

## Microwave theory 2016: Problems for week 5

### 1

- a) Show analytically that the time averages of the stored electric and magnetic energies in a resonance cavity are equal.
- b) Check with COMSOL for a cavity that you choose.

### 2

Estimate the number of resonances with a wavelength larger than 500 nm in a cubic vacuum cavity with volume one cubic meter.

### 3

- a) Determine the fundamental mode in a circular waveguide with radius  $a$  with vacuum inside the waveguide.
- b) Plot the transverse electric and magnetic field of the fundamental mode in the cross section of the waveguide.
- c) Determine the surface current density for the fundamental mode.

### 4

- a) Determine the fundamental mode for a rectangular waveguide.
- b) Give the  $x, y$ -dependence of the different components of the transverse electric and magnetic fields for the fundamental mode without doing any calculations.
- c) Determine the  $z$ -component of the Poynting vector  $\mathbf{S}(x, y)$ .
- d) Determine the attenuation of the fundamental mode in dB/m when  $a = 3$  cm,  $b = 1$  cm in the frequency interval  $[0, f_{c10}]$  where  $f_{c10}$  is the cut-off frequency for the fundamental mode.

### 5

Around 1970 there were attempts to use circular waveguides for communication. Such a system was developed by Bell Telephone Laboratories in USA that managed to send 234 000 two-way telephone channels in one waveguide. One then used the waveguide mode  $TE_{01}$  in a waveguide with radius 25 mm in the frequency band 40-110 GHz. Determine the attenuation in dB/m for the fundamental mode  $TE_{11}$  and for the mode  $TE_{01}$  in a circular waveguide with radius 25 mm as a function of frequency in the interval  $[f_c, 40 \text{ GHz}]$ , where  $f_c$  is the cut-off frequency for the  $TE_{11}$  mode. Explain why they used the mode  $TE_{01}$  and not the fundamental mode. The waveguide is made out of copper.

**6**

Determine the three lowest resonance frequencies for a coaxial cable that is terminated by perfectly conducting plates at  $z = 0$  and  $z = h = 10$  cm. The radius of the inner and outer conductor is  $a = 3$  mm and  $b = 10$  mm, respectively. The material between the conductors is a plastic material with  $\sigma = 0$  and  $\varepsilon = 4$ .

**7**

- a) Determine the three lowest resonance frequencies for a rectangular parallelepiped with sides  $a = 4$  cm,  $b = 3$  cm and height  $h = 5$  cm. All of the walls are perfectly conducting and there is vacuum in the cavity.
- b) Use COMSOL 3D to check your result.
- c) Use COMSOL with 3D and extrude to check your result.

**8**

- a) Design a resonator in the shape of a circular cylinder such that it has the two lowest resonance frequencies at  $f_1 = 10$  GHz and  $f_2 = 15$  GHz.
- b) Use COMSOL 2D axisymmetric mode to check your result.

**9**

Let COMSOL determine the ten lowest resonance frequencies for a spherical cavity with radius  $a = 1$  m, filled with vacuum and with perfectly conducting wall and compare with the exact values in table 6.1 in the book.

- a) First use Axially symmetry with three different mesh sizes.
- b) Use 3D with three different meshes.

**10**

Use COMSOL to determine the Q-values for the four lowest resonances in a spherical cavity with radius  $a = 1$  m, filled with vacuum and with copper walls. Compare the Q-values with the analytic expressions in Eqs. 6.41 and 6.44,

$$Q = \begin{cases} \frac{a}{\delta} & \text{TE } (\tau = 1) \\ \frac{a}{\delta} \left( 1 - \frac{l(l+1)}{(ka)^2} \right) & \text{TM } (\tau = 2) \end{cases}$$

where  $\delta = \sqrt{\frac{2}{\omega\sigma\mu_0}}$  is the skin depth of metal and  $l$  is the index for the polar variation, see section 6.3.2.

- a) First use Axially symmetry with three different mesh sizes.
- b) Use 3D with three different meshes.

**11**

Check the result from Slater's theorem in Eq. 6.17 (or 6.16) by entering a sphere into a cylindrical cavity. Let the cylindrical cavity have radius 10 cm and height 10 cm. Let the sphere have radius  $a = 0.5$  cm and be located in the middle of the cylinder. Use 2D axial symmetry in Comsol.

