Selected answers*
Problem set 4

Wireless Communications, 2nd Ed.

13.1 Impact of diversity

Correction: The problem erroneously points to Eq. (13.35) in the textbook. It should be Eq. (13.38).

The general approximation of average BER (called SER in the book, but for binary modulation it is the same) is

$$\overline{BER} \approx \left(\frac{1}{4E_b/N_0}\right)^{N_r} \binom{2N_r - 1}{N_r} \tag{1}$$

where N_r is the number of (independently fading) receive antennas. Our average SNR (average E_b/N_0) is 20 dB.

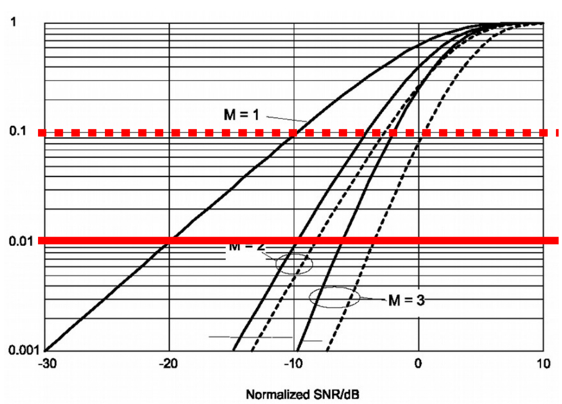
a) Using only one receive antenna ($N_r = 1$) we get $\overline{BER} \approx 1/(4\overline{E_b/N_0}) = 2.5 \times 10^{-3}$.

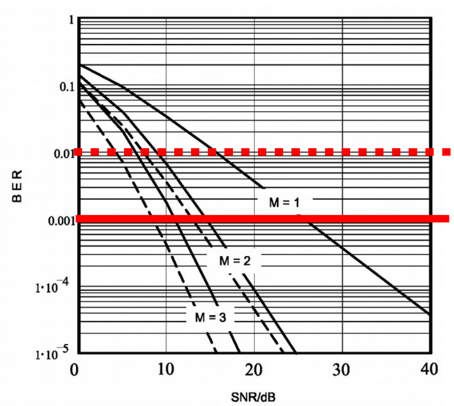
b) With three receive antennas ($N_r = 3$) we get $\overline{BER} \approx 10/(4\overline{E_b/N_0})^3 = 1.6 \times 10^{-7}$

c) Reaching a BER of 1.6×10^{-7} with only one receive antenna would require an average SNR of $\overline{E_b/N_0} \approx 1/(4 \times 1.6 \times 10^{-7}) = 1.56 \times 10^6 = 62\ \mathrm{dB}$, i.e. 42 dB more than if we use three receive antennas.

[13.2] Adding extra antennas

This entire problem can be solved quickly by studying figures 13.10 and 13.11 in the textbook. The latter figure is for MSK but, as our investigations show, MSK has the same BER as QPSK and BPSK … of which the latter is the one used in this problem. For simplicity, these figures are repeated here (solid line RSSI, dashed lines MRC):





a) Inspecting the outage figure (left – solid horizontal line) we see that 1% outage is reached at **fading margins** (how far down is the 1% level) 21 dB for 1 antenna, 10 dB (RSSI) or 8 dB (MRC) for 2 antennas, and 7 dB (RSSI) or 4 dB (MRC) for 3 antennas. This constitutes **diversity gains** (difference compared to only one antenna) of 11 dB (RSSI) or 13 dB (MRC) for two antennas

* Note: Solutions provided here are less detailed than the ones expected during the exam. Many steps are excluded.

and 14 dB (RSSI) or 17 dB (MRC) for three antennas.

b) Here we inspect the average BER figure at $\overline{BER} = 10^{-3}$ (right – solid horizontal line). We find that the required **average SNR**, $\overline{E_b/N_0}$, has to be 25 dB for one antenna, 14 dB (RSSI) or 13 dB (MRC) for 2 antennas, and 11 dB (RSSI) or 8 dB (MRC) for 3 antennas. This constitutes **diversity gains** (improvement over the 1-antenna case) of 11 dB (RSS) or 12 dB (MRC) for 2 antennas and 14 dB (RSSI) or 17 dB (MRC) for 3 antennas.

c) If we perform a) and b) for 10% outage and average BER of 10^{-2} (dashed horizontal lines in the figures), respectively, we see that the corresponding diversity gains are smaller than in a) and b). Since higher diversity orders means a steeper slope on the error curves, the gap (diversity gains) become larger for higher SNRs or, equivalently, for lower outages and lower average BERs.

