Electromagnetic Wave Propagation
Lecture 7: Pulse propagation in dispersive media

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1. Propagation of narrow-band pulses
2. Group velocity as velocity of peak: linear approximation
3. Pulse broadening: quadratic approximation
4. Slow, fast, and negative group velocities
5. Application: chirp radar and pulse compression
6. Conclusions
Outline

1 Propagation of narrow-band pulses

2 Group velocity as velocity of peak: linear approximation

3 Pulse broadening: quadratic approximation

4 Slow, fast, and negative group velocities

5 Application: chirp radar and pulse compression

6 Conclusions
Key questions

- How does a narrow-band pulse propagate?
- How can pulse distortion be characterized?
- How can pulse distortion be compensated?

Lots of text in Orfanidis, especially when discussing applications. We concentrate on the fundamental ideas rather than the details.
The time domain pulse $E(z, t)$ is decomposed in Fourier components, $\hat{E}(\omega)e^{j(\omega t - k(\omega)z)}$.

Each component is propagated according to its wavenumber $k(\omega)$.

The resulting pulse is synthesized as

$$E(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{E}(\omega)e^{j(\omega t - k(\omega)z)} \, d\omega$$

To simplify the computation of the Fourier integral, various approximations of $k(\omega)$ and $\hat{E}(\omega)$ are introduced.
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2. **Group velocity as velocity of peak: linear approximation**
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A pulse with carrier frequency $\omega_0$ and envelope $F(0, t)$ can be written $E(0, t) = e^{j\omega_0 t} F(0, t)$, corresponding to the frequency shift $\hat{E}(0, \omega) = \hat{F}(0, \omega - \omega_0)$.

This can be seen as the translation of a low-frequency spectrum $\hat{F}(0, \omega)$ to be centered around $\omega_0$, $\hat{F}(0, \omega - \omega_0)$. $F$ (or $\hat{F}$) is called the envelope of the carrier signal $\omega_0$. 

(Fig. 3.5.1 in Orfanidis)
Propagation in the frequency domain corresponds to
\[ \hat{E}(z, \omega) = e^{-jk(\omega)z} \hat{E}(0, \omega), \] which implies (where \( k_0 = k(\omega_0) \))

\[
E(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\omega t - k(\omega)z)} \hat{F}(0, \omega - \omega_0) \, d\omega_0
\]

\[ = e^{i(\omega_0 t - k_0 z)} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\omega - \omega_0) t - j(k(\omega) - k_0)z} \hat{F}(0, \omega - \omega_0) \, d\omega
\]

\[ = e^{i(\omega_0 t - k_0 z)} F(z, t) \]

with

\[ F(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\omega - \omega_0) t - j(k(\omega) - k_0)z} \hat{F}(0, \omega - \omega_0) \, d\omega \]

The factor \( e^{i(\omega_0 t - k_0 z)} \) is the carrier wave, propagating with the
\textit{phase velocity} \( v_p = \omega_0/k_0 \).
Taylor expansion of the wavenumber

Making the Taylor expansion

\[ k(\omega) = k(\omega_0) + k'(\omega_0)(\omega - \omega_0) + \cdots \]

implies (with \( \omega' = \omega - \omega_0 \), \( k_0 = k(\omega_0) \), \( k'_0 = k'(\omega_0) \))

\[
F(z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j((\omega-\omega_0)t-(k(\omega)-k_0)z)} \hat{F}(0,\omega-\omega_0) \, d\omega \\
\approx \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega'(t-k'_0z)} \hat{F}(0,\omega') \, d\omega' = F(0,t-k'_0z)
\]

Thus, the envelope \( F \) is propagating with the group velocity

\[
v_g = \frac{1}{k'(\omega_0)}
\]

If \( k(\omega) = \beta(\omega) - j\alpha(\omega) \) is complex, replace by \( v_g = 1/\beta'(\omega_0) \).
The impulse response can be written

\[ h(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\omega t - k(\omega)z)} \, d\omega = e^{i(\omega_0 t - k_0 z)} g(z, t) \]

where

\[ g(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\omega - \omega_0) t - j(k - k_0) z} \, d\omega \]

We can define transfer functions for the envelope \( F \) as

\[ \hat{F}(0, \omega) \xrightarrow{F(0, t)} G(z, \omega) \xrightarrow{g(z, t)^*} \hat{F}(z, \omega) \]

with \( G(z, \omega') = e^{-j(k(\omega' + \omega_0) - k(\omega_0))z} \).
Making the approximation

\[ k(\omega) = k_0 + k'_0 \omega' \]

implies the envelope impulse response is

\[
g(z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j(\omega - \omega_0)t - j(k-k_0)z} \, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega'(t - k'_0z)} \, d\omega' = \delta(t - k'_0z)
\]

