Digital Signal Processing
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Digital Signal Processing, examples of applications

Audio processing
Image and Video processing
ECG, Ultrasound
Compressing images (JPG), sound (MP3) or video (H.264)
Sound generation (synthesizers)
Measurements
Process Control
Reducing noise and vibrations

GSM, 3G, 4G, LTE
ADSL (OFDM)
Data networks

The processing can be performed in software in a PC (CPU)
dedicated Signal processors
graphics processor
specialized equipment like CD or DVD players or modems
embedded systems, in for example cars

In many cases real-time is demanded

Example of analog and digital circuits

Analog RC-circuit

\[ y'(t) + ay(t) = b x(t) \]

Digital circuit

\[ y(n) = -ay(n-1) + bx(n) \]

Example of program code executed when a new value \( x \) is available at the A/D input \( (a=0.9, b=1) \)

\[ x = \text{ADinput}; \]
\[ y = 0.9 \times y\text{old} + x; \]
\[ y\text{old} = y; \]
\[ \text{DAoutput} = y; \]

Example: MP3 coding of music

Figure 9.12 Simplified block diagram of a ISO-MPEG1 coder.

Figure 9.13 Simplified block diagram of a ISO-MPEG1 decoder.

(decoding is easier!)
Example of reverberation (echoes)

Example of echo suppression

What is a time discrete signal?

Example of time discrete signals

Temperature

Example of Digital Signal Processing

Example of speech and time-frequency analyses, spectrogram

Goal of the course:
To understand the properties of these, especially the relation between the properties in the time domain and the properties in the frequency domain.
Example of perceptual video quality
(sharper but more blocks)

(less blocking, but more blurred)

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Sampling  page 21

\[ x(t) = 20 \cos \left( 2 \pi \frac{440}{F_0} t - 0.4 \pi \right) \]

Sample the signal using the rate

\[ F_T = 1000 \text{ Hz} \]

or \[ T = \frac{1}{F_T} = \frac{1}{1000} = 0.001 \text{ s} \]

between the samples

\[ x(n) = x(t) \big|_{t=nT} = 20 \cos \left( 2 \pi \frac{440}{1000} n - 0.4 \pi \right) \]

\[ f_0 = \frac{440}{1000} = 0.44 \]

Notations:

\[ \Omega = 2 \pi F \quad \text{frequency in Hertz or radians, respectively, for time continuous signals.} \]

\[ \omega = 2 \pi f \quad \text{frequency in Hertz or radians, respectively, for time discrete signals.} \]

Time discrete sinusoid page 23

\[ x(n) = \cos \left( 2 \pi f_0 n \right) = \frac{1}{2} \left( e^{j 2 \pi f_0 n} + e^{-j 2 \pi f_0 n} \right) = \cos \left( 2 \pi f_0 n \right) + \cos \left( 2 \pi f_0 (n+1) \right) = \frac{1}{2} \left( e^{j 2 \pi (f_0+1) n} + e^{-j 2 \pi (f_0+1) n} \right) \]

\[ \text{n integer, } f_0 = 1/8 = 0.125 \quad (f_0 < 0.5 \text{ gives at least 2 sample/period}) \]

How to plot the frequency contents?

Listen to the signal by playing the signal using a D/A converter.

We select the period \[-0.5 < f < 0.5\]

and play using \[ F_s = 10000 \text{ Hz} \]

\[-5000 < F < 5000 \quad \text{(real frequencies)} \]

\[ y(n) = \cos \left( 2 \pi \frac{1}{8} 10000 n \right) = \cos \left( 2 \pi \ 1250 \ n \right) \]

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Some math’s used in the course

Complex numbers:

\[ z = a + j b = r e^{j \phi} \quad \text{with} \quad r = \sqrt{a^2 + b^2}, \]

\[ \Phi = \arctan(b/a) \quad \text{if} \quad a > 0 \]

\[ re^{j \phi} = r \cos \Phi + j r \sin \Phi \]

Euler formula:

\[ \cos \omega = \frac{1}{2} (e^{j \omega} + e^{-j \omega}) \]

\[ \sin \omega = \frac{1}{2j} (e^{j \omega} - e^{-j \omega}) \]