13.3 Fading margin

We have independent fading with an exponential exponential distribution

$$f(\gamma) = \frac{1}{\Gamma} e^{-\frac{\gamma}{\Gamma}}, \gamma \geq 0, \quad (1)$$

on the received SNRs, γ, and the average SNR, Γ, is the same for both antennas. The specified outage probability is P_{out}.

a) When one antenna is used, the outage probability for a fading margin M becomes (the outage threshold is M times lower than the average SNR, according to the definition of fading margin)

$$P_{\mathrm{out},1}(M) = \int_0^{\Gamma/\mathrm{M}} f(\gamma) d\gamma = \frac{1}{\Gamma} \int_0^{\Gamma/\mathrm{M}} e^{-\frac{\gamma}{\Gamma}} d\gamma = 1 - e^{-1/M} \quad (2)$$

b) Since the fading is independent, and RSSI selection is used, the probability that the received signal is below the outage threshold is

$$P_{\mathrm{out},2}(M) = P^2_{\mathrm{out},1}(M) = \left(1 - e^{-1/M}\right)^2 \quad (3)$$

c) For an outage probability of 1% in both cases, $P_{\mathrm{out},1} = P_{\mathrm{out},2} = 1\%$ the required fading margins become

$$M_1 = \frac{-1}{\ln\left(1 - P_{\mathrm{out},1}\right)} = 99.5 = 20 \text{ dB} \quad (4)$$

and

$$M_2 = \frac{-1}{\ln\left(1 - \sqrt{P_{\mathrm{out},2}}\right)} = 9.49 = 9.8 \text{ dB}. \quad (5)$$

The diversity gain (how much less fading margin we need) when using two antennas instead of one is $M_1 - M_2 = 20 - 9.8 = 10.2$ dB.

13.6 The Alamouti scheme

a) The output at the receiver side in the first interval is

$$r_1 = h_1 s_1 + h_2 s_2 + n_1 \tag{1}$$

and in the second interval

$$r_2 = h_1 s_2^* - h_2 s_1^* + n_2. \tag{2}$$

b) The proposed operations on r_1 and r_2 gives

$$\hat{s}_1 = h_1^* r_1 - h_2 r_2^* = \left(|h_1|^2 + |h_2|^2\right) s_1 + h_1^* n_1 - h_2 n_2^* \tag{3}$$

and

$$\hat{s}_2 = h_2^* r_1 + h_1 r_2^* = \left(|h_1|^2 + |h_2|^2\right) s_2 + h_2^* n_1 + h_1 n_2^*, \tag{4}$$

which means that we have achieved **diversity order TWO** (by adding the channel powers from both h_1 and h_2 in the expressions) without the need to know the channel at the transmitter side and using only a single receive antenna.

[13.11] Maximal SNR antenna combining

We know that received power is proportional to signal amplitude squared, but we do not know the proportionality constant, which depends on the impedance. Let us therefore just assume that the proportionality constant is A (positive and non-zero) and that the signal power on the kth antenna is $A|s_k|^2$. With noise power N_0 (this is not the noise power spectral density) on all antennas, the **SNR on antenna** k becomes $A|s_k|^2/N_0$.

Adding the received signals, with weight α_k^* on the signal from the kth antenna[1], we get a total signal amplitude

$$s = \sum_{k=1}^{N_r} \alpha_k^* s_k, \tag{1}$$

which corresponds to an equivalent signal power

$$P = A \left| \sum_{k=1}^{N_r} \alpha_k^* s_k \right|^2, \tag{2}$$

where the proportionality constant A has the same meaning as before. Since the noise from the different antennas is independent and weighted by the same coefficients, α_k^*, we get a total noise power

$$N = N_0 \sum_{k=1}^{N_r} |\alpha_k|^2 \tag{3}$$

and an **SNR of the combined** signal as

$$\frac{P}{N} = \frac{A \left| \sum_{k=1}^{N_r} \alpha_k^* s_k \right|^2}{N_0 \sum_{k=1}^{n_r} |\alpha_k|^2} \tag{4}$$

1 It may seem strange to use the complex conjugate α_k^* of our coefficients as weights, but it does not change the result and will simplify notation in the coming calculations.

