Microwave theory 2015: Problems week 3 and 4

Problems

Problems 5.1, 5.2, 5.4–5.8 in the book.

Problem V3.1

Consider a waveguide filled with air and with perfectly conducting walls.

a) Show that the complex Poynting vector $\boldsymbol{S}(\boldsymbol{r}) = \boldsymbol{E}(\boldsymbol{r}) \times \boldsymbol{H}^*(\boldsymbol{r})$ for a waveguide mode is purely imaginary for frequencies below the cut-off frequency of the mode.

b) Show that the z-component of the complex Poynting vector is real and the transverse components are imaginary for frequencies above the cut-off frequency.

c) Give a physical explanation of the statements in a) and b).

Problem V3.2

A circular waveguide of length $0 < z < L$ and radius a has a non-reflecting termination at $z = L$. In the region $\frac{L}{4} < z < \frac{L}{2}$ one has inserted a circular metal cylinder with radius $b < a$ and very thin wall.

Determine the radius b such that it is only the TM_{02} mode that can propagate from $z = 0$ to $z = L$ without reflections from the inner cylinder.

Solutions

5.1

We analyze a TM-mode here. The analysis of the TE-mode is done in the same manner.

For a TM-mode in a planar waveguide the electric field reads, cf section 5.5.1:

$$\boldsymbol{E}_n(\boldsymbol{r}) = (\boldsymbol{E}_{Tn}(y) + v_n(y)\hat{z})e^{\mathrm{i}k_{zn}z}$$

where

$$v_n(y) = \sqrt{\frac{2}{b}} \sin k_{tn} y$$

$$\boldsymbol{E}_{Tn}(y) = \mathrm{i}\frac{k_{zn}}{k_{tn}^2}\nabla_T v_n(y) = \mathrm{i}\hat{y}\frac{k_{zn}}{k_{tn}}\sqrt{\frac{2}{b}} \cos k_{tn} y$$

The electric field then reads

$$\begin{aligned}\boldsymbol{E}_n(\boldsymbol{r}) &= \sqrt{\frac{2}{b}}\frac{1}{k_{tn}}\left(\hat{y}\mathrm{i}k_{zn}\cos k_{tn}y + \hat{z}k_{tn}\sin k_{tn}y\right)e^{\mathrm{i}k_{zn}z} \\ &= \sqrt{\frac{2}{b}}\frac{\mathrm{i}}{2k_{tn}}\left((0, k_{zn}, -k_{tn})e^{\mathrm{i}(k_{tn}y + k_{zn}z)} + (0, k_{zn}, k_{tn})e^{\mathrm{i}(-k_{tn}y + k_{zn}z)}\right)\end{aligned}$$

We notice that the field consists of two planar transverse waves with directions of propagation $\hat{k}_1 = (0, k_{tn}, k_{zn})/k$ and $\hat{k}_2 = (0, -k_{tn}, k_{zn})/k$. The electric fields are directed along $\hat{k}_1 \times \hat{x}$ and $\hat{k}_2 \times \hat{x}$, respectively.

We need to determine the corresponding magnetic fields by using the plane wave relation $\boldsymbol{H} = \eta_0^{-1}\hat{k} \times \boldsymbol{E}$ for each of the two plane waves. This gives

$$\begin{aligned}\boldsymbol{H}_n(\boldsymbol{r}) &= -\eta_0^{-1}\sqrt{\frac{2}{b}}\frac{\mathrm{i}}{2k_{tn}}\frac{1}{k}\left((k_{tn}^2 + k_{zn}^2, 0, 0)e^{\mathrm{i}(k_{tn}y+k_{zn}z)} + (k_{tn}^2 + k_{zn}^2, 0, 0)e^{\mathrm{i}(-k_{tn}y+k_{zn}z)}\right)\\ &= -\hat{x}\eta_0^{-1}\sqrt{\frac{2}{b}}\frac{\mathrm{i}}{k_{tn}}k\cos k_{tn}ye^{\mathrm{i}k_{zn}z}\end{aligned}$$

