Microwave theory 2015: Exercises for week 1 and 2

1–11

Problems 3.6–3.16 in the book.

12

We know that a time harmonic wave that propagates in the positive $z-$direction along a loss-less transmission line can be expressed in terms of the voltage as

$$V(z) = V_p e^{-\mathrm{j}\beta z}$$

The corresponding current is given by

$$I(z) = I_p e^{-\mathrm{j}\beta z} = Z_0^{-1} V_p e^{-\mathrm{j}\beta z}$$

The wave can also be expressed in terms of the electric field as

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}_p(x, y) e^{-\mathrm{j}\beta z}$$

The electric field is a plane wave that propagates in the positive $z-$direction. The relation between the electric and magnetic field is the usual plane wave relation, *i.e.*,

$$\boldsymbol{H}(\boldsymbol{r}) = (\eta_0 \eta)^{-1} \hat{\boldsymbol{z}} \times \boldsymbol{E}(\boldsymbol{r})$$

where $\eta_0 \eta = \sqrt{\frac{\mu_0}{\varepsilon_0 \varepsilon}}$ is the wave impedance.
Now consider two coaxial cables that are identical except that they have different materials between the conductors. The material between the conductors is nonmagnetic and have permittivity ε_1 in cable one and ε_2 in cable two. Cable one extends from $z = 0$ to $z = \ell$ and is at $z = \ell$ connected to cable two, that extends from $z = \ell$ to $z = 2\ell$. At $z = 2\ell$ cable two is connected to a matched load. At $z = 0$ there is a time harmonic source.
a) Determine the reflection coefficient at $z = \ell$ using the voltage description of the line.
b) Determine the reflection coefficient at $z = \ell$ using the electric field description of the line.
c) Show that the reflection coefficients are the same.
d) Show that the expressions for the two reflection coefficients are independent of the geometry of the conductors, as long as the two cables are identical, except for the material between the cables.

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Assume that we can either have the source at $z = 0$ and the matched load at $z = 2\ell$ or the source at $z = 2\ell$ and the matched load at $z = 0$ in the case above. Find the scattering matrix S for the cables when $\ell = \lambda$ and show that the scattering matrix is a unitary matrix ($S^t S^* = U$).

14

The line parameters for a transmission line can often be determined from electro- and magnetostatics. The electrostatic and magnetostatic energies per unit length of a transmission line with air (same as vacuum) between the conductors are given by the surface integrals

$$W_e = \frac{\epsilon_0}{2} \int\limits_S |\boldsymbol{E}(x,y)|^2 \, dS$$
$$W_m = \frac{\mu_0}{2} \int\limits_S |\boldsymbol{H}(x,y)|^2 \, dS$$

where S is the cross-section surface between the cables and $\boldsymbol{E}$, $\boldsymbol{H}$ denote the electrostatic and magnetostatic fields.
a) Give an expression for the capacitance per unit length for a transmission line. The expression should contain W_e and the static voltage V between the cables.
b) Use the expression to determine the capacitance per unit length for a coaxial cable.
c) Give an expression for the inductance per unit length for a transmission line. The expression should contain W_m and the current I of the inner conductor.
d) Use the expression to determine the inductance per unit length for a coaxial cable. Assume that the material is non-magnetic.

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The time average of the electric and magnetic energies per unit length for a time-harmonic wave that travels in the positive $z-$direction along a loss-less transmission line are

$$W_e = \frac{1}{4} C |V(z)|^2$$
$$W_m = \frac{1}{4} L |I(z)|^2$$

Show that these two energies are the same.
Do the same by using the electric and magnetic fields.

16

Derive the analytic expressions for R, L, G and C for the parallel plate on page 62. The medium around the plates is homogeneous with permittivity ε and conductivity σ_s. The conductivity of the metal in the plates is σ_c and the thicknesses of the plates are much larger than the skin depth.

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Determine the analytic expressions for R, L, G and C for a microstrip above a ground plane. The microstrip has width b and the distance to the ground plane is a. The material around the microstrip is homogeneous with permittivity ε and conductivity σ_s. The conductivity of the metal in the plates is σ_c and the thicknesses of the strip and the ground plane are much larger than the skin depth.

18

Sketch the electric and magnetic fields around the two-wire line. Do not make any explicit calculations.