To find the set of coefficients that maximize the combined SNR we can, without loss of generality, assume that their square sum is one and that the other constants (A and N_0) are absorbed into a single constant, which we also assume to be one. The SNR-maximizing coefficients α_k are then given by

$$[\alpha_1, \ldots, \alpha_{N_r}] = \arg \max_{\sum |\alpha_k|^2 = 1} \left| \sum_{k=1}^{N_r} \alpha_k^* s_k \right|^2 . \tag{5}$$

This is a classical maximization problem that can be solved by, *e.g.*, applying the method of Lagrange multipliers. To do that, we formulate the Lagrangian

$$J(\alpha_1, \ldots, \alpha_{N_r}, \lambda) = \left| \sum_{k=1}^{N_r} \alpha_k^* s_k \right|^2 + \lambda \left(1 - \sum_{k=1}^{N_r} |\alpha_k|^2 \right). \tag{6}$$

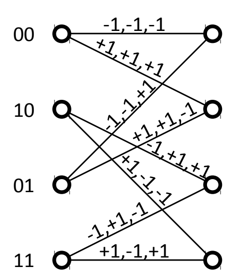
After performing the optimization, we obtain the optimal SNR as

$$\left(\frac{P}{N}\right)_{\mathrm{opt}} = \frac{A\left(\sum_{k=1}^{N_r} |s_k|^2\right)^2}{N_0 \sum_{k=1}^{N_r} |s_k|^2} = \frac{A \sum_{k=1}^{N_r} |s_k|^2}{N_0} = \sum_{k=1}^{N_r} \frac{A|s_k|^2}{N_0} \tag{7}$$

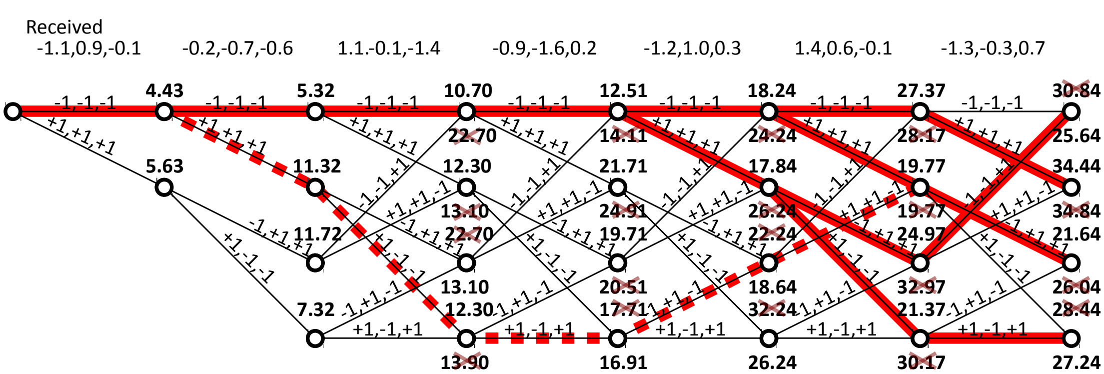
where we recognize the terms $A|s_k|^2/N_0$ in the last sum as the SNRs we calculated for the individual antennas (see above).

14.10 Soft decoding of convolutional code

a) Using -1 and +1 instead of 0 and 1 on trellis labels gives the following trellis stage:



b) When performing soft decoding, we get the following trellis, metrics, and survivors (red):



The survivors are not the same as for the hard decoding presented in Fig. 14.5 in the textbook. The dashed path represents a path equally good as one of the other paths. In the second to last trellis stage, two incoming metrics had the same value ($d^2 = 19.77$) and one was discarded by the toss of a fair coin.

