TM-modes rect. waveguide

$$H_z = 0 , \quad E_z = v(\bar{\rho})\, e^{i k_z z}$$

$$\begin{cases} \nabla_T^2 v(\bar{\rho}) + k_t^2 v(\bar{\rho}) = 0 \\ v(0,y) = v(a,y) = v(x,0) = v(x,b) = 0 \end{cases}$$

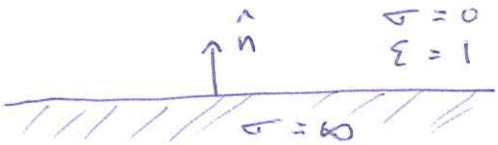
$$\Rightarrow \quad v_{mn}(x,y) = \frac{2}{\sqrt{ab}} \sin \frac{m\pi x}{a} \cdot \sin \frac{n\pi y}{b}$$

$$k_{tmn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \qquad \begin{matrix} m = 1, 2 \ldots \\ n = 1, 2 \ldots \end{matrix}$$

$\Rightarrow$ lowest cut-off frequency for $m = 1$, $n = 1$

$$\Rightarrow \quad f_{c_{11}} = \frac{c}{2\pi} \sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2}$$

Surface charges and currents



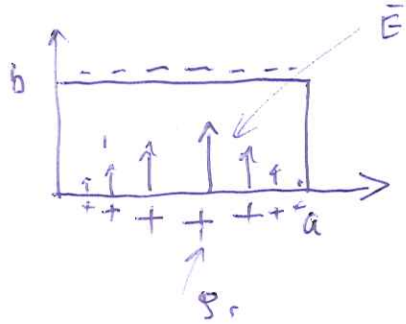
$$\rho_s = \hat{n} \cdot \bar{D} = \varepsilon_0\, \hat{n} \cdot \bar{E}$$

$$\bar{J}_s = \hat{n} \times \bar{H}$$

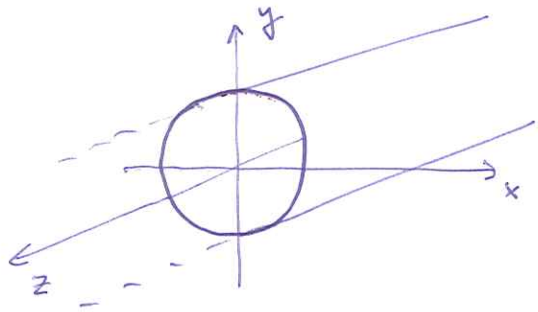
Ex $\quad TE_{10} \Rightarrow$

$$\bar{E} = \sqrt{\frac{2}{ab}} \sin \frac{\pi x}{a}\, e^{i k_z z}\, \hat{y}$$

$$\Rightarrow \quad \rho_s = \begin{cases} \varepsilon_0 \sqrt{\frac{2}{ab}} \sin \frac{\pi x}{a}\, e^{i k_z z} & \text{on } y = 0, \\ -\varepsilon_0 \sqrt{\frac{2}{ab}} \sin \frac{\pi x}{a}\, e^{i k_z z} & \text{on } y = b \\ 0 & \text{on } x = 0, a \end{cases}$$

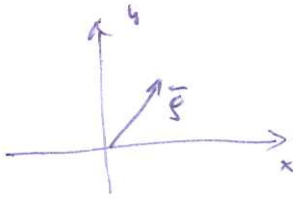


5.5.3 Circular waveguides



Cylindrical coordinates (ρ, φ, z)

$$\Rightarrow \quad \bar{r} = \rho\,\hat{\rho} + z\,\hat{z}$$



TM-modes

$$H_z = 0\,, \qquad E_z = v(\rho, \varphi)\, e^{i k_z z}$$

Eigenvalue problem

$$\begin{cases} \nabla_T^2 v + k_t^2 v = 0 \\ v(a, \varphi) = 0 \\ |v(0, \varphi)| < \infty \\ v(\rho, \varphi) = v(\rho, \varphi + 2\pi) \end{cases}$$

Cylindrical coordinates $\Rightarrow$ (see App. B.2)

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial v}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 v}{\partial\varphi}+k_t^2 v=0$$

Separation of variables

$$v(\rho,\varphi)=f(\rho)\cdot g(\varphi)$$

$$\Rightarrow \quad g\cdot\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial f}{\partial\rho}+f\cdot\frac{1}{\rho^2}\frac{\partial^2 g}{\partial\varphi^2}+k_t^2 f\cdot g=0$$

Divide by $\frac{f\cdot g}{\rho^2}$

$$\Rightarrow \quad \frac{1}{f}\cdot\rho\cdot\frac{\partial}{\partial\rho}\rho\frac{\partial f}{\partial\rho}+k_t^2\rho^2=-\frac{1}{g}\frac{\partial^2 g}{\partial\varphi^2}$$

