{x}}(b) - F_{\tilde{x}}(a) \qquad a \leq b$$

Kontinuerligt	**Diskret**
Frekvensfunktion:	Täthetsfunktion:
$f_{\tilde{x}}(x) = \frac{d}{dx} F_{\tilde{x}}(x)$	$p_{\tilde{x}}(k) = F_{\tilde{x}}(k) - F_{\tilde{x}}(k-1)$
$F_{\tilde{x}}(x) = \int_{-\infty}^{x} f_{\tilde{x}}(x) dx$	$F_{\tilde{x}}(k) = \sum_{i=-\infty}^{k} p_{\tilde{x}}(i)$
$\int_{-\infty}^{\infty} f_{\tilde{x}}(x)\, dx = 1$	$\sum_{k=-\infty}^{\infty} p_{\tilde{x}}(k) = 1$
Medelvärde:	Medelvärde:
$E\{\tilde{x}\} = \int_{-\infty}^{\infty} x f_{\tilde{x}}(x)\, dx$	$E\{\tilde{x}\} = \sum_{k=-\infty}^{\infty} k\, p_{\tilde{x}}(k)$
2:a nollpunktsmomentet:	2:a nollpunktsmomentet:
$E\{\tilde{x}^2\} = \int_{-\infty}^{\infty} x^2 f_{\tilde{x}}(x)\, dx$	$E\{\tilde{x}^2\} = \sum_{k=-\infty}^{\infty} k^2 p_{\tilde{x}}(k)$
Varians:	Varians:
$V\{\tilde{x}\} = E\{\tilde{x}^2\} - E\{\tilde{x}\}^2$	$V\{\tilde{x}\} = E\{\tilde{x}^2\} - E\{\tilde{x}\}^2$

1:a moment av funktion av s.v. $\tilde{y} = g(\tilde{x})$

$$E\{\tilde{y}\} = E\{g(\tilde{x})\} =$$

$$= \int_{-\infty}^{\infty} g(x) f_{\tilde{x}}(x) dx$$

1:a moment av funktion av s.v. $\tilde{l} = h(\tilde{k})$

$$E\{\tilde{l}\} = E\{h(\tilde{k})\} =$$

$$= \sum_{k=-\infty}^{\infty} h(k) p_{\tilde{k}}(k)$$

Variationskoefficient: $C^2\{\tilde{x}\} = \frac{V\{\tilde{x}\}}{E\{\tilde{x}\}^2}$

Betingad sannolikhet: $P(A|B) = \frac{P(AB)}{P(B)} \quad P(B) \neq 0$

Satsen om total sannolikhet:

$$P(A) = \sum_i P(A|H_i) \cdot P(H_i), \quad f_{\tilde{y}}(y) = \int_x f_{\tilde{y}}(y|\tilde{x} = x) f_{\tilde{x}}(x) dx$$

2 NÅGRA SPECIELLA FUNKTIONER

Taylorserie:

$$f(a+h) = f(a) + \sum_{n=1}^{\infty} \frac{h^n}{n!}\, f^n(a)$$

Aritmetrisk serie:

$$a_n = a_1 + (n-1)h$$

$$s_n = a_1 + a_2 + \ldots + a_n = \frac{n}{2}\,(a_1 + a_n)$$

Geometrisk serie:

$$a_n = r^n$$

$$s_n = \sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}$$

Derivation:

$$\frac{d}{dx} f(x) g(x) = \frac{df(x)}{dx} g(x) + f(x) \frac{dg(x)}{dx}$$

$$\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{\frac{df(x)}{dx} g(x) - f(x) \frac{dg(x)}{dx}}{[g(x)]^2}$$