This is the same expression as in equation 5.24 *i.e.*,

$$\begin{aligned}\boldsymbol{H}_n(\boldsymbol{r}) &= \eta_0^{-1}\frac{\mathrm{i}}{k_{tn}^2}k\epsilon\hat{z} \times \nabla_T v_n(\boldsymbol{\rho})\\ &= -\hat{x}\eta_0^{-1}\sqrt{\frac{2}{b}}\frac{\mathrm{i}}{k_{tn}}k\cos k_{tn}ye^{\mathrm{i}k_{zn}z}\end{aligned}$$

5.2

Assume a waveguide mode that is propagating in the positive z-direction in a conducting material with $\mathrm{Im}\epsilon\mu > 0$. The time average of the power transported by the modes is, *cf.*, page 117

$$P = \mathrm{Re}P_{n\nu}^E|a_{n\nu}^+|^2e^{-2\mathrm{Im}k_{zn}z}$$

where $P_{n\nu}^E$ is given by equation 5.36

$$P_{n\nu}^E = \frac{\omega}{k_{tn}^2(\omega)}\begin{cases} k_{zn}\epsilon_0\epsilon^* & \nu = \mathrm{TM}\\ k_{zn}^*\mu_0\mu & \nu = \mathrm{TE}\end{cases}$$

For a passive material $\mathrm{Re}\epsilon\mu > 0$, $\mathrm{Im}\epsilon\mu > 0$ and

$$k_{zn} = \sqrt{k^2 - k_{tn}^2} = \sqrt{\omega^2(\mathrm{Re}\{\epsilon\mu\} + \mathrm{i}\mathrm{Im}\{\epsilon\mu\})/c_0^2 - k_{tn}^2} = \alpha + \mathrm{i}\beta$$

where $\alpha > 0$ and $\beta > 0$. This implies $\mathrm{Re}k_{zn}\epsilon^* = \alpha\mathrm{Re}\epsilon + \beta\mathrm{Im}\epsilon > 0$ and $\mathrm{Re}P_{nTM}^E > 0$ and hence $P > 0$ for TM-modes. In the same manner we can show that $\mathrm{Re}Y_{nTE}^E > 0$.

The power transported through a cross section $z = z_0$ of the waveguide for frequencies below the cut-off frequency is transferred to heat in the region $z > z_0$.

5.4

We only consider TE-modes. For a TE-mode $H_z(\boldsymbol{r}) = w(\rho, \phi)e^{ik_zz}$ where

$$\begin{aligned}&\nabla_T^2 w + k_t^2 w = 0\\ &\frac{\partial w}{\partial\rho}(a, \phi) = \frac{\partial w}{\partial\phi}(\rho, 0) = \frac{\partial w}{\partial\phi}(a, 2\pi) = 0\\ &w(\rho, \phi) \text{ finite}\end{aligned}$$

Separation of variables $w = f(\rho)g(\phi)$ inserted in the Helmholtz equation $\nabla_T^2 w + k_t^2 w = 0$ gives the eigenvalue problem

$$g''(\phi) + \gamma g(\phi) = 0 \tag{0.1}$$
$$g'(0) = g'(2\pi) = 0 \tag{0.2}$$

Equation (0.1) gives $g(\phi) = A\sin\sqrt{\gamma}\phi + B\cos\sqrt{\gamma}\phi$ where equation (0.2) gives $A = 0$ and $\gamma = (m/2)^2$, $m = 0, 1, \ldots$. In the ρ-direction we get the Bessel differential equation

$$\begin{aligned}&\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial f(\rho)}{\partial\rho} + \left(k_t^2 - \left(\frac{m}{2\rho}\right)^2\right) f(\rho) = 0\\ &f'(a) = 0\end{aligned}$$

with the finite solutions

$$f_{mn}(\rho) = J_{m/2}(k_t\rho) = J_{m/2}(\eta_{m/2,n}\rho/a)$$

where $J'_{m/2}(\eta_{m/2,n}) = 0$. The normalized modes are given by

$$w_{mn} = BJ_{m/2}(k_{tmn}\rho)\cos m\phi/2$$

where $k_{tmn} = \eta_{m/2,n}/a$ and

$$B = \sqrt{\frac{\epsilon_m}{2\pi}}\left(\int_0^a \left(J_{m/2}(k_{tmn}\rho)\right)^2 \rho d\rho\right)^{-1/2}$$

with $\epsilon_m = 2 - \delta_{m,0}$.