19

Consider a two-wire line with circular conductors with radius a and with distance c between the centers of the circles.
a) What limiting values do R, L, G and C get when $c \to 2a$? Try to give some physical explanation to the limiting values.
b) Assume that there is air (same as vacuum) between the wires. What limiting value does the characteristic impedance get when $c \to 2a$?
c) Determine c expressed in a such that the characteristic impedance is 50 Ω when there is air between the wires.

20

Sketch the electric and magnetic fields around the parallel plate. Do not make any explicit calculations.

21

Assume that $a/b \to 0$ in the parallel plate. What values do R, L, G and C approach? Try to give some physical explanation.

22

Assume the two cases of two-port matching on page 53. Describe how these matching can be done by using the Smith charts (both impedance and admittance charts are needed). The explicit expressions for the impedances should not be used.

Solutions to problems 12-22

Solution problem 12

For convenience we use $z' = z - \ell$ as our $z-$coordinate. That means that the interface between the coaxial cables is at $z' = 0$.
a) Let the characteristic impedance be Z_1 for the coaxial cable to the left and Z_2 for the coaxial cable to the right. The input impedance at $z' = 0$ of the coaxial cable to the right is then Z_2. Thus the reflection coefficient is

$$\Gamma = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$

When we insert the expressions for the characteristic impedance for the coaxial cable we get

$$\Gamma = \frac{\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}}{\sqrt{\varepsilon_2} + \sqrt{\varepsilon_1}}$$

b) The electric and magnetic fields along the cables are written as

$$\begin{aligned}
\boldsymbol{E}_1 &= \boldsymbol{E}_p e^{-\mathrm{j}\beta_1 z'} + \boldsymbol{E}_n e^{\mathrm{j}\beta_1 z'} = \boldsymbol{E}_p e^{-\mathrm{j}\beta_1 z'} + R\boldsymbol{E}_p e^{\mathrm{j}\beta_1 z'} \\
\boldsymbol{H}_1 &= \boldsymbol{H}_p e^{-\mathrm{j}\beta_1 z'} + \boldsymbol{H}_n e^{\mathrm{j}\beta_1 z'} = (\eta_0\eta_1)^{-1}(\hat{z} \times \boldsymbol{E}_p e^{-\mathrm{j}\beta_1 z'} - R\hat{z} \times \boldsymbol{E}_p e^{\mathrm{j}\beta_1 z'})
\end{aligned}$$

for $z' < 0$ and

$$\begin{aligned}
\boldsymbol{E}_2 &= T\boldsymbol{E}_p e^{-\mathrm{j}\beta_2 z'} \\
\boldsymbol{H}_2 &= (\eta_0\eta_2)^{-1} T\hat{\boldsymbol{z}} \times \boldsymbol{E}_p e^{-\mathrm{j}\beta_2 z'}
\end{aligned}$$

where we have introduced the transmission coefficient T. The reflection and transmission coefficients are obtained from the boundary conditions at $z' = 0$, *i.e.*, that the electric and magnetic fields are continuous. We then get

$$\begin{aligned}
&1 + R = T \\
&(\eta_0\eta_1)^{-1}(1 - R) = (\eta_0\eta_2)^{-1} T
\end{aligned}$$

This gives

$$\begin{aligned}
R &= \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}} \\
T &= \frac{2\eta_2}{\eta_2 + \eta_1} = \frac{2\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}}
\end{aligned}$$

c) It is clear that the reflection and transmission coefficients for the electromagnetic fields are valid for all types of transmission lines. To show that this is the case when voltages and currents are used we write the reflection coefficient as

$$\Gamma = \frac{\sqrt{\frac{C_1}{C_2}} - \sqrt{\frac{L_1}{L_2}}}{\sqrt{\frac{C_1}{C_2}} + \sqrt{\frac{L_1}{L_2}}} \tag{0.1}$$

From the derivation of the line parameters we see that when the two lines are identical except for the permittivity then $C_1/C_2 = \varepsilon_1/\varepsilon_2$ and $L_1 = L_2$. Thus

$$\Gamma = \frac{\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}}$$

An alternative is to multiply the nominator and denominator of Eq. (0.1) with $\sqrt{\frac{L_1}{L_2}}$ and use $v_p = \dfrac{1}{\sqrt{LC}} = \dfrac{1}{\sqrt{\epsilon_0 \epsilon \mu_0}}$

$$\Gamma = \frac{\sqrt{\dfrac{L_1 C_1}{L_2 C_2}} - \dfrac{L_1}{L_2}}{\sqrt{\dfrac{L_1 C_1}{L_2 C_2}} + \dfrac{L_1}{L_2}} = \frac{\dfrac{v_{p2}}{v_{p1}} - \dfrac{L_1}{L_2}}{\dfrac{v_{p2}}{v_{p1}} + \dfrac{L_1}{L_2}} = \frac{\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}}$$

