Microwave theory 2017: Problems week 1 and 2

Problem 1

The TE_{10} mode in a rectangular waveguide with cross section $0 < x < a$, $0 < y < b$ ($a > b$), filled with air, has the complex electric field

$$\boldsymbol{E}(\boldsymbol{r}) = E_0 \sin\left(\frac{\pi x}{a}\right) e^{\mathrm{i}k_z z}\hat{\boldsymbol{y}}$$

where $k_z = \sqrt{k^2 - (\pi/a)^2}$ is the $z-$component of the wave vector and $k = \omega/c$ is the wavenumber.

a) What is the time-domain electric field $\boldsymbol{E}(\boldsymbol{r}, t)$? (Time dependence is $e^{-\mathrm{i}\omega t}$)

b) Below a certain frequency (the cut-off frequency) the wave will attenuate with increasing z. What is this frequency if $a = 10$ cm?

c) What is the phase speed in the $z-$direction of the TE_{10} mode when $a = 10$ cm and $f = 2$ GHz?

d) Is the phase speed larger or smaller than the speed of light? Is there a contradiction with the theory of special relativity?

e) Assume that you like to feed the TE_{10} in the waveguide by a coaxial cable. You drill a small hole in the waveguide and attach the coaxial cable. The inner conductor of the coaxial cable extends straight into the waveguide. Suggest a suitable position (x, y) for the hole.

Problem 2

The boundary condition for electromagnetic waves at a perfectly conducting surface is that the tangential component of the electric field is zero. From this, and the Maxwell equations, one can derive the following boundary conditions at the surfaces of a rectangular waveguide:

1. The normal component of the magnetic field is zero on the surfaces ($\hat{\boldsymbol{n}}\cdot\boldsymbol{H} = 0$).
2. The normal derivative of the normal component of the electric field is zero on the surfaces ($\hat{\boldsymbol{n}} \cdot \nabla(\hat{\boldsymbol{n}} \cdot \boldsymbol{E}) = 0$)).
3. The normal derivative of the tangential components of the magnetic field are zero on the surfaces ($\hat{\boldsymbol{n}} \cdot \nabla(\hat{\boldsymbol{n}} \times \boldsymbol{H}) = \boldsymbol{0}$).

This imply that all components of the electric and magnetic fields have the $x-$dependence

$$\sin\left(\frac{m\pi x}{a}\right) \text{ or } \cos\left(\frac{m\pi x}{a}\right)$$

and $y-$dependence

$$\sin\left(\frac{n\pi y}{b}\right) \text{ or } \cos\left(\frac{n\pi y}{b}\right).$$

The $z-$dependence is $e^{\mathrm{i}k_z z}$ for all componants.

a) The TE_{mn} modes have $E_z = 0$ whereas H_z has the space dependence

$$\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{\mathrm{i}k_z z}.$$

Determine the space dependences of E_x, E_y, H_x, and H_y.

b) The TM_{mn} modes have $H_z = 0$ whereas $E_z \not\equiv 0$. Determine the space dependences of E_x, E_y, E_z, H_x, and H_y.

c) Can there be TM_{m0} and TM_{0n} modes in a rectangular waveguide?

Problem 3

A circular waveguide of length $0 < z < L$ and radius a has a non-reflecting termination at $z = L$. In the region $\frac{L}{4} < z < \frac{L}{2}$ one has inserted a circular metal cylinder with radius $b < a$ and very thin wall.

Determine b/a such that it is only the TM_{02} mode that can propagate from $z = 0$ to $z = L$ without reflections from the inner cylinder.

Problem 5.6 in the book

A circular hollow waveguide has the radius $a = 3$ cm.

a) Determine the modes that can propagate at the frequency $f = 5$ GHz when the waveguide is filled with air.

b) Assume that the waveguide is filled with a plastic material with relative permittivity $\epsilon = 3$ and conductivity $\sigma = 10^{-11}$ S/m. Determine the attenuation of the dominant mode in dB/km as a function of frequency.

Problem 5.7 in the book

Consider a rectangular waveguide with size $0 < x < a$, $0 < y < b$, $a = 6$ cm, $b = 4$ cm. In the region $z > 0$, a metallic plate is inserted, see figure 1. The plate i parallel with the y-z-plane and is placed at $x = x_0$. The walls of the waveguide and the plate are perfect conductors. For a certain value of x_0 we measure P_i and P_r, where P_i and P_r are the time averages of the power flow in positive and negative z-direction, respectively, in the region $z < 0$. If only the fundamental mode TE_{10} propagates in positive z-direction for $z < 0$ we have $P_r/P_i = 1$ for all frequencies below 3.75 GHz and $P_r/P_i < 1$ for frequencies above 3.75 GHz.