- a) Consider the  $TM_{010}$  mode. Enter the sphere but let it be air. Check that the frequency of the  $TM_{010}$  mode is given by  $f = \frac{c_0}{2\pi} \frac{2.405}{a}$ . Then normalize the electric field such it satisfies Eq. 6.1. Now let Comsol calculate the RHS of Eq. 6.17.
- b) Let the sphere be PEC. Calculate the new resonance frequency and check if 6.17 is satisfied.

## Solutions

### S1

Let  $\mathbf{E}_n$  be the electric field of a resonance mode in a cavity. Since  $\mathbf{E}_n$  satisfies Maxwell's equations it follows that

$$\nabla \times (\nabla \times \mathbf{E}_n) - k_n^2 \mathbf{E}_n = \mathbf{0}$$

Take the scalar product with  $\mathbf{E}_n^*$  and use

$$\mathbf{E}_n^* \cdot \nabla \times (\nabla \times \mathbf{E}_n) = (\nabla \times \mathbf{E}_n^*) \cdot (\nabla \times \mathbf{E}_n) - \nabla \cdot (\mathbf{E}_n^* \times (\nabla \times \mathbf{E}_n)) = \frac{1}{\epsilon_0 \epsilon} k_n^2 \mathbf{B}_n \cdot \mathbf{H}_n^* - \nabla \cdot (\mathbf{E}_n^* \times (\nabla \times \mathbf{E}_n))$$

Thus

$$\frac{1}{\epsilon_0 \epsilon} k_n^2 \mathbf{B}_n \cdot \mathbf{H}_n^* - \nabla \cdot (\mathbf{E}_n^* \times (\nabla \times \mathbf{E}_n)) = k_n^2 |\mathbf{E}_n|^2$$

The volume integral of this relation and the use of Gauss theorem give

$$k_n^2 \iiint_V \mathbf{B}_n \cdot \mathbf{H}_n^* dV + \epsilon_0 \epsilon \oint_S \hat{\mathbf{n}} \cdot (\mathbf{E}_n^* \times (\nabla \times \mathbf{E}_n)) dS = k_n^2 \epsilon_0 \epsilon \iiint_V |\mathbf{E}_n|^2 dV$$

Due to the boundary condition  $\hat{\mathbf{n}} \times \mathbf{E}_n = \mathbf{0}$  we get  $\hat{\mathbf{n}} \cdot (\mathbf{E}_n^* \times (\nabla \times \mathbf{E}_n)) = \nabla \times \mathbf{E}_n \cdot (\hat{\mathbf{n}} \times \mathbf{E}_n^*) = 0$ . Since  $\mathbf{D} = \epsilon_0 \epsilon$  we get

$$\frac{1}{4} \iiint_V \mathbf{B}_n \cdot \mathbf{H}_n^* dV = \frac{1}{4} \iiint_V \mathbf{E}_n \cdot \mathbf{D}_n^* dV$$

The volume integral to the left is the time average of the magnetic energy in the cavity and the volume integral to the right is the time average of the electric energy.

### S2

The resonance wavenumbers (TE or TM) are given by  $k_{nml}^2 = \left(\frac{2\pi}{\lambda_{nml}}\right)^2 = (n^2 + m^2 + l^2) \left(\frac{\pi}{a}\right)^2$  where  $a = 1$  m. Now  $\lambda_{nml} > \lambda = 500$  nm. This means that the  $n, m, l$  have to satisfy

$$n^2 + m^2 + l^2 \leq \left(\frac{2a}{\lambda}\right)^2 = 16 \cdot 10^{12}$$

Only positive  $n, m, l$  are allowed. The total number of combinations  $n, m, l$  that satisfies this relation equals the volume of an eighth of a sphere with radius  $4 \cdot 10^6$  units. This means that there are  $\frac{1}{8} \cdot \frac{4}{3} \pi (4 \cdot 10^6)^3$  resonances. Since there are both TE and TM modes this should be multiplied by two. The total number is then  $6.7 \cdot 10^{19}$ .