Thus, in the linear approximation the envelope propagates undistorted with the group velocity.
Simulation illustration
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Quadratic expansion

Keeping one more term in the Taylor expansion

\[ k(\omega) = k(\omega_0) + k'(\omega_0)(\omega - \omega_0) + k''(\omega_0)(\omega - \omega_0)^2/2 + \cdots \]

implies (with \( \omega' = \omega - \omega_0 \), \( k_0 = k(\omega_0) \), \( k'_0 = k'(\omega_0) \), \( k''_0 = k''(\omega_0) \))

\[ F(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i[(\omega-\omega_0)t-(k(\omega)-k_0)z]} \hat{F}(0, \omega - \omega_0) \, d\omega \]

\[ \approx \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega'[t-(k'_0+k''_0\omega'/2)z]} \hat{F}(0, \omega') \, d\omega' \]

The quadratic term leads to pulse spreading. Frequency dependent velocity:

\[ v(\omega') = \frac{1}{k'_0 + k''_0\omega'/2} \approx \frac{1}{k'_0} - \frac{k''_0\omega'}{2k_0} \]
Assume a Gaussian envelope,

\[ F(0, t) = e^{-t^2/(2\tau_0^2)} \quad \Leftrightarrow \quad \hat{F}(0, \omega) = \sqrt{2\pi\tau_0^2} e^{-\tau_0^2\omega^2/2} \]
Assume a Gaussian envelope,

\[ F(0, t) = e^{-t^2/(2\tau_0^2)} \iff \hat{F}(0, \omega) = \sqrt{2\pi\tau_0^2} e^{-\tau_0^2 \omega^2/2} \]

The propagated envelope is then

\[ F(z, t) = \frac{\sqrt{2\pi\tau_0^2}}{2\pi} \int_{-\infty}^{\infty} e^{j\omega'(t-(k'_0+k''_0\omega'/2)z)} e^{-\tau_0^2 (\omega')^2/2} \, d\omega' \]

\[ = \cdots = \sqrt{\frac{\tau_0^2}{\tau_0^2 + jk''_0 z}} \exp \left[ -\frac{(t - k'_0 z)^2}{2(\tau_0^2 + jk''_0 z)} \right] \]
Assume a Gaussian envelope,

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The propagated envelope is then

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\[ = \cdots = \sqrt{\frac{\tau_0^2}{\tau_0^2 + jk''_0z}} \exp\left[-\frac{(t - k'_0 z)^2}{2(\tau_0^2 + jk''_0z)}\right] \]

Since \( \exp[-\frac{1}{a+jb}] = \exp[-\frac{a}{a^2+b^2} + j\frac{b}{a^2+b^2}] \), the width of \( |F(z, t)| \) in time is proportional to

\[ \tau(z) = \left( \frac{\tau_0^4 + (k''_0 z)^2}{\tau_0^2} \right)^{1/2} = \left[ \tau_0^2 + \left( \frac{k''_0 z}{\tau_0} \right)^2 \right]^{1/2} \]
Making the approximation

\[ k(\omega) = k_0 + k'_0 \omega' + k''_0 (\omega')^2 / 2 \]

implies the envelope impulse response is

\[ g(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\omega - \omega_0)t - j(k - k_0)z} \, d\omega \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega'(t-k'_0 z) - jk''_0 z (\omega')^2 / 2} \, d\omega' \]

This has the exact solution

\[ g(z, t) = \frac{1}{\sqrt{2\pi jk''_0 z}} \exp \left[ - \frac{(t - k'_0 z)^2}{2jk''_0 z} \right] \rightarrow \delta(t - k'_0 z), \quad k''_0 \rightarrow 0 \]

Thus, the delta pulse has width \( \sim \sqrt{k''_0 z} \) in time after propagating a distance \( z \).
Simulation illustration

Pulse dispersion demo

Pulse propagation in dispersive media

Interval width
3

Number of time steps
100

Number of points
1000

Center angular frequency
50

Group refractive index
2

k0bis
-0.03

Pulse width
0.1

Run!
A Gaussian pulse of initial width $\tau_0$ has extra width after propagation from 0 to $z$:

$$\Delta t = \frac{k''_0 z}{\tau_0}$$
A Gaussian pulse of initial width $\tau_0$ has extra width after propagation from 0 to $z$:

$$\Delta t = \frac{k''_0 z}{\tau_0}$$

A typical value of $k''_0$ for a standard single mode optical fiber at $\lambda = 1.55 \, \mu m$ is

$$k''_0 = -21.67 \, (ps)^2 / km$$

and attenuation about 0.2 dB/km. With a data rate of 40 Gbit/s, the pulses are spaced by 25 ps. How long can a 10 ps pulse propagate before it blends with its neighbors?
Example: pulse broadening in optical fibers