Rewriting using Euler formula:

\[ 1 + e^{-j \omega} = e^{-j \omega/2} (e^{j \omega/2} + e^{-j \omega/2}) = 2 \cos \left( \frac{\omega}{2} \right) e^{-j \omega/2} \]

\[ 1 - e^{-j \omega} = e^{-j \omega/2} (e^{j \omega/2} - e^{-j \omega/2}) = 2 \sin \left( \frac{\omega}{2} \right) e^{-j \omega/2} e^{j \pi/2} \]

Integral:

\[ \int_{t=0}^{T} e^{-j 2 \pi f t} dt = \int_{j 2 \pi f} \frac{e^{-j 2 \pi f T} - e^{-j 2 \pi f 0}}{j 2 \pi f} \]

\[ = \left( \frac{e^{-j 2 \pi f T}}{j 2 \pi f} - e^{-j 2 \pi f 0} \right) = \frac{\sin(2 \pi f T)}{2 \pi f} \]

\[ e^{-j \pi f T} \int e^{-2 j \pi f} = \frac{\sin(2 \pi f T)}{2 \pi f} e^{-j \pi f T} \]
The geometrical sum:
\[
S_1 = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \frac{1}{1 - \frac{1}{2}} = 2
\]

infinite sum

\[
S_2 = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n+1} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = \frac{1}{1 - \frac{1}{2}} = 2
\]

finite sum

Derivation of the finite geometrical sum:

\[
\text{Sum} = \sum_{n=0}^{N} a^n = 1 + a + a^2 + \ldots + a^N
\]

Multiply with \(a\):

\[
a \cdot \text{Sum} = a + a^2 + a^3 + \ldots + a^{N+1}
\]

Form the difference:

\[
\text{Sum} - a \cdot \text{Sum} = 1 - a^{N+1}
\]

This gives the finite sum:

\[
\text{Sum} = \frac{1 - a^{N+1}}{1 - a}
\]

The infinite sum

\[
\text{Sum} = \sum_{n=0}^{\infty} a^n = 1 + a + a^2 + \ldots = \frac{1}{1 - a} \quad \text{if} \quad |a| < 1
\]

Appendix Chapter 1

Sinusoids (notations)

\[
x(t) = 10 \cos(2\pi \frac{440}{F_0} t - 0.4\pi)
\]

Amplitude: \(A\)

Frequency: \(F_0\)

Phase: \(\Phi\)

Period: \(T_0 = \frac{1}{F_0}\)

\(\Omega = 2\pi F\)

Trigonometric relations:

\[
\cos \Omega t = \frac{e^{i\Omega t} + e^{-i\Omega t}}{2}
\]

Euler formula:

\[
\sin \Omega t = \frac{e^{i\Omega t} - e^{-i\Omega t}}{2j}
\]

Appendix Chapter 1 continued

Example of synthesized sounds: Example 2

(upper: waveform, lower: frequency contents)

Sinusoid

\[
x(t) = \sin(2\pi \frac{F_1}{220} t)
\]

Additive synthesis (sum of sinusoids)

\[
x(t) = \sum_{k} a_k \sin(2\pi k \frac{F_0}{220} t)
\]

Example of synthesized sounds: Example 3

(upper: waveform, lower: frequency contents)

AM-synthesis

\[
x(t) = (1 + 0.8 \sin(2\pi \frac{F_0}{220} t)) \sin(2\pi \frac{3F_0}{660} t)
\]

FM-synthesis (Yamaha)

\[
x(t) = \sin \{2\pi \frac{F_0}{220} t + 3 \sin(2\pi \frac{F_0}{220} t)\}
\]
Periodical signals, signals based on harmonic sinusoids

Signals which are periodical, i.e. the same waveform is repeated with the period \( T \) can be written as a sum of harmonically sinusoids with frequencies

\[ F_0, \ 2F_0, \ 3F_0, \ 4F_0, \ \ldots \] and \( F_0 = \frac{1}{T} \) is called the fundamental frequency.