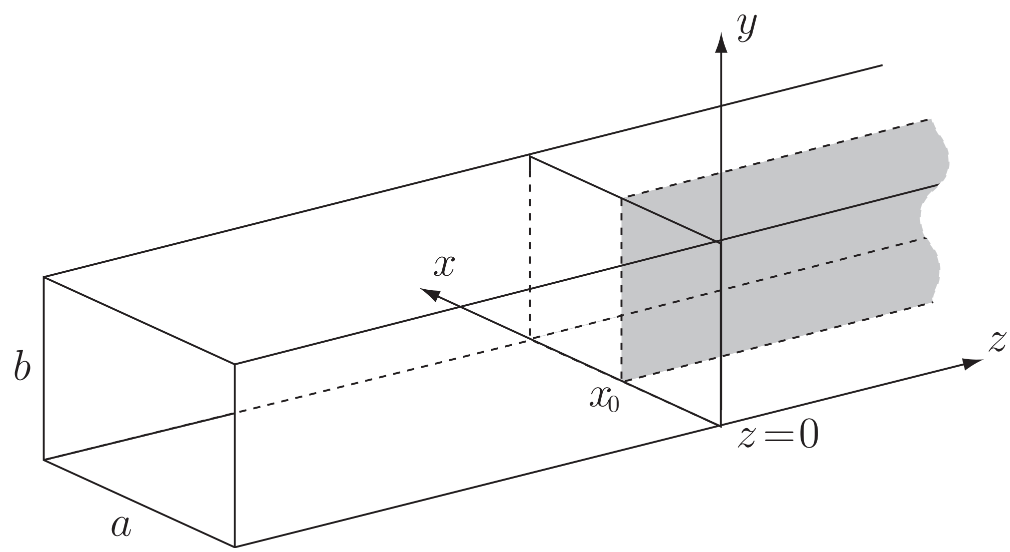

Figure 1: Geometry for problem 5.6.

a) Determine x_0.

b) What is the quotient P_r/P_i when a TE_{03} mode propagates in the positive $z-$direction for $z < 0$ at the frequency 20 GHz?

c) What is P_r/P_i when a TE_{30} mode propagates in positive $z-$direction for $z < 0$ at the frequency 10 GHz?

Problem 5.8 in the book

A waveguide has a cross section in the shape of a quarter circle with radius R. Determine all TE- and TM-modes for the waveguide.

Solutions

Solution 1

a) $\boldsymbol{E}(\boldsymbol{r}, t) = \mathrm{Re}\{\boldsymbol{E}(\boldsymbol{r})e^{-\mathrm{i}\omega t}\}$ and then

$$\boldsymbol{E}(\boldsymbol{r}, t) = E_0 \sin\left(\frac{\pi x}{a}\right)\cos(\omega t - k_z z)\hat{\boldsymbol{y}}$$

b) Cut-off when $k_z = 0$ gives $k = \pi/a$ and $f_c = c/(2a)$. Since $a = 10$ cm

$$f_c = 1.5\,\mathrm{GHz}$$

c) Phase speed$=\dfrac{\omega}{k_z}$, where $\omega = 4\pi \cdot 10^9$ rad/s and $k_z = \sqrt{k^2 - (\pi/a)^2}$. Thus $v_p = 4.54 \cdot 10^8$ m/s.

d) The phase speed is 50% larger than the speed of light. The special theory of relativity is not violated since the phase speed is not the speed the information and power move with.

e) The charges on the inner conductor couples to the electric field of the TM_{10} mode. It should then be placed where the electric field is strong. That is either on the lower surface at $(x, y) = (a/2, 0)$ or on the upper surface at $(x, y) = (a/2, b)$.

Solution 2

a)

$$\begin{aligned}
E_x &\sim \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{\mathrm{i}k_z z}\\
E_y &\sim \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{\mathrm{i}k_z z}\\
H_x &\sim \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{\mathrm{i}k_z z}\\
H_y &\sim \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{\mathrm{i}k_z z}
\end{aligned}$$

b)

$$\begin{aligned}
E_x &\sim \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{\mathrm{i}k_z z}\\
E_y &\sim \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{\mathrm{i}k_z z}\\
E_z &\sim \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{\mathrm{i}k_z z}\\
H_x &\sim \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{\mathrm{i}k_z z}\\
H_y &\sim \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{\mathrm{i}k_z z}
\end{aligned}$$