$\Rightarrow$ RHS = LHS = constant = λ

$$\Rightarrow \quad g''(\varphi)+\lambda\, g(\varphi)=0$$

$$g(\varphi)=g(\varphi+2\pi)$$

$$\Rightarrow \quad g(\varphi)=A\cos\sqrt{\lambda}\,\varphi+B\sin\sqrt{\lambda}\,\varphi$$

$$\Rightarrow \quad \lambda=m^2 \qquad m=0,1,2..$$

$$\therefore \quad g(\varphi)=A_m\cos m\varphi+B_m\sin m\varphi$$

$$\rho\frac{\partial}{\partial\rho}\rho\frac{\partial f}{\partial\rho}+k_t^2\rho^2 f-m^2 f=0$$ Bessel's diff. eq see App. A

$$\Rightarrow \quad f(\rho)=C_m J_m(k_t\rho)+D_m N_m(k_t\rho)$$

Bessel fcn — Neumann fcn

$f(a)=0$, $|f(0)|<\infty$

$|N_m(k_t\rho)| \longrightarrow \infty$ when $\rho \to 0 \Rightarrow D_m = 0$

$\Rightarrow \quad f(\rho) = C_m J_m(k_t \rho)$

$f(a) = 0 \Rightarrow \quad k_t \cdot a = \xi_{mn}, \quad m = 0, 1, 2, \ldots \quad n = 1, 2 \ldots$

where $\xi_{mn} =$ n:th zero of $J_m(x)$

$$\boxed{\begin{aligned} &\therefore \quad E_{z\,mn} = J_m(k_{tmn}\,\rho)\left(C_{mn}\cos m\varphi + D_{mn}\sin m\varphi\right) e^{i k_{zmn} z} \\ &k_{tmn} = \frac{\xi_{mn}}{a} \end{aligned}}$$

Ex $\quad m = 0$, see App. A

$\Rightarrow$

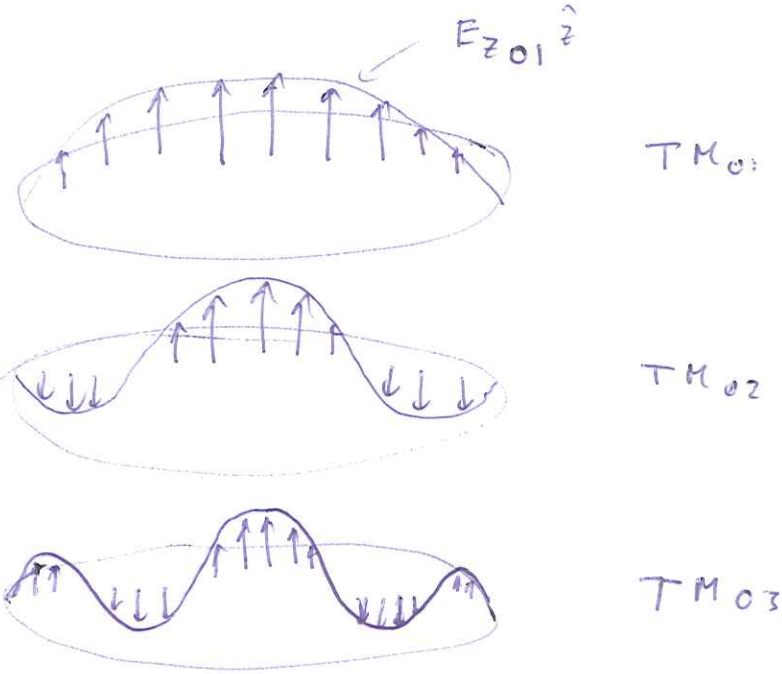


Table A.1 says that the fundamental TM-mode is TM_{01} $\Rightarrow$ $k_t = \frac{2{,}405}{a}$ $\Rightarrow$ $f_c = \frac{c}{2\pi} \cdot \frac{2{,}405}{a}$

TE-modes

$E_z = 0$, $H_z = w(\rho, \varphi)\, e^{ik_z z}$

$$\begin{cases} \nabla_T^2 w + k_t^2 w = 0 \\ \frac{\partial w}{\partial \rho}(a, \varphi) = 0 \qquad |w(0, \varphi)| < \infty \\ w(\rho, \varphi) = w(\rho, \varphi + 2\pi) \end{cases}$$

$\Rightarrow$ $w(\rho, \varphi) = J_m(k_{tmn}\rho)\left(F_{mn} \cos m\varphi + G_{mn} \sin m\varphi\right)$

$m = 0, 1, 2 \ldots$
$n = 1, 2, 3 \ldots$

where $k_{tmn} = \frac{\eta_{mn}}{a}$

and η_{mn} = n:th zero of $J_m'(x)$

Table A:2 says that the fundamental TE-mode is TE_{11} $\Rightarrow$ $k_{t11} = \frac{1.841}{a}$; $f_c = \frac{c}{2\pi} \cdot \frac{1.841}{a}$

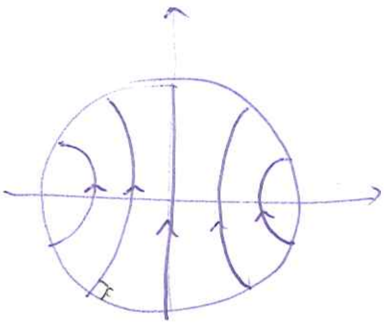
$\therefore$ The fundamental mode for a circular waveguide is TE_{11}

TE_{11}

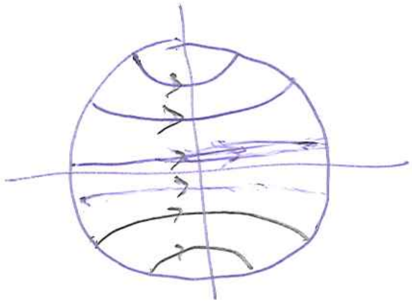
$$5.24 \Rightarrow \bar{E}_{10}(\bar{r}) = -\frac{i\omega}{k_{t_{10}}^2}\mu_0 \, \hat{z} \times \bar{\nabla}_T w_{11}(\rho,\varphi)$$

where $\nabla_T = \hat{\rho}\frac{\partial}{\partial\rho} + \hat{\varphi}\frac{1}{\rho}\frac{\partial}{\partial\varphi}$

$\Rightarrow$



or



Compare with TE_{10}