### S3

a) For the TE modes  $E_z(\mathbf{r}) = 0$  and  $H_z(\mathbf{r}) = w(\rho, \phi)e^{ik_z z}$  where

$$w_{mn}(\rho, \phi) = A_{mn} J_m(\eta_{tmn} \rho/a) \cos m\phi$$

where  $A_{mn}$  denotes the normalization constant and where  $\eta_{mn}$  is the  $n$ :th zero of the derivative  $J'_m(k_t a)$ . The cut-off frequency for the  $mn$ -mode is given by

$$f_{cmn} = \frac{c_0 \eta_{mn}}{2\pi a}$$

The lowest zero is  $\eta_{11} = 1.841$  which gives the cut-off frequency

$$f_{c11} = \frac{3 \cdot 10^8 \cdot 1.841}{2\pi a} = \frac{8.79 \cdot 10^7}{a} \text{ Hz}$$

where  $a$  is in meter.

For the TM-modes  $H_z(\mathbf{r}) = 0$  and  $E_z(\mathbf{r}) = v(\rho)e^{ik_z z}$  where

$$v_{mn}(\rho, \phi) = B_{mn} J_m(\xi_{mn} \rho/a) \cos m\phi$$

where  $B_{mn}$  denotes the normalization constant and  $\xi_{mn}$  is the  $n$ th zero of  $J_m(k_t a)$ . The lowest cut-off frequency is the  $\text{TM}_{01}$  mode which has the cut-off frequency

$$f_c = \frac{3 \cdot 10^8 \cdot 2.405}{2\pi a} = \frac{1.14 \cdot 10^8}{a} \text{ Hz}$$

where  $a$  is in meter. The fundamental mode is the mode with the lowest cut-off frequency and in this case it is the  $\text{TE}_{11}$  mode.

b) The transverse fields for the TE-modes are given by Eqs. (6.23) and (6.24)

$$\mathbf{E}_T = -\frac{i}{k_{tmn}^2} \frac{\omega}{c_0} \hat{\mathbf{z}} \times \nabla_T w_{mn} = -\frac{i A_{mn} \omega}{\eta_{mn}^2 c_0} (\hat{\phi} \eta_{mn} J'_m(\eta_{mn} \rho/a) \cos m\phi + \hat{\rho} \frac{m a}{\rho} J_m(\eta_{mn} \rho/a) \sin m\phi)$$

$$\mathbf{H}_T = Z_{mn}^{-1} \hat{\mathbf{z}} \times \mathbf{E}_T = Z_{mn}^{-1} \frac{i A_{mn} \omega}{\eta_{mn}^2 c_0} (\hat{\rho} \eta_{mn} J'_m(\eta_{mn} \rho/a) \cos m\phi - \hat{\phi} \frac{m a}{\rho} J_m(\eta_{mn} \rho/a) \sin m\phi)$$

where  $Z_{mn} = \frac{\omega \mu_0}{k_{zmn}}$  is the TE-mode impedance. For the fundamental mode we get

$$\mathbf{E}_T = -\frac{i}{k_{t11}^2} \frac{\omega}{c_0} \hat{\mathbf{z}} \times \nabla_T w_{11} = -\frac{i A_{11} \omega}{\eta_{11}^2 c_0} (\hat{\phi} \eta_{11} J'_1(\eta_{11} \rho/a) \cos \phi + \hat{\rho} \frac{a}{\rho} J_1(\eta_{11} \rho/a) \sin \phi)$$

$$\mathbf{H}_T = Z_{11}^{-1} \hat{\mathbf{z}} \times \mathbf{E}_T = Z_{11}^{-1} \frac{i A_{11} \omega}{\eta_{11}^2 c_0} (\hat{\rho} \eta_{11} J'_1(\eta_{11} \rho/a) \cos \phi - \hat{\phi} \frac{a}{\rho} J_1(\eta_{11} \rho/a) \sin \phi)$$

When these vectors are plotted in Matlab we get the patterns in figure 1.