[14.11] Diversity and hard decoding

Diversity is seen in the slope of the BER curve as SNR increases. If we rewrite the given BER expression as

$$\overline{\mathrm{BER}} = \sum_{i=t+1}^{N} K_i \left(\frac{1}{\bar{\gamma}_B}\right)^i \left(\frac{1}{2/\bar{\gamma}_B + 2}\right)^i \left(1 - \frac{1}{2+2\bar{\gamma}_B}\right)^{N-i} \tag{1}$$

we see that when the SNR $\bar{\gamma}_B$ increases

$$\left(\frac{1}{2/\bar{\gamma}_B + 2}\right) \to \frac{1}{2} \tag{2}$$

and

$$\left(1 - \frac{1}{2+2\bar{\gamma}_B}\right) \to 1. \tag{3}$$

The behaviour of the BER is therefore dominated by the lowest power of $(1/\bar{\gamma}_B)$. This term has the power $t+1$, which gives us the diversity order. Now we only need to combine this with the fact that the error correcting capability of a block code with minimum distance $d_{\min}$ is calculated as

$$t = \left\lfloor \frac{d_{\min} - 1}{2} \right\rfloor. \tag{4}$$

This gives us the final result that a code with minimum distance $d_{\min}$ achieves **diversity order**

$$\left\lfloor \frac{d_{\min} - 1}{2} \right\rfloor + 1. \tag{5}$$

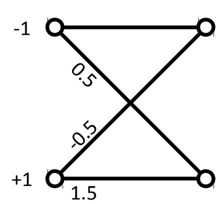
16.6 Channel equalization

a) If the channel is equalized with a ZF equalizer, the inverse of the channel is used as a filter, namely

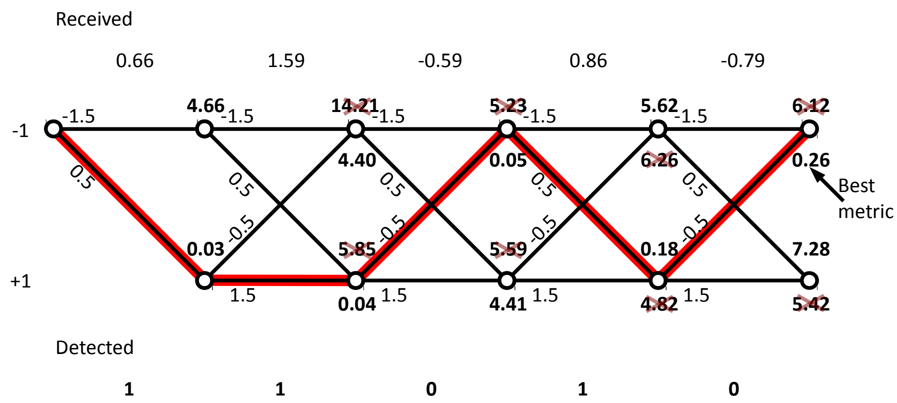
$$G(z) = \frac{1}{1 + 0.5z^{-1}}. \tag{1}$$

b) The memory of the channels is 1, since there is only one delay element.

c) A trellis stage for the given channel is



d) Decoding of the five stages gives this trellis, with metrics and survivor(s):



The maximum-likelihood detected sequence becomes **1 1 0 1 0**.

Extra Problems – Batch 2

1. Encoding & decoding

(a) With 64 kbit/sec data rate and $R = 1/2$ coding, the coded bit rate becomes 128 kbit/sec.

(b) Each 4QAM symbol carries two bits, which gives a symbol rate of 64 ksymb/sec.

(c) Rectangular basis pulses and 99% of the energy gives (Table 11.1 in textbook 4QAM same as 4PSK) a bandwidth $B = 128/0.1 = 1280$ kHz.

(d) If the channel is memoryless, the only memory in the system is the one in the convolutional code, which is of depth 3. With memory 3, we need $2^3 = 8$ states in the Viterbi decoder.

(e) Fully compensating for the doubling of the number of bits by the code would require a signal constellation carrying twice as many bits/symbol as the 4QAM, namely 4 bits/symbol. This indicates the use of 16QAM.

2. The Viterbi algorithm

(a) For error correcting codes it is the *memory of the encoder* and the *generator sequences*. For channel equalization it is the *impulse response* of the channel (which implicitly defines the memory).

(b) The VA finds the most likely transmitted path/sequence, given the received (noisy) signal.