Solution 3

The TM_{02} mode has $E_z(\boldsymbol{r}) = v_{02}(\rho)e^{ik_z z}$ where $v_{02}(\rho) = A_{02}J_0\left(\frac{\xi_{02}\rho}{a}\right)$. However $J_0\left(\frac{\xi_{02}\rho}{a}\right)$ is also zero when $\left(\frac{\xi_{02}\rho}{a}\right) = \xi_{01}$. This means that E_z is zero at $\rho = \frac{\xi_{01}}{\xi_{02}}a$. The boundary condition that the tangential component of $\boldsymbol{E}$ is zero at $\rho = b$ is then satisfied if $b = \frac{\xi_{01}}{\xi_{02}}a = 2.405a/5.520 = 0.437a$. The transverse part of the electric field is directed in the radial direction and is not affected by the boundary at $\rho = b$.

Solution 5.6

a) We first determine the modes that can propagate when $a = 3$ cm and $f = 5$ GHz. The lowest cut-off frequencies are obtained from the tables of zeros for $J_m(x)$ (för TM) and $J_m'(x)$ (för TE) in appendix A

$$f_{11}^{TE} = \frac{c_0}{2\pi}\frac{1.841}{3}10^2 = 2.93\,\text{GHz} < 5\,\text{GHz} \tag{0.1}$$

$$f_{21}^{TE} = \frac{c_0}{2\pi}\frac{3.053}{3}10^2 = 4.86\,\text{GHz} < 5\,\text{GHz} \tag{0.2}$$

$$f_{01}^{TM} = \frac{c_0}{2\pi}\frac{2.405}{3}10^2 = 3.83\,\text{GHz} < 5\,\text{GHz} \tag{0.3}$$

The next modes are f_{01}^{TE} and f_{11}^{TM} which both are non-propagating modes since they have cut-off frequency 6.1 GHz.

b) The waveguide is filled with a plastic material with $\sigma = 10^{-11}$ S and $\varepsilon = 3$. The z-dependence of the fundamental mode TE_{11} is given by $e^{ik_z z}$ where $k_z = \sqrt{k^2 - k_{t11}^2}$. The wave number k is given by

$$k^2 = \left(\frac{\omega}{c_0}\right)^2 \epsilon_{ny} = \left(\frac{\omega}{c_0}\right)^2 \left(\epsilon + i\frac{\sigma}{\omega\epsilon_0}\right)$$

It is seen that $\sigma/(\epsilon\epsilon_0) \approx 10^{-11}/(3 \cdot 8.854\,10^{-12})$ In the microwave region $\sigma/(\omega\epsilon\epsilon_0) \ll 1$ and the following approximations are valid

$$\begin{aligned}
k_z = (k^2 - k_{t11}^2)^{1/2} &= \left(\left(\frac{\omega}{c_0}\right)^2 \epsilon - k_{t11}^2\right)^{1/2} \left(1 + i\frac{\sigma\omega\mu_0}{((\omega/c_0)^2\epsilon - k_{t11}^2)}\right)^{1/2} \\
&\approx \left(\left(\frac{\omega}{c_0}\right)^2 \epsilon - k_{t11}^2\right)^{1/2} \left(1 + i\frac{\sigma\omega\mu_0}{2((\omega/c_0)^2\epsilon - k_{t11}^2)}\right)
\end{aligned}$$

and hence $k_z = \mathrm{Re}(k_z) + i\,\mathrm{Im}(k_z)$ where

$$\mathrm{Im}(k_z) = \frac{\sigma\omega\mu_0}{2}\left((\omega/c_0)^2\epsilon - k_{t11}^2\right)^{-1/2} = \frac{\sigma\eta}{2}\left(1 - (f_c/f)^2\right)^{-1/2}$$

where $f_c = c_0\xi_{11}/(2\pi a\sqrt{\epsilon})$ och $\eta = \sqrt{\mu_0/(\epsilon\epsilon_0)}$ =wave impedance. The numerical value is $f_c = 1.7$ GHz.

Solution 5.7

For the TE_{10}-mode the electric field in the region $z<0$ is

$$\boldsymbol{E}(\boldsymbol{r})=\hat{y}E_0\sin\frac{\pi x}{a}e^{ik_z z}$$

This is the fundamental mode with cut-off frequency $f_c=0.5c_0/b=2.5$ GHz. When this mode hits the plate it couples to the TE_{m0}-modes in $z>0$ and to the reflected TE_{m0}-modes in $z<0$.

Assume that $x_0>a/2$. The fundamental mode in $z>0$, $x<x_0$ is TE_{10}. This mode has the cut-off frequency $f_c=0.5c_0/x_0$.