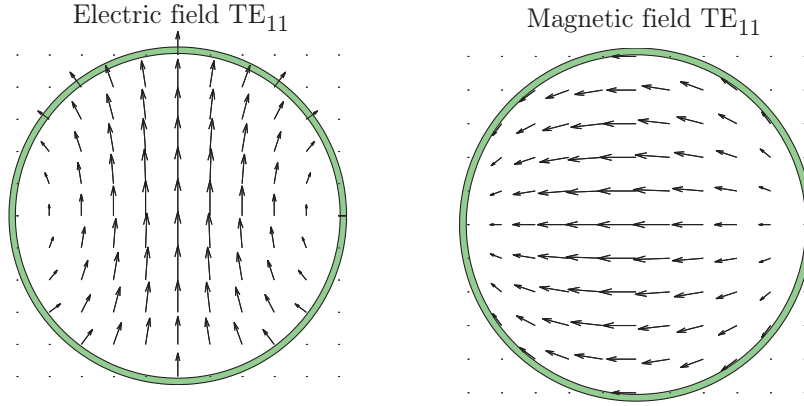
c) The surface current density is given by the condition

$$\mathbf{J}_s = \hat{\mathbf{n}} \times \mathbf{H}(\mathbf{r}) = -\hat{\rho} \times \mathbf{H}(a, \phi, z) = \hat{\phi} H_z(a, \phi, z) - \hat{\mathbf{z}} H_\phi(a, \phi, z)$$

This gives

$$\mathbf{J}_s(\rho, \phi, z) = \left( \hat{\phi} A_{11} J_1(\eta_{11}) \cos \phi - \hat{\mathbf{z}} Z_{11}^{-1} \frac{i A_{11} \omega}{\eta_{11} c_0} J_1(\eta_{11}) \sin \phi \right) e^{ik_z z}$$

Notice that the  $\phi$  and  $z$ -components are  $90^\circ$  out of phase.



**Figure 1:** The electric field and the transverse magnetic field for the  $TE_{11}$  mode

## S4

a) The fundamental mode is the mode with the lowest cut-off frequency. From the book we know that this is the  $TE_{10}$ -mode. For this mode

$$H_z(\mathbf{r}) = \sqrt{\frac{2}{ab}} \cos \frac{\pi x}{a} e^{ik_z z}$$

where  $k_z = \sqrt{(\omega/c)^2 - (\pi/a)^2}$ .

b) We do not have any  $y$ -dependence and the  $x$ -dependence is either  $\sin \frac{\pi x}{a}$  or  $\cos \frac{\pi x}{a}$ . The boundary conditions are that the tangential components of the electric field are zero at all walls, that the normal components of  $\mathbf{H}$  is zero on all walls, and that the normal derivative of the tangential component of the tangential component of  $\mathbf{H}$  is zero on all walls. Then:

- $E_z = 0$  since it is a TE-mode. It also follows from that  $E_z$  must be zero at all walls and hence also for all  $y$ -values.
- $E_x = 0$  since it must be zero at  $y = 0$  and  $y = b$  and hence for all  $y$
- $E_y \sim \sin \frac{\pi x}{a}$  since  $E_y$  must be zero at  $x = 0$  and  $x = a$
- $H_x \sim \sin \frac{\pi x}{a}$  since  $H_x$  is zero at  $x = 0$  and  $x = a$
- $H_y = 0$  since it must be zero at  $y = 0$  and  $y = b$  and hence for all  $y$

c) We now have to calculate the explicit expressions for the transverse electric and magnetic fields. We use Eqs. (6.23) and (6.24) for this

$$\mathbf{E}_T(x, y) = -\frac{i\omega}{k_{t10}^2 c_0} \hat{\mathbf{z}} \times \nabla_T \left( \sqrt{\frac{2}{ab}} \cos \frac{\pi x}{a} \right) = \hat{\mathbf{y}} \frac{i\omega}{k_{t10}^2 c_0} \frac{\pi}{a} \sqrt{\frac{2}{ab}} \sin \frac{\pi x}{a} = \hat{\mathbf{y}} \frac{i\omega a}{\pi c_0} \sqrt{\frac{2}{ab}} \sin \frac{\pi x}{a}$$