Partiell integration:

$$\int_x f(x)\; g(x)\; dx = [F(x)\; g(x)] - \int_x F(x)\; g'(x)\; dx$$

En användbar formel:

$$\int_0^{\infty} t^k e^{-at} dt = \frac{k!}{a^{k+1}}$$

Viktiga summor:

$$\sum_{k=0}^{N} \alpha^k = \begin{cases} \frac{1-\alpha^{N+1}}{1-\alpha} & \alpha \neq 1 \\ N+1 & \alpha = 1 \end{cases}$$

$$\left(\begin{array}{ll} \text{Specialfall, låt} N \to \infty & \\ \sum_{k=0}^{\infty} \alpha^k = \frac{1}{1-\alpha} & |\alpha| < 1 \end{array} \right)$$

$$\sum_{k=0}^{\infty} \frac{\alpha^k}{k!} = e^{\alpha}$$

$$\sum_{k=1}^{\infty} k\alpha^{k-1} = \frac{1}{(1-\alpha)^2}$$

$$\sum_{k=0}^{N} \binom{N}{k} a^k b^{N-k} = (a+b)^N$$

Binomialkoefficienter:

$$n! = 1 \cdot 2 \cdots (n-1) \cdot n$$

$$0! = 1$$

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!} \qquad n > 0 \quad r \geq 0$$

$$\binom{n}{r} = \frac{n(n-1)\ \ldots\ (n-r+1)}{r!}$$

$$\binom{n}{r} = \binom{n}{n-r}$$

$$\binom{n}{0} = 1\ , \binom{n}{1} = n \qquad \binom{n}{n} = 1$$

$$\binom{n}{r} = 0 \text{ för } r > n \text{ och } r < 0$$

a. Antal olika sätt att ordna n element $= n!$

b. Antal olika sätt att välja r element av n med tillåtelse att repetera element $= n^r$

c. Antalet permutationer av r element ur $n = (n)_r =$ $n(n-1)\dots(n-r+1) = n!/(n-r)!$

d. Antalet kombinationer av r element ur n = binomialkoefficienterna $(n)_r/r! = \binom{n}{r}$

   Def: En <u>kombination</u> utgöres av alla permutationer inom en grupp specificerade element.

3 TRANSFORMER

3.1 Definition av z-transform och Laplace-transform

Diskret	**Kontinuerligt**
z-transform[1]	Laplace-transform
Def. $\tilde{x}$ diskret, icke-negativ s.v.	Def. $\tilde{x}$ kontinuerlig, icke-negativ s.v.
$P(z) = E\{z^{\tilde{x}}\} =$	$F^*_{\tilde{x}}(s) = E\{e^{-s\tilde{x}}\} =$
$= \sum_k P(\tilde{x} = k) z^k = \sum_k p_k z^k$	$= \int_0^\infty e^{-sx} f_{\tilde{x}}(x) dx$
OBS! $P(1) = \sum_k p_k = 1$	**OBS!** $F^*_{\tilde{x}}(0) = \int_0^\infty f_{\tilde{x}}(x) dx = 1$
$\frac{\partial P(z)}{\partial z} = \sum_k k\, p_k\, z^{k-1}$	$\frac{\partial F^*_{\tilde{x}}(s)}{\partial s} = -\int_0^\infty x\, e^{-sx} f_{\tilde{x}}(x) dx$
$E\{\tilde{x}\} = \lim_{z \to 1} \frac{\partial P(z)}{\partial z}$	$E\{\tilde{x}\} = -\lim_{s \to 0} \frac{\partial F^*_{\tilde{x}}(s)}{\partial s}$
$\frac{\partial^2 P(z)}{\partial z^2} = \sum_k k(k-1) p_k\, z^{k-2} =$	$\frac{\partial^2 F^*_{\tilde{x}}(s)}{\partial s^2} = \int_0^\infty x^2\, e^{-sx} f_{\tilde{x}}(x) dx$
$= (\sum_k k^2\, p_k - \sum_k k\, p_k) z^{k-2}$	
$E\{\tilde{x}^2\} = \lim_{z \to 1} \frac{\partial^2 P(z)}{\partial z^2} + E\{\tilde{x}\}$	$E\{\tilde{x}^2\} = \lim_{s \to 0} \frac{\partial^2 F^*_{\tilde{x}}(s)}{\partial s^2}$
	$E\{\tilde{x}^k\} = \lim_{s \to 0} (-1)^k \frac{\partial^k F^*_{\tilde{x}}(s)}{\partial s^k}$