Solution problem 13

Since the characteristic impedances of the two lines are not equal the scattering matrix is given by Eq. (4.39) in the book. Thus

$$\begin{pmatrix} V_1^- \\ V_2^- \end{pmatrix} = \begin{pmatrix} S_{11} & \sqrt{\dfrac{Z_1}{Z_2}} S_{12} \\ \sqrt{\dfrac{Z_2}{Z_1}} S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} V_1^+ \\ V_2^+ \end{pmatrix}$$

where $Z_2/Z_1 = \sqrt{\epsilon_1/\epsilon_2}$. It is clear that S_{11} is the reflection coefficient from an incident wave from the left, $\sqrt{\dfrac{Z_2}{Z_1}} S_{21}$ is the transmission coefficient from left to right, S_{22} is the reflection coefficient for a wave that is incident from right, and $\sqrt{\dfrac{Z_1}{Z_2}} S_{12}$ is the transmission coefficient from the right to left. Since the transmission lines are one wavelength long the reflection and transmission coefficient are the same as at $z' = 0$. Then we get

$$\begin{aligned} S_{11} &= \frac{\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}} \\ S_{21} &= \sqrt{\frac{Z_1}{Z_2}} \frac{2\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}} = \frac{2(\varepsilon_1 \varepsilon_2)^{1/4}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}} \\ S_{22} &= -S_{11} \\ S_{12} &= S_{21} \end{aligned}$$

Note: The unitary condition $[S]^t [S]^* = U$ is still satisfied in this case.

Solution problem 14

a) The electrostatic energy in a capacitor C with DC voltage V is $\frac{1}{2}C|V|^2$. Thus

$$\frac{1}{2}C|V|^2 = W_e$$

and

$$C = \frac{2W_e}{|V|^2}$$

b) To get the electric field in the coaxial cable we let the inner conductor have potential V and the outer conductor have potential 0. Due to axial symmetry the electrostatic potential Φ is only dependent on the radial coordinate r_c. In cylindrical coordinates we get

$$\nabla^2\Phi(r_c) = \frac{1}{r_c}\frac{\partial}{\partial r_c} r_c \frac{\partial \Phi(r_c)}{\partial r_c}$$

with boundary conditions $\Phi(a) = V$ and $\Phi(b) = 0$. By integrating two times and by using the boundary conditions we get the solution

$$\Phi(r_c) = V\frac{\ln(r_c/b)}{\ln(a/b)}$$

the electric field is

$$\boldsymbol{E}(r_c) = -\nabla\Phi(r_c) = -\frac{\partial\Phi(r_c)}{\partial r_c} = -\frac{V}{r_c\ln(a/b)} = \frac{V}{r_c\ln(b/a)}$$

The energy per unit length along the coaxial cable is

$$W_e = \frac{\epsilon_0}{2}\int_S |\boldsymbol{E}(r_c)|^2\, dS = \frac{1}{2}\epsilon_0\left(\frac{V}{\ln(b/a)}\right)^2 2\pi\int_a^b \frac{1}{r_c^2} r_c\, dr_c = \epsilon_0\frac{V^2\pi}{\ln(b/a)}$$

The capacitance per unit length becomes

$$C = \frac{2W_e}{V^2} = \varepsilon_0\frac{2\pi}{\ln(b/a)}$$

c) The magnetic energy in an inductor with current I is $\frac{1}{2}LI^2$. Thus

$$L = \frac{2W_m}{I^2}$$

d) Due to the axial symmetry the magnetic field between the conductors is given by Amperes law as $\boldsymbol{H} = \frac{I}{2\pi r_c}\hat{\boldsymbol{\phi}}$. Since the material is non-magnetic we get

$$W_m = \frac{\mu_0}{2}2\pi\int_a^b \frac{I^2}{4\pi^2 r_c^2} r_c\, dr_c = \frac{\mu_0}{4\pi}I^2\ln\left(\frac{b}{a}\right)$$