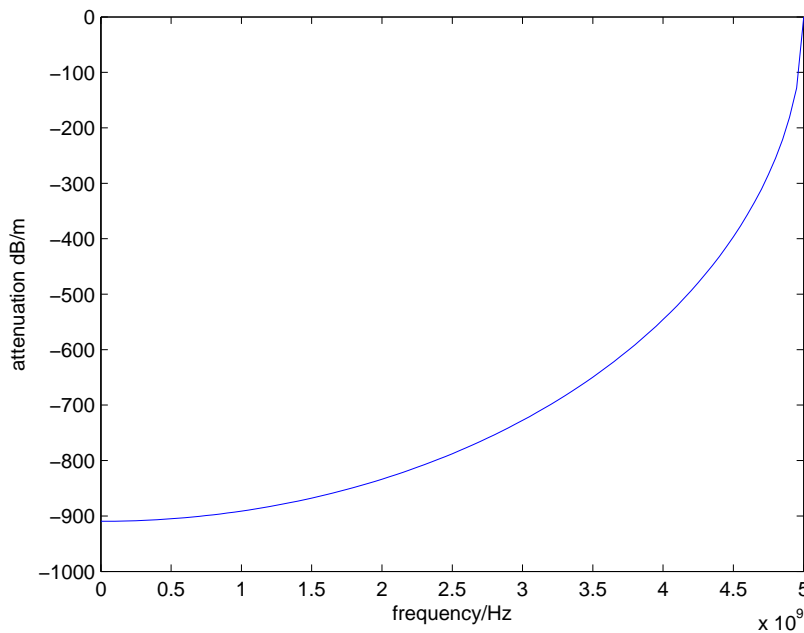
$$\mathbf{H}_T(x, y) = Z_{11}^{-1} \hat{\mathbf{z}} \times \mathbf{E}_T = -\hat{\mathbf{x}} Z_{11}^{-1} \frac{i\omega a}{\pi c_0} \sqrt{\frac{2}{ab}} \sin \frac{\pi x}{a}$$

since  $k_{t10} = \frac{\pi}{a}$ . Here  $Z_{11} = \frac{\omega\mu_0}{k_{z11}}$  is the TE-mode impedance. The  $z$ -component of the complex Poynting vector is

$$S_z = \hat{z} \cdot \frac{1}{2} \mathbf{E}_T \times \mathbf{H}_T^* = (Z_{11}^*)^{-1} \frac{1}{ab} \left( \frac{\omega a}{\pi c_0} \sin \frac{\pi x}{a} \right)^2$$

The time average of the Power flow density is  $\text{Re}\{S_z\}$ . This is zero for frequencies below the cut-off frequency since then  $k_{z11}$  is imaginary but the rest of the factors are real.

d) For frequencies below the cut-off frequencies then  $k_{z10} = i\sqrt{\left(\frac{\pi}{a}\right)^2 - k^2}$  and  $e^{ik_{z10}z} = e^{-\alpha z}$  where  $\alpha = \sqrt{\left(\frac{\pi}{a}\right)^2 - k^2}$ . When  $a = 3$  cm the cut-off frequency is 5 GHz. The attenuation is given in figure 3.