- a) According to the text, power propagates in $z>0$ for frequencies above 3.75 GHz. This means that 3.75 GHz is the cut-off frequency for the fundamental mode TE_{10} in $x<x_0$. Hence the plate is placed at $x_0=0.5c_0/f_c=0.5\cdot 3\cdot 10^8/3.75\cdot 10^9=4$ cm.

- b) The electric for the TE_{03}-mode in $z<0$

  $$\boldsymbol{E}(\boldsymbol{r})=\hat{x}E_0\sin\frac{3\pi y}{b}e^{\mathrm{i}k_z z}$$

  and then the boundary condition at $x=x_0$ is already satisfied since the tangential component is zero. Thus $P_r/P_i=0$. The corresponding cut-off frequency is at $f_c=3\cdot 0.5\cdot c_0/b=15$ GHz and hence the mode propagates at 20 GHz.

- c) The electric field for the TE_{30}-mode in $z<0$ is

  $$\boldsymbol{E}(\boldsymbol{r})=\hat{y}E_0\sin\frac{3\pi x}{a}e^{\mathrm{i}k_z z}$$

  The corresponding cut-off frequency is at $f_c=3\cdot 0.5\cdot c_0/a=7.7$ GHz and hence the mode propagates at 10 GHz. At $x=x_0=4$ cm we see that $\boldsymbol{E}(x_0,z)=\boldsymbol{0}$ for the TE_{30}-mode. The electric field satisfies the correct boundary conditions on the plate $x=x_0$. this means that this mode is not affected by the plate and it continues to propagate in $z>0$, without a reflected wave.Hence $P_r/P_i=0$.

  Comment In $z>0$ the mode TE_{30} splits up in a TE_{20}-mode in the region $x<x_0$ and one TE_{10}-mode in $x_0<x<a$.

Solution 5.8

A quarter circle

<u>TM-modes:</u>

$E_z(\boldsymbol{r})=v(\boldsymbol{\rho})e^{ik_z z}$ where v satisfies

$$\begin{cases}\nabla_T^2 v(\boldsymbol{\rho})+k_t^2 v(\boldsymbol{\rho})=0\\ v(R,\phi)=v(\rho,0)=v(\rho,\pi/2)=0\\ v(\boldsymbol{\rho})\text{ begränsad}\end{cases}$$

Separation of variables $v(\boldsymbol{\rho}) = f(\rho)g(\phi)$ gives

$$g''(\phi) + \gamma g(\phi) = 0$$
$$g(0) = g(\pi/2) = 0$$

$$\Rightarrow g(\phi) = \sin(2m\phi), \quad \gamma = 4m^2$$

In the ρ-direction we get the Bessel differential equation of order $2m$

$$\begin{cases} \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial f(\rho)}{\partial\rho} + \left(k_t^2 - \left(\frac{2m}{\rho}\right)^2\right) = 0 \\ f(R) = 0 \quad |f(0)| < \infty \end{cases}$$

this gives $f(\rho) = J_{2m}(\xi_{2m,n}\rho/R)$ and $k_t^2 = (\xi_{2m,n}/R)^2$, where $J_{2m}(\xi_{2m,n}) = 0$.

The normalized eigenfunctions for the TM-modes are given by

$$v_{2m,n}(\boldsymbol{\rho}) = \sqrt{\frac{2}{\pi}}\frac{J_{2m}(\xi_{2m,n}\rho/R)}{RJ'_{2m}(\xi_{2m,n})}\sin 2m\phi$$

<u>TE-modes:</u>

$H_z(\boldsymbol{r}) = w(\boldsymbol{\rho})e^{ik_z z}$

We get the same problem as in the TE-case except that the boundary conditions are

$$\frac{\partial w(R,\phi)}{\partial\rho} = 0, \; \frac{\partial w(\rho,0)}{\partial\phi} = \frac{\partial w(\rho,\pi/2)}{\partial\phi} = 0$$

This gives the eigenfunctions

$$w_{2m,n}(\boldsymbol{\rho}) = \sqrt{\frac{\epsilon_m}{\pi}}\frac{\eta_{2m,n}J_{2m}(\eta_{2m,n}\rho/R)}{\sqrt{\eta_{2m,n}^2 - 4m^2}RJ_{2m}(\eta_{2m,n})}\cos 2m\phi$$

and the eigenvalues $k_t^2 = (\eta_{2m,n}/R)^2$ where $J'_{2m}(\eta_{2m,n}) = 0$, $m = 0, 1, 2..$, $n = 1, 2, ...$