The inductance per unit length is given by

$$L = \frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right)$$

Solution problem 15

Since $\dfrac{V(z)}{I(z)} = \dfrac{V^+(z)}{I^+(z)} = Z_0 = \sqrt{\dfrac{L}{C}}$ for a wave traveling in the positive $z-$direction we get

$$W_e = \frac{1}{4}C|V(z)|^2 = \frac{1}{4}C\frac{L}{C}|I(z)|^2 = W_m$$

We use the plane wave relation $\boldsymbol{E} = \eta_0\eta\boldsymbol{H}\times\hat{\boldsymbol{z}}$, where $\eta_0\eta = \sqrt{\mu_0/\epsilon_0\epsilon}$ is the wave impedance, to get

$$W_e = \frac{\epsilon\epsilon_0}{4}\int_S \boldsymbol{E}\cdot\boldsymbol{E}^*\,dS = \frac{\epsilon_0\epsilon}{4}\frac{\mu_0}{\epsilon_0\epsilon}\int_S \boldsymbol{H}\cdot\boldsymbol{H}^*\,dS = W_m$$

Solution problem 16

The capacitance for a plate capacitor with plate area A and distance a between the plates is

$$C = \frac{\varepsilon_0\varepsilon A}{a}$$

Thus the capacitance per unit length for a parallel plate line with width b and distance a between the plates is

$$C = \frac{\varepsilon_0\varepsilon b}{a}$$

We now use the relations

$$L = \frac{1}{v_p^2 C} = \frac{\mu_0 a}{b}$$
$$G = \frac{\sigma_s}{\varepsilon_0\varepsilon}C = \frac{\sigma_s b}{a}$$

For the resistance we utilize the surface resistance is R_s. The surface current is only floating in the upper surface of the lower plate and on the lower surface of the upper plate. Each surface can be viewed as parallel coupled resistors with width dx and resistance per unit length R_s/dx. The resistance per unit length of each conductor is then given by

$$\frac{1}{R_c} = \int_0^b \frac{1}{R_s}dx = \frac{b}{R_s}$$

and then the total resistance is

$$R = 2R_c = \frac{2R_s}{b}$$

where $R_s = \sqrt{\dfrac{\omega\mu_0}{2\sigma_c}}$. An alternative is to use the relation $P = \dfrac{1}{2}R_s|\boldsymbol{J}_s|^2$ =dissipated power per unit area. The current density is constant on the upper surface of the lower plate and on the lower surface of the upper plate, but zero everywhere else.

Thus it equals $\boldsymbol{J}_s = I/b\hat{\boldsymbol{z}}$ on the upper plate. The dissipated power per unit length is then

$$\frac{1}{2}R|I|^2 = 2bP = R_s\frac{|I|^2}{b}$$

and hence

$$R = \frac{2R_s}{b}$$

Solution problem 17

The analysis is exactly the same as for the parallel plate transmission line and the parameters are also the same.

Solution problem 18

The magnetic field lines are given in figure 4.16 in the book. The electric field lines are everywhere perpendicular to the magnetic field lines. They start at the surface of one conductor and end at the surface of the other.

Solution problem 19

a) When $c \to 2a$ we see from the analytical expressions for the two-wire that $R \to \infty$, $L \to 0$, $C \to \infty$ and $G \to \infty$. R goes to infinity since the surface current density is confined to the spot where the two cylinders almost touch. We can use the plate capacitor formula to understand that the capacitance goes to infinity. The distance between the conductors goes to zero but the surface is finite and hence the capacitance goes to infinity. The conductance goes to infinity since the resistance between the conductors goes to zero. It is harder to see why the inductance is zero. The surface current density will be finite even when the distance between the circles goes to zero. One way to see this is to magnify the region where the wires are closest to each other. This region will look like two parallell planes very close to each other. Hence the surface current density is finite. Thus the magnetic flow density is finite everywhere and the magnetic flow goes to zero.