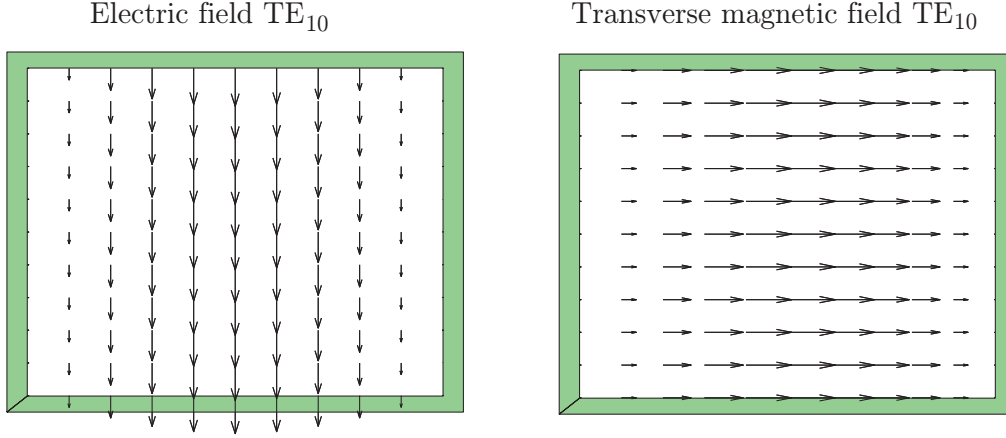


**Figure 2:** The attenuation of the fundamental mode in a rectangular waveguide. The cut-off frequency is 5 GHz.

## S5

The power is attenuated as  $P(z) = P(0)e^{-\alpha_p z}$  in the waveguide. The attenuation constant is for TE-modes given by the formula on page 124 in the book. Let  $a$  denote the radius.

$$\alpha_p = \frac{k_{tmn}^2}{\omega\sigma\delta\mu_0 k_{zmn}} \oint \left( \frac{k_{zmn}^2}{k_{tmn}^4} |\nabla_T w_{mn}|^2 + |w_{mn}|^2 \right) dl$$



**Figure 3:** The electric field and the transverse component of the magnetic field for the fundamental mode in a rectangular waveguide.

Here  $\sigma = 5.8 \cdot 10^7$  is the conductivity of copper and  $\delta = \sqrt{2/(\omega\mu_0\sigma)}$  is the skin depth. For the  $TE_{01}$  mode we have

$$w_{mn}(\rho, \phi) = A_{mn} J_m(k_{tmn}\rho) \cos m\phi$$

where the normalization constant is give by the table on page 112

$$A_{mn} = \sqrt{\frac{\epsilon_m}{\pi(\eta_{mn}^2 - m^2)}} \frac{\eta_{mn}}{J_m(\eta_{mn})}$$

where  $\epsilon_m$  is 1 for  $m = 0$  and 2 otherwise and  $\eta_{mn} = k_{tmn}a$  is the  $n$  :th zero of  $J'_m$ . The gradient of  $w_{mn}$  is given by

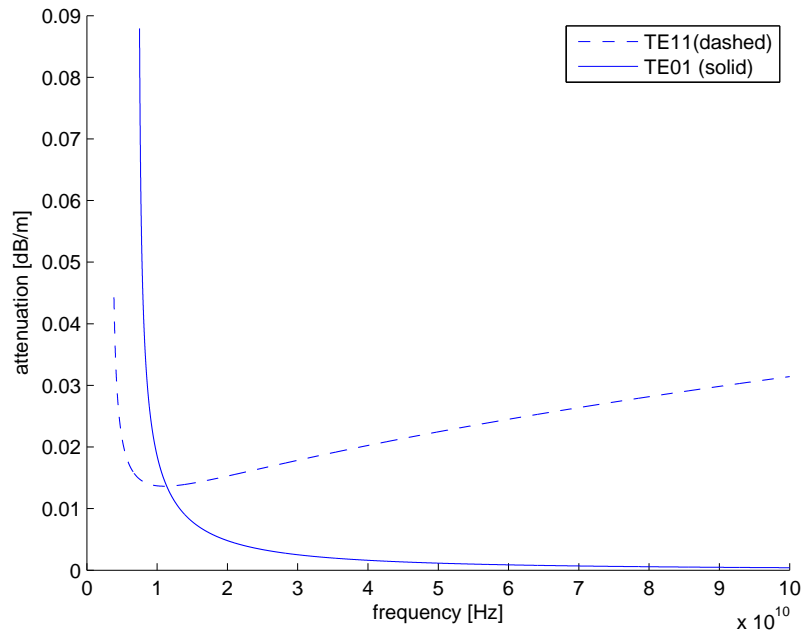
$$\nabla_T w_{mn}(\rho, \phi) = A_{mn} \left( \hat{\rho} k_{tmn} J'_m(k_{tmn}\rho) \cos m\phi - \hat{\phi} \frac{m}{\rho} J_m(k_c\rho) \sin n\phi \right)$$