[1] Observera den definition av z-transformen som används inom detta ämnesområde. En del andra använder $Y(z) = \sum_k y_k\, z^{-k}$

3.2 Några egenskaper hos Laplace-transformen

Funktion	Transform
1. $f(t) \qquad t \geq 0$	$F^*(s) = \int_0^\infty f(t)e^{-st}dt$
2. $af(t) + bg(t)$	$aF^*(s) + bG^*(s)$
3. $f(\frac{t}{a})$	$aF^*(as)$
4. $f(t-a)$	$e^{-as}F^*(s)$
5. $e^{-at}f(t)$	$F^*(s+a)$
6. $t\ f(t)$	$-\frac{dF^*(s)}{ds}$
7. $t^n f(t)$	$(-1)^n \frac{d^n F^*(s)}{ds^n}$
8. $\frac{f(t)}{t}$	$\int_{s_1=s}^\infty\ F^*(s_1)ds_1$
9. $f(t)\ \otimes\ g(t)$ (faltning)	$F^*(s)G^*(s)$
10. $\frac{df(t)}{dt}$	$sF^*(s) - f(0^-)$

3.3 Några Laplace-transformer

Funktion	Transform
1. $f(t) \qquad t \geq 0$	$F^*(s) = \int_0^\infty f(t) e^{-st} dt$
2. $\delta(t)$	1
3. $\delta(t-a)$	e^{-as}
4. Ae^{-at}	$\frac{A}{s+a}$
5. $t\ e^{-at}$	$\frac{1}{(s+a)^2}$
6. $\frac{t^n}{n!}\ e^{-at}$	$\frac{1}{(s+a)^{n+1}}$

3.4 Några egenskaper hos z-transformen

Sekvens	**z-transform**
1. $f_n \qquad n = 0, 1, 2, \ldots$	$F(z) = \sum_{n=0}^{\infty} f_n \; z^n$
2. $a \; f_n + b \; g_n$	$a \; F(z) + b \; G(z)$
3. $a^n f_n$	$F(az)$
4. f_{n+1}	$\frac{1}{z}(F(z) - f_0)$
5. f_{n-1}	$zF(z)$
6. $n \; f_n$	$z \; \frac{d}{dz} \; F(z)$
7. $f_n \; \otimes \; g_n$ (faltning)	$f(z)G(z)$
8. $f_n - f_{n-1}$	$(1 - z)F(z)$
9. $\sum_{k=0}^{n} \; f_k \qquad n = 0, 1, 2, \ldots$	$\frac{F(z)}{1-z}$

3.5 Några z-transformer

Sekvens	**z-transform**
1. $f_n \qquad n = 0, 1, 2, \ldots$	$F(z) = \sum_{n=0}^{\infty} f_n z^n$
2. $\delta_n = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$	1
3. δ_{n-k}	z^k
4. $A\alpha^n$	$\frac{A}{1-\alpha z}$
5. $n\alpha^n$	$\frac{\alpha z}{(1-\alpha z)^2}$
6. $n^2\ \alpha^n$	$\frac{\alpha z(1+\alpha z)}{(1-\alpha z)^3}$

4 KÖTEORI

4.1 Beteckningar

C_n n:te kunden som kommer till systemet

τ_n ankomsttidpunkt för C_n

$t_n = \tau_n - \tau_{n-1}$ ankomstavstånd mellan kunderna C_{n-1} och C_n

$A(t) = P(t_n \leq t)$

x_n betjäningsbehov för C_n

$B(x) = P(x_n \leq x)$

w_n väntetid i kö för C_n

s_n total tid i systemet för C_n

$s_n = w_n + x_n$

Ankomstintervallen:

$$t_n \rightarrow \tilde{t},\ A(t) = P(\tilde{t} \leq t)$$

$$E\{t_n\} = \bar{t} = \frac{1}{\lambda}$$

$$a(t) = \frac{dA(t)}{dt}$$

$$A^*(s) = \int_0^\infty e^{-st}\, a(t)dt$$

Betjäningstiderna:

$$x_n \to \tilde{x},\ B(x) = P(\tilde{x} \leq x)$$

$$E\{x_n\} = \overline{x}_n \to \overline{x} = 1/\mu$$

$$b(x) = \frac{dB(x)}{dx}$$

$$B^*(s) = \int_0^\infty e^{-sx}\, b(x)\, dx$$

Väntetiderna:

$$w_n \to \tilde{w},\ W(y) = P(\tilde{w} \leq y)$$

$$E\{w_n\} = \overline{w}_n \to W$$

$$w(y) = \frac{dW(y)}{dy}$$

$$W^*(s) = \int_0^\infty e^{-sy}\, w(y) dy$$

Total tid i systemet:

$$s_n \to \tilde{s},\ S(y) = P(\tilde{s} \leq y)$$

$$E\{s_n\} = \overline{s}_n \to T$$

$$s(y) = \frac{dS(y)}{dy}$$

$$S^*(s) = \int_0^\infty e^{-sy}\, s(y) dy$$

4.2 Upptagetsystem

Erlangfördelning:

$$A(t) = 1 - e^{-\lambda t}$$

$$B(x) = 1 - e^{-\mu x}$$

$$p_k = \frac{\rho^k/k!}{\sum_0^m \rho^i/i!} \qquad \rho = \lambda/\mu \qquad p_m = E_m(\rho)$$

Tidsspärr E = Anropsspärr $B = p_m = E_m(\rho)$

Rekursiv formel:

$$E_m(\rho) = \frac{\rho\, E_{m-1}(\rho)}{m + \rho\, E_{m-1}(\rho)} \; ; \; E_0(\rho) = 1$$

4.3 Väntsystem

M/M/1

$$p_k = \rho^k(1-\rho) \qquad \rho = \lambda/\mu$$

$$\overline{N} = \frac{\rho}{1-\rho} \quad \text{medelantal kunder i systemet}$$

$$W = \frac{\rho\overline{x}}{1-\rho}$$

$$T = \frac{\overline{x}}{1-\rho}$$

5 MARKOVKEDJOR

Diskret tid

$$P = (P_{ij})$$

$$p_{ij} = P(X_n = j | X_{n-1} = i)$$

$$\underline{p} = (p_1, p_2, \ldots)$$

$$\underline{p} = \underline{p} \cdot P$$

$$\sum p_i = 1$$

Absorberande kedjor:

$$P = \begin{pmatrix} S & R \\ O & I \end{pmatrix}$$

$$N = (I - S)^{-1} \quad \text{Fundamentalmatrisen}$$

Kontinuerlig tid

$$P(X(t + \Delta t) = j | X(t) = i) = \begin{cases} 1 - q_{ii}\Delta t & \\ q_{ij}\Delta t & i \neq j \end{cases}$$

$$\underline{p}(t) = (p_1(t), p_2(t), \ldots)$$

$$\frac{d\underline{p}}{dt} = \underline{p} \cdot Q$$

$$Q = \begin{bmatrix} -q_{11} & q_{12} & \cdots \\ q_{21} & -q_{22} & \cdots \\ \vdots & \vdots & \end{bmatrix}$$

$$\underline{0} = \underline{p} \cdot Q$$

$$\sum_i p_i = 1$$

Absorberande kedjor:

$$Q = \begin{bmatrix} \hat{Q} & R \\ & \\ 0 & 0 \end{bmatrix}$$

$N = -\hat{Q}^{-1}$ Fundamentalmatrisen

Räkneregel

$$A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A)$$

$\mathrm{adj}(A)$: Element j, i beräknas som $(-1)^{i+j}\det(A'_{ij})$.

A'_{ij} är A med rad i och kolonn j strukna.

Exempel

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Rightarrow$$

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Födelse-dödsprocesser:

$$q_{ij} = \begin{cases} \lambda_i & j = i+1 \\ & \\ \lambda_i + \mu_i & j = i \\ & \\ \mu_i & j = i-1 \\ & \\ 0 & \text{för övrigt} \end{cases}$$

$$p_k = \frac{\lambda_0 \lambda_1 \cdots \lambda_{k-1}}{\mu_1 \mu_2 \cdots \mu_k} \, p_0$$

$$\sum_k p_k = 1$$