3. Coding gain in Rayleigh fading channels

(a) Using $C/N_0 = E_b/N_0 + d_b$, in combination with a required average SNR $E_b/N_0 \approx 13.5$ dB (using Slide 37, Lecture 6, or knowing that binary antipodal signalling has the same BER characteristic as 4QAM and using eq. (12.52)), we get an average required

$$C/N_0 = 13.5 + 50 = 63.5 \text{ dB}.$$

(b) The 6 dB coding gain directly translates to a 6 dB reduction in the C/N_0 requirement, i.e.,

$$C/N_0 = 63.5 - 6 = 57.5 \text{ dB}.$$

(c) A rate $R = 1/2$ code expands the transmission bandwidth by a factor two, hence the bandwidth in (b) is twice that of in (a).

4. Mobile radio link in Rayleigh fading

(a) Solved in Extra Problems Batch 1, but for a BER requirement of 10^{-3}. Following the same strategy, but for a 10^{-4} requirement, gives $\text{SNR} = E_b/N_0 \approx 36$ dB and, at $d_b = 10$ kbit/sec = 40 dB,

$$C = 36 + 40 + (-199) = -123 \text{ dBW or } -93 \text{ dBm.}$$

(b) Solved in Extra Problems Batch 1, but for a BER requirement of 10^{-3}. Following the same strategy, but for a 10^{-4} requirement, gives $\text{SNR} = E_b/N_0 \approx 19$ dB with two antennas and a diversity gain of $36 - 19 = 17$ dB. The corresponding requirement on received power is reduced by the same amount to

$$C = -123 - 17 = -140 \text{ dBW or } -110 \text{ dBm.}$$

(c) To determine the gain by using the Golay code, we need to find the channel bit error rate p that results in a 10^{-4} bit-error probability after (hard) decoding. Given the parameters of the Golay code, the requirement becomes (see Lecture 7, slide 20):

$$10^{-4} \approx \frac{7}{23} \sum_{m=4}^{23} \binom{23}{m} p^m (1-p)^{23-m}.$$

This is a sum with 20 terms in it and finding p directly would be quite difficult. Using the fact that the total probability (of all terms in th sum from 0 to 23) is 1, we can use

$$10^{-4} \approx \frac{7}{23} \left(1 - \sum_{m=0}^{3} \binom{23}{m} p^m (1-p)^{23-m} \right).$$

This expression "only" contains four terms in the sum and it is quite possible to search for a reasonable value on p using a pocket calculator (starting at a guess, e.g., $p = 0.01$ and refining the result from there). Using this strategy, a reasonable value would be $p \approx 0.015 = 1.5\%$.

Another, less labor intensive but also less accurate, approach would be to exploit the fact that the BER expression is a polynomial with lowest degree 4. With a small p, the entire expression will be dominated by that lowest-degree term (which appears when $m = 4$ in the first sum) and the approximation becomes

$$10^{-4} \approx \frac{7}{23} \binom{23}{4} p^4 = \frac{7}{23} \frac{23 \times 22 \times 21 \times 20}{4!} p^4.$$

This yields

$$10^{-4} \approx \frac{7}{23} 8855 p^4 \rightarrow p \approx 0.014 = 1.4\%.$$

To obtain this uncoded BER over a Rayleigh fading channel with 2ASK, we need $\bar{E}_c/N_0 \approx 12$dB, where $\bar{E}_c$ is the average energy per code bit on the channel. Since there are 23 code bits per 11 information bits in the code, the corresponding

$$\bar{E}_b/N_0 = \bar{E}_c/N_0 + 10 \log(23/11) \approx 12 + 3.2 = 15.2 \text{ dB}.$$

Comparing this to the 36 dB's required without coding (in a), the gain is about 21 dB. This also translates directly to a 21 dB reduction in required average received power. Hence, the required average received power becomes $C = -123 - 21 = -144$ dB or -114 dBm.

(d) Using the Golay code, our point-of-operation in terms of bit-error performance on the channel is around 1.4%. At this level the required $\bar{E}_c/N_0$ is only about 8 dB (see Fig 13.11 in textbook), with two-antenna RSSI selection diversity included. The corresponding $\bar{E}_b/N_0 \approx 11$ dB (correcting for the 11/23 code rate as in (c)). This means that the combined coding and antenna diversity gain (compared to a) is $36 - 11 = 25$ dB, which gives a required average received power of $C = -123 - 25 = -148$ dBW or -118 dBm.