Since  $J'_m(k_{tmn}a) = 0$  it is seen that

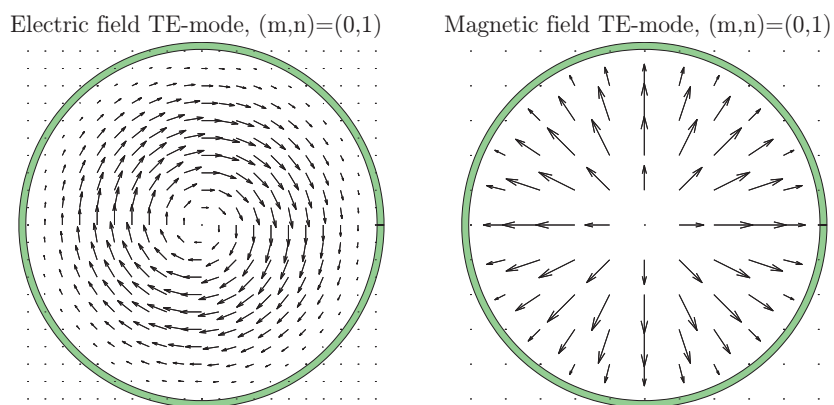
$$\begin{aligned} \alpha_p &= \frac{A_{mn}^2 k_{tmn}^2}{\omega\sigma\delta\mu_0 k_{zmn}} \int_0^{2\pi} \left( \frac{m^2 k_{zmn}^2}{k_{tmn}^4 a^2} \sin^2 m\phi + \cos^2 m\phi \right) (J_m(k_c a))^2 a d\phi \\ &= \frac{A_{mn}^2 k_{tmn}^2 2\pi a}{\epsilon_m \omega\sigma\delta\mu_0 k_{zmn}} \left( \frac{m^2 k_{zmn}^2}{k_{tmn}^4 a^2} + 1 \right) (J_m(\eta_{mn}))^2 \\ &= \frac{2}{\omega\sigma\delta\mu_0 k_{zmn} a} \left( \frac{m^2 k_{zmn}^2 + k_{tmn}^4 a^2}{k_{tmn}^2 a^2 - m^2} \right) \end{aligned}$$

When the values for the  $TE_{01}$  and  $TE_{11}$  modes are inserted in this expression figure 5 is obtained. We see that the attenuation of  $TE_{01}$  decreases with frequency and gives a very small attenuation at high frequencies.





**Figure 4:** The attenuation of the  $TE_{01}$  and  $TE_{11}$  modes as a function of frequency



**Figure 5:** The electric field and the transverse magnetic field for the  $TE_{01}$

## S6

In the coaxial cable the TEM-mode propagates at all frequencies. The  $z$ -dependence is given by

$$\mathbf{E}(\boldsymbol{\rho}, z) = \mathbf{E}(\boldsymbol{\rho})(a^+ e^{ikz} + a^- e^{-ikz})$$

The boundary conditions says that the electric field is zero at  $z = 0$  and  $z = h$ . Thus

$$\begin{aligned} (a^+ + a^-) &= 0 \\ (a^+ e^{ikh} + a^- e^{-ikh}) &= 0 \end{aligned}$$

From this we get  $a^+ = -a^-$  and  $a^+ \sin kh = 0$  and then  $kh = \ell\pi$ ,  $\ell = 1, 2, 3, \dots$ . The resonance frequencies are given by  $f_r = \frac{c\ell}{2h} = \frac{c_0\ell}{2h\sqrt{\varepsilon}}$ . That gives the three lowest resonant frequencies

$$\begin{aligned} f_1 &= \frac{c_0}{2h\sqrt{\varepsilon}} = 0.75 \text{ GHz} \\ f_2 &= \frac{2c_0}{2h\sqrt{\varepsilon}} = 1.5 \text{ GHz} \\ f_3 &= \frac{3c_0}{2h\sqrt{\varepsilon}} = 2.25 \text{ GHz} \end{aligned}$$

There are also TE and TM waveguide modes in the coaxial cables that will have resonances. However, their resonances are much higher than 2.25 GHz.