b) $Z_0 \to 0$

c) $c = a\dfrac{e^{10/12} - 1}{e^{5/12}} = 2.176a$

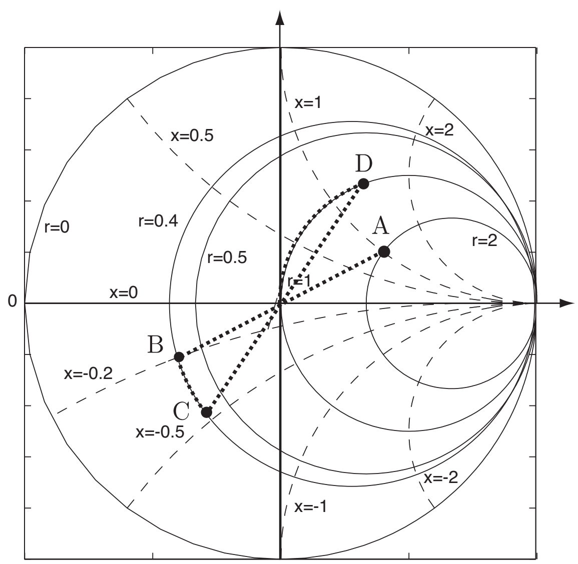
Solution problem 20

The magnetic field lines are given in figure 4.18 in the book. The electric field lines are everywhere perpendicular to the magnetic field lines. They start at the surface of one conductor and end at the surface of the other. Most of the field lines start from the upper surface of the lower conductor and goes straight up to the lower surface of the upper conductor.

Solution problem 21

From the formulas on page 62 we see that R is still finite with the value $2R_s/b$, $L \to 0$, $C \to \infty$ and $G \to \infty$. The magnetic field between the plates is given by the surface current density I/b ($\boldsymbol{H} = I/b\hat{\boldsymbol{x}}$ between the plates due to Amperes law). Thus the magnetic flow between the plates is $\mu_0 aI/b$ and hence L goes to zero. C goes to infinity according to the parallel plate capacitor formula and G goes to infinity since the resistance between the plates goes to zero.

Solution problem 22



The Smith chart: Assume that we like to match a load Z_L to a transmission line with characteristic impedance Z_0. First we introduce the normalized impedance $z_L = Z_L/Z_0$. The trick is to go from normalized impedance z_L to normalized impedance 1. The rules are that we are only aloud to move along curves with constant resistance or constant conductance. As an example consider a load impedance $Z_L = 2Z_0 + \mathrm{j}Z_0$. Then $z_L = 2 + \mathrm{j}$. This is point A in the Smith chart. It is useless to add reactance. Instead we go to the admittance $y_L = Y_L/Y_0 = Z_0/Z_L = 1/z_L$ at point B. We find this point in the Smith chart by drawing a line from z_l through $z = 1$ and let the length between z_L and $z = 1$ be the same as between $z = y = 1$ and y_L. From y_L we add a susceptance $b = -0.3$ and reach C. Then we are at a point y for which the impedance z is on the circle that goes through $z = y = 1$. We go back to z (point D) and add a reactance $x = -1.27$ and end up at $z = 1$. The load is now matched to the line. We first added a susceptance $B = bY_0 \approx -0.3/50 = -0.006$ A/V parallel to Z_L. We then added a reactance $X = xZ_0 \approx -1.27 \cdot 50 = -63.5$ V/A in series with the parallel coupled impedances.

If we start with an impedance with $R_L < Z_0$ then we have to add a reactance first and then a susceptance. As an example we start with the impedance $Z_L = 0.5Z_0 + \mathrm{j}Z_0$. Then $z_L = 0.5 + \mathrm{j}$. We find this impedance in the Smith chart. We add $x_L = -0.5$ which gives $z = 0.5 + \mathrm{j}0.5$. From there we switch to admittance $y_l = 1 - \mathrm{j}$. We add a susceptance $b = 1$ to get to $y = z = 1$ and the goal is reached. Thus we have first put a reactance $X = -0.5 \cdot 50 = -25$ V/A in series with Z_L and then put a susceptance $B = \mathrm{j}1/50 = \mathrm{j}0.02$ A/V in parallel with the two series coupled impedances. We can compare with the formulas on page 7. When $Z_L = 2Z_0 + \mathrm{j}Z_0$ we should first put a susceptance B

$$B = \pm\sqrt{\frac{G_L(1 - Z_0G_L)}{Z_0}} - B_L$$

in parallel with Z_L and then a reactance

$$X = \pm Z_0\sqrt{\frac{1 - Z_0G_L}{Z_0G_L}}$$

in series. The values are $B = -0.0058$ A/V (- sign used) and $X = -61.2$ V/A. A more careful analysis in the Smith chart gives the same values as the formulas.
When $Z_L = 0.5Z_0 + \mathrm{j}Z0$ we should first put a reactance

$$X = \pm\sqrt{R_L(Z_0 - R_L)} - X_L$$

in series and then

$$B = \pm\frac{\sqrt{(Z_0 - R_L)/R_L}}{Z_0}$$

in parallel. The values are $X = -25$ V/A (+ sign used) and $B = 0.02$ A/V. Which is exactly what we got from the Smith chart.