A Gaussian pulse of initial width $\tau_0$ has extra width after propagation from 0 to $z$:

$$\Delta t = \frac{k''_0 z}{\tau_0}$$

A typical value of $k''_0$ for a standard single mode optical fiber at $\lambda = 1.55 \, \mu$m is

$$k''_0 = -21.67 \frac{\text{ps}^2}{\text{km}}$$

and attenuation about 0.2 dB/km. With a data rate of 40 Gbit/s, the pulses are spaced by 25 ps. How long can a 10 ps pulse propagate before it blends with its neighbors?

$$z \leq \frac{\Delta t \tau_0}{k''_0} = \frac{25 \, \text{ps} \cdot 10 \, \text{ps}}{21.67 \, (\text{ps})^2 / \text{km}} = 11.5 \, \text{km}$$

This limits the length of individual fibers, and compensations must be inserted at regular intervals.
Dispersion compensation

The dispersion due to propagation in a dispersive medium can be compensated by a suitable filter at the receiver or transmitter:

\[ F(0,t) \xrightarrow{G(z,\omega)} F(z,t) \xrightarrow{H_{\text{comp}}(\omega)} F_{\text{comp}}(z,t) = F(0,t - t_d) \]

\[ F(0,t) \xrightarrow{H_{\text{comp}}(\omega)} F_{\text{comp}}(0,t) \xrightarrow{G(z,\omega)} F_{\text{comp}}(z,t) = F(0,t - t_d) \]

(Fig. 3.8.1 in Orfanidis)

With *a priori* knowledge of the dispersion, the pulse can be predistorted to compensate for the propagation effects.

Naively we would expect \( H_{\text{comp}} = 1/G \), but this violates causality. With additional delay, \( H_{\text{comp}} = e^{-j\omega t_d}/G \), causality is preserved.
Dispersion compensation

In the frequency domain, the compensation filter is given by

\[ H_{\text{comp}}(z, \omega) = \frac{e^{-j\omega t_d}}{G(z, \omega)} \]
Dispersion compensation

In the frequency domain, the compensation filter is given by

\[ H_{\text{comp}}(z, \omega) = \frac{e^{-j\omega t_d}}{G(z, \omega)} \]

Realize the filter via an additional propagation medium:

\[ k'_0, k''_0 \quad k'_1, k''_1 \]

\[ z \quad z_1 \]

(Fig. top p. 105 in Orfanidis)
Dispersion compensation

In the frequency domain, the compensation filter is given by

\[ H_{\text{comp}}(z, \omega) = \frac{e^{-j\omega t_d}}{G(z, \omega)} \]

Realize the filter via an additional propagation medium:

\[ k'_0, k''_0 \quad \text{and} \quad k'_1, k''_1 \]

\[
G_0(z, \omega)G_1(z_1, \omega) = e^{-jk'_0 z \omega} e^{-jk''_0 z \omega^2/2} e^{-jk'_1 z_1 \omega} e^{-jk''_1 z_1 \omega^2/2} \\
= e^{-j(k'_0 z + k'_1 z_1) \omega} e^{-j(k''_0 z + k''_1 z_1) \omega^2/2} \\
= \text{delay} \quad \text{and} \quad \text{dispersion}
\]

Dispersionless (only delay) if

\[ k''_0 z + k''_1 z_1 = 0 \]
For a narrow-band pulse with carrier frequency $\omega_0$ and envelope $F(z,t)$, we have $E(z,t) = e^{j(\omega_0 t - k_0 z)} F(z,t)$.

- The envelope $F(z,t)$ propagates with the group velocity $v_g = 1/k'_0$.
- The width of the pulse increases with propagation length as $\tau(z) = \sqrt{\tau_0^2 + (k''_0 z/\tau_0)^2}$. This limits the propagation length in communication applications.
- The local frequency varies with $z$ (or $t$), which is called *chirping*.
- The propagation dispersion can be compensated for by the use of filters.
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When can we trust the group velocity?

The concept of group velocity relies on **narrow-band pulses**. With $\tau_0$ being the duration of the pulse we have implicitly assumed:

$$|k''_0 z| \ll \tau_0^2 \quad \text{and} \quad |\text{Im}(k_0) z| \ll 1$$

Typical risks:

- Fast frequency variations (typical at resonance).
- Active/gain medium.
A resonant medium is typically described by

\[ \chi(\omega) = \frac{f \omega_p^2}{\omega_r^2 - \omega^2 + j\omega\gamma} \]

where \( f = +1 \) for an absorbing medium, and \( f = -1 \) for a gain medium. The refractive index is

\[ n = \sqrt{1 + \chi} \approx 1 + \frac{1}{2} \chi = 1 + \frac{f \omega_p^2 / 2}{\omega_r^2 - \omega^2 + j\omega\gamma} \]

At resonance \( \omega = \omega_r \):

\[ n = 1 - j \frac{f \omega_p^2}{2\gamma\omega_r}, \quad n_g = 1 - \frac{f \omega_p^2}{\gamma^2} \]

The group refractive index can be just about anything, depending on the choice of parameters!
Slow, fast, and negative group velocities

Default values: $\omega_p = 1$, $\omega_r = 5$, $\gamma = 0.4$. Top row: $f = +1$. Middle row: $f = -1$. Bottom row: $f = -1$, $\gamma = 0.2$. $n = n_r - jn_i$.