## S7

We view the parallelepiped as a waveguide in the  $z$ -direction with cross section  $0 < x < a$ ,  $0 < y < b$ . The general theory for finite waveguide cavities gives the resonance frequencies

$$f_{mnl} = \frac{c}{2\pi} \sqrt{k_{tmn}^2 + \left(\frac{\ell\pi}{h}\right)^2} = \frac{c}{2\pi} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{\ell\pi}{h}\right)^2}$$

The  $mnl$  values are

$$\begin{cases} m = 1, 2, 3, \dots, n = 1, 2, 3, \dots, \ell = 0, 1, 2, 3, \dots, \text{ TM-modes} \\ m = 0, 1, 2, 3, \dots, n = 0, 1, 2, 3, \dots, (m, n) \neq (0, 0), \ell = 1, 2, 3, \dots, \text{ TE-modes} \end{cases}$$

Only one of  $mnl$  can be zero. Hence the three lowest resonance frequencies are

$$\begin{aligned} f_{TE101} &= \frac{c}{2\pi} \sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{h}\right)^2} = 4.80 \text{ GHz} \\ f_{TE011} &= \frac{c}{2\pi} \sqrt{\left(\frac{\pi}{b}\right)^2 + \left(\frac{\pi}{h}\right)^2} = 5.83 \text{ GHz} \\ f_{TM110} &= \frac{c}{2\pi} \sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2} = 6.25 \text{ GHz} \end{aligned}$$

## S8

We let the waveguide be directed in the  $z$ -direction with perfectly conducting plates at  $z = 0$  and  $z = h$ . The radius of the waveguide is  $a$ . The general theory for finite waveguide cavities gives the resonance frequencies

$$f_{mn\ell} = \frac{c}{2\pi} \sqrt{k_{tmn}^2 + \left(\frac{\ell\pi}{h}\right)^2}$$

The transverse wavenumbers and the  $\ell$ -values are given by

$$\begin{cases} k_{tmn} = \xi_{mn}/a, & \ell = 0, 1, 2, 3 \dots \text{ TM-modes} \\ k_{tmn} = \eta_{mn}/a, & \ell = 1, 2, 3 \dots \text{ TM-modes} \end{cases}$$

where  $\xi_{mn}$  is the  $n$ :th zero of the Bessel function  $J_m(k_t a)$  and  $\eta_{mn}$  is the  $n$ :th zero of the derivative  $J'_m(k_t a)$ . The lowest resonance frequencies depend on the height  $h$  and the radius  $a$ . The candidates are

1.  $\text{TM}_{010}$  with  $f_c = \frac{c\xi_{01}}{2\pi a} = \frac{3 \cdot 10^8 \cdot 2.405}{2\pi a}$
2.  $\text{TM}_{110}$  with  $f_c = \frac{c\xi_{11}}{2\pi a} = \frac{3 \cdot 10^8 \cdot 3.832}{2\pi a}$
3.  $\text{TE}_{111}$  with  $f_c = \frac{c}{2\pi} \sqrt{\left(\frac{\eta_{11}}{a}\right)^2 + \left(\frac{\pi}{h}\right)^2} = \frac{c}{2\pi} \sqrt{\left(\frac{1.841}{a}\right)^2 + \left(\frac{\pi}{h}\right)^2}$
4.  $\text{TE}_{112}$  with  $f_c = \frac{c}{2\pi} \sqrt{\left(\frac{\eta_{11}}{a}\right)^2 + \left(\frac{2\pi}{h}\right)^2} = \frac{c}{2\pi} \sqrt{\left(\frac{1.841}{a}\right)^2 + \left(\frac{2\pi}{h}\right)^2}$

We have two parameters to determine, namely  $a$  and  $h$ . There are several options. One is to have  $f_{c\text{TM}010} = 10$  GHz and  $f_{c\text{TE}111} = 15$  GHz. This gives  $a = 1.15$  cm and  $h = 1.16$  cm. The next resonance frequency is  $f_{c\text{TM}110} = 15.93$  GHz.

## S9

See Table 6.2.

**S10**

The expressions in the book and the evaluations in COMSOL using Extremely fine mesh give

frequency/MHz	Q-value/ $10^5$ book	Q-value/ $10^5$ COMSOL
130.911744010408	1.2928833946	1.3182895043
184.662441148350	1.2534662007	1.2576461408
214.396074654639	2.2531573086	2.2550594492
237.299051157491	1.2204405958	1.22064638087
274.994531395624	2.5517911552	2.55179068083