The signal can be written

\[ x(t) = A_1 \sin(2\pi F_0 t + \phi_1) + A_2 \sin(2\pi 2F_0 t + \phi_2) + \ldots \]

Example

![Waveform and Frequency Contents](image)

Energy, power page 45 (44)

Energy

\[ E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \]

Power:

\[ P = \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 \]

if \( 0 < E < \infty \), \( x(n) \) is called "energy signal"

if \( 0 < P < \infty \), \( x(n) \) is called "power signal"

Even, odd signals

\[ x(n) = x(-n) \]

\[ x(n) = -x(-n) \]

Folding

Folding \( x(n) \) (reflection) around origin gives \( y(n) = x(-n) \)

Definition of Discrete-Time Systems (LTI systems)

FIR, IIR

FIR: Circuit with finite length memory

\[ y(n) = x(n) + x(n-1) \]

IIR: Circuit with infinite length memory

\[ y(n) = 0.5 \ y(n-1) + x(n) \]

Linearity

if \( x(n) = \alpha x_1(n) + \beta x_2(n) \)

gives

\[ y(n) = \alpha y_1(n) + \beta y_2(n) \]

Shift invariance

if \( x(n) \Rightarrow y(n) \)

gives

\[ y(n-m) \Rightarrow y(n-m), \ \text{any } m \]

BIBO-stability

Bounded input \( \Rightarrow \) bounded output

if \[ |x(n)| \leq M_x \]

then

\[ |y(n)| \leq M_y < \infty \]
Chapter 2 Discrete-Time Signals

Page 43 (40)

Notations: \( x(n) \) (some textbooks use the notation \( x[n] \))

Impulse:
\[
\delta(n) = \begin{cases} 
1 & n = 0 \\
0 & \text{elsewhere}
\end{cases} = \{ \ldots 0 0 0 1 1 0 0 0 \ldots \} \quad (n=0)
\]

Step:
\[
u(n) = \begin{cases} 
1 & n \geq 0 \\
0 & n < 0
\end{cases} = \{ \ldots 0 0 0 1 1 1 1 1 \ldots \}
\]

\[x(n) = a^n u(n)\]
\[x(n) = \cos(2\pi f_n n) = \frac{1}{2}(e^{j2\pi f_n n} + e^{-j2\pi f_n n})\]

Definition: Causal signal = signal which is 0 for negative index

Using the impulse, we can write
\[x(n) = \{1 4 1\} = \delta(n) + 4 \delta(n-1) + \delta(n-2) = \sum_{k} x(k) \delta(n-k)\]

Chapter 2 Convolution pp 71-80 (68-)

The most important relation between input signal and output signal is called convolution.

If we know the impulse response \( h(n) \) we can determine the output for any other input signal. The only properties needed are linearity and time invariance (LTI).

Input signal \( x(n) \)  
Output signal \( y(n) \)

Definition

Impulse \( \delta(n) \rightarrow \) impulse response \( h(n) \)

\[\delta(n-k) \rightarrow h(n-k)\]
\[x(k) \delta(n-k) \rightarrow x(k)h(n-k)\]

\[\sum_{k} x(k) \delta(n-k) \rightarrow \sum_{k} x(k)h(n-k)\]

\[y(n) = \sum_{k} x(k)h(n-k) = \sum_{k} h(k)x(n-k) = h(n) \ast x(n)\]

This is called convolution and is the most general relation given in this course.

Example of convolution

Given: Input signal \( x(n) \) och impulse response \( h(n) \)
\[x(n) = \ldots 0 0 2 4 6 4 2 0 \ldots \quad \text{and} \quad h(n) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\]

Task: Determine the output signal (convolution)
\[y(n) = \sum_{k} h(n-k)x(k) = \sum_{k} h(k)x(n-k) = h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) = 3x(n) + 2x(n-1) + x(n-2)\]

Solution: We solve it 'graphically' by the method below.

\(h(n)\) backwards:
\[h(n-2) = \begin{bmatrix} 1 & 2 & \frac{1}{2} \end{bmatrix} \quad x(k) = \begin{bmatrix} \ldots 0 & 0 & 2 & 4 & 6 & 4 & 2 & 0 \end{bmatrix}\]

gives the result \(y(n) = \begin{bmatrix} 0 & 0 & 6 & 16 & 28 & 28 & 20 & 8 & 2 \end{bmatrix}\)

Multiply and add. Then shift the impulse response to the right one step and repeat.
Properties of the convolution (normal properties), page 81 (79)

\[ x_1(n) * x_2(n) = x_1(n) * x_2(n) \]

\( (x_1(n) * x_2(n)) * x_3(n) = x_1(n) * (x_2(n) * x_3(n)) \)