Slow, $n_g = 1.48 + 0.39j$

Fast, $n_g = 0.52 - 0.39j$

Negative, $n_g = -0.58 - 1.02j$

Fig. 3.9.1 in Orfanidis. Observe active media are used!
Slow, fast, and negative group velocities

Using two nearby resonances:

\[ n_g = 8.104 + 0.063j \]

\[ n_g = 0.208 - 0.021j \]

\[ n_g = -0.778 - 0.032j \]

Fig. 3.9.2 in Orfanidis. Observe active media are used!

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Propagation through slow and fast media

Matlab functions `grvmovie1.m` and `grvmovie2.m`. Movies plot only envelope propagation.
Stenner, Gauthier, Neifeld, Nature 2003

Diagram a shows a setup with an AOM, K vapour, and waveform generator. The diagram b illustrates the power (µW) over time (ns) with two curves: one for Advanced and one for Vacuum, where $t_{adv} = 27.4$ ns.
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Radar system

RADAR = RAdio Detection And Ranging, was developed mainly in the 1940s. The principles have been transferred to many other areas, for instance LIDAR = Light Detection and Ranging, atmospheric measurements, and speed measurements.
Weather radar (www.smhi.se)

Frequency 5.6 GHz, wavelength 5 cm, range 240–250 km.

Radar in Vilebo  BALTEX stations  Resulting image
Simple estimate for signal-to-noise ratio SNR, range resolution $\Delta R$, and Doppler resolution (velocity resolution) $\Delta v$:

$$SNR = \frac{E_{\text{rec}}}{N_0} = \frac{P_{\text{rec}} T}{N_0}, \quad \Delta R = \frac{c}{2B}, \quad \Delta v = \frac{c}{2f_0 T}$$

For good range and Doppler resolution, the product $BT$ must be large.

So how to design a pulse with large bandwidth and long duration?
The bats have a suggestion!

Wavelength \( \approx (340 \text{ m/s}) \cdot (20 \mu\text{s}) = 7 \text{ mm}. \)

Change the frequency linearly with time (chirp).
Chirp radar system

Stretch the pulse before sending it out, compress it when coming back.

(Fig. 3.10.1 in Orfanidis)
A chirp filter can be realized by a surface acoustic wave device.

The device couples electromagnetic energy to acoustic waves, where the coupling is strongest when the distance between the metal fingers correspond to $\lambda/2$ for the acoustic wave. Chirping is obtained by different acoustic propagation lengths. Works up to about 3 GHz.
Pulse compression

Chirped pulse (input)

\[ E(t) = F(t)e^{j\omega_0 t + j\dot{\omega}_0 t^2/2} \]

Compression filter

\[ h_{\text{compr}}(t) = \sqrt{\frac{j\dot{\omega}_0}{2\pi}} e^{j\omega_0 t - j\dot{\omega}_0 t^2/2} \]

Compressed pulse

\[ E_{\text{compr}}(t) = [h_{\text{compr}} * E](t) = \cdots = \sqrt{\frac{j\dot{\omega}_0}{2\pi}} e^{j\omega_0 t - j\dot{\omega}_0 t^2/2} \hat{F}(-\dot{\omega}_0 t) \]

Compression since if \( F(t) \) is wide, then \( \hat{F}(\omega) \) is narrow.
Rectangular pulse

\[ \hat{E}(\omega) = \sqrt{\frac{2\pi j}{\omega_0}} e^{-j(\omega-\omega_0)^2/(2\omega_0)} D^*(\omega) \]

(Figs. 3.10.3–3.10.4 in Orfanidis)
Pulse compression

A smoother window function like the Hamming
\[ w(t) = 1 + 2\alpha \cos\left(\frac{2\pi t}{T}\right) \] for \(-T/2 < t < T/2\) and zero otherwise, can reduce the sidelobes.

Even better with matched filters, \( h(t) = E^*(-t) \) and \( H(\omega) = \hat{E}^*(\omega) \). Implemented digitally after recording impulse response (time reversal) or by SAW.
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Conclusions

- Narrow-band pulses propagate with the group velocity.
- Dispersive media leads to pulse broadening.
- High and low frequencies are spread throughout the pulse.
- Dispersion leads to finite lengths of propagation in a communication application.
- The dispersion can to some extent be counteracted by suitable filters.
- Pulses can be tailored to specific needs.