Input - output relations

\[ x(n) \rightarrow h(n) \rightarrow y(n) = x(n) * h(n) \]

Cascade circuits (serial)

\[ x(n) \rightarrow h_1(n) \rightarrow h_2(n) \rightarrow y(n) = (x(n) * h_1(n)) * h_2(n) \]

Parallel circuits

\[ x(n) \rightarrow h_1(n) \rightarrow h_2(n) \rightarrow y(n) = h_1(n) + h_2(n) \]

Difference equation pp 93-95 (88-)

General difference equation:

\[ y(n) + \sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{N} b_k x(n-k) \]

Example: FIR filter

\[ y(n) = 0.5 \, x(n) + 0.25 \, x(n-1) + 0.25 \, x(n-2) \]

gives the impulse response directly

\[ h(n) = [0.5 \, 0.25 \, 0.25] \]

Example: First order IIR filter (recursive filter)

\[ y(n) = 0.5 \, y(n-1) + 2 \, x(n) \]

Example: Second order IIR filter

\[ y(n) = 0.5 \, y(n-1) + 0.5 \, y(n-2) + x(n) \]

We have to solve the difference equation to get the impulse response.

We solve the first order difference equation page 94 (92)

\[ y(n) = -a_1 \, y(n-1) + b_0 \, x(n) \]

Solve iteratively for \( n \geq 0 \)

\[ y(0) = -a_1 \, y(-1) + b_0 \, x(0) \]

\[ y(1) = -a_1 \, y(0) + b_0 \, x(1) = (-a_1)^2 \, y(-1) + b_0 \, x(1) + b_0 \, (-a_1) \, x(0) \]

\[ y(2) = -a_1 \, y(1) + b_0 \, x(2) = (-a_1)^3 \, y(-1) + b_0 \, x(2) + b_0 \, (-a_1) \, x(1) \]

\[ + b_0 \, (-a_1)^2 \, x(0) \]

\[ y(n) = \sum_{k=0}^{n} b_0 \, (-a_1)^k \, x(n-k) \]

Impulse response

If \( x(n) = \delta(n) \), then we have

\[ y(n) = h(n) = b_0 \, (-a_1)^n \quad \text{for} \ n \geq 0 \]

For a general input signal we then determine the output using the convolution

\[ y(n) = \sum_{k} h(k) \, x(n-k) \]

For higher order systems, we will use the Z transform (chapter 3).
Correlation functions deterministic page 118 (117).

We end the chapter 2 by defining correlation functions.

Autocorrelation function
\[ r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n) x(n-l) \]

Cross correlation function
\[ r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n) x(n-l) \]

Can be written using these convolution notations.
\[ r_{xx}(l) = x(l) * x(-l) \]
\[ r_{yx}(l) = y(l) * x(-l) \]

Input – Output Relations using correlation functions

Autocorrelation function for the input
\[ r_{xx}(l) = x(l) * x(-l) \]

Autocorrelation function for the output
\[ r_{yy}(l) = y(l) * y(-l) = h(l) * x(l) * h(-l) * x(-l) = r_{hh}(l) * r_{xx}(l), \]
where \( r_{hh}(l) = h(l) * h(-l) \)

Cross correlation function for input-output signal
\[ r_{yx}(l) = y(l) * x(-l) = h(l) * x(l) * x(-l) = h(l) * r_{xx}(l) \]

We can use this to estimate an unknown system \( h(n) \) by choosing an input \( x(n) \) with the autocorrelation function \( r_{xx}(l) = \delta(l) \)

Then we directly has
\[ h(l) = r_{yx}(l) \]

Example of correlation functions for the delay in GSM systems.

Definition of stability page 85 (84).

A circuit is BIBO-stable (bounded input-bounded output) if
\[ x(n) \leq M \quad \text{yields} \quad |y(n)| \leq M \]

In the impulse response, we have the demand of stability
\[ |y(n)| = | \sum_{k=-\infty}^{\infty} h(n-k)x(n-k)| \leq M \sum_{k=-\infty}^{\infty} |h(k)||x(n-k)| \leq M \sum_{k=-\infty}^{\infty} |h(k)| \]

i.e. stable if
\[ \sum_{k=-\infty}^{\infty} |h(k)| \leq \infty \]