Cut-off frequencies

For even m we get the zeros of $J'_{m/2}$ from appendix A. For odd m we can utilize that $J'_\nu(x) = \frac{\nu}{x}J_\nu(x) - J_{\nu+1}(x)$ and determine the zeros numerically in *e.g.*, Matlab. This gives the following values of $\eta_{m/2,n}$

	n=1	n=2	n=3	n=4
m=0	3.832	7.016	10.17	13.32
m=1	1.1656	4.60	7.79	19.95
m=2	1.841	5.331	8.536	11.71
m=3	2.46	6.03	9.26	12.44

We see that $TE_{1/2,1}$ has the lowest cut-off frequency $f_{1/2,1} = c_0 1.1656/(2\pi a)$. Without the metal plate, according to table 3.4 in the book, the TE_{11} mode has the lowest cut-off frequency $f_{1,1} = c_0 1.841/(2\pi a)$. The cut-off frequency for the fundamental mode is then reduced by almost 40% when the metal plate is introduced.

5.5

We utilize the solution to Example 5.8 in the book. For $z<0$ there is an incident an a reflected TM-mode and for $z>0$ a transmitted TM-mode.

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{r}) &= \boldsymbol{E}^{+}_{nTM}(\boldsymbol{r}) + r_n\boldsymbol{E}^{-}_{nTM}(\boldsymbol{r})\\ \boldsymbol{H}(\boldsymbol{r}) &= \boldsymbol{H}^{+}_{nTM}(\boldsymbol{r}) + r_n\boldsymbol{H}^{-}_{nTM}(\boldsymbol{r})\end{aligned} \qquad z\le 0$$

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{r}) &= t_n\boldsymbol{E}^{+}_{nTM}(\boldsymbol{r})\\ \boldsymbol{H}(\boldsymbol{r}) &= t_n\boldsymbol{H}^{+}_{nTM}(\boldsymbol{r})\end{aligned} \qquad z\ge 0$$

where r_n is the reflection coefficient and t_n is the transmission coefficient. We let the amplitude of the incident wave be 1 V/m. The boundary conditions at $z=0$ imply that the transverse components of $\boldsymbol{E}$ and $\boldsymbol{H}$ are continuous. Since $v_n(\boldsymbol{\rho})$ and k_{tn} are independent of z it follows that

$$\begin{aligned}&k_{zn}(1+r_n) = \tilde{k}_{zn}t_n\\ &1-r_n = \epsilon t_n\end{aligned}$$

where $k_{zn}=\sqrt{(\omega/c_0)^2-k_{tn}^2}$ and $\tilde{k}_{zn}=\sqrt{(\omega/c_0)^2\epsilon-k_{tn}^2}$ are the longitudinal wave numbers for $z<0$ and $z>0$, respectively. The solution is given by

$$r_n = \frac{\tilde{k}_{zn}-\epsilon k_{zn}}{\tilde{k}_{zn}+\epsilon k_{zn}}$$

If the TM mode number n is above cut-off then k_{zn} and the mode power Y^E_{nTM} are real, see equation 5.36. The power transport in the waveguide in the region $z<0$ is given by equation (5.36)

$$P_i - P_r = \iint_\Omega \hat{z}\cdot < \boldsymbol{S}(t) > (\boldsymbol{r},\omega)dxdy = P^E_{nTM}|a^+_{0TM}|^2(1-|r_n|^2)$$

where $P_i = P^E_{nTM}|a^+_{0TM}|^2$ is the power of the incident mode and $P_r = P^E_{nTM}|a^+_{0TM}|^2|r_n|^2$ is the power of the reflected mode. Thus

$$\frac{P_r}{P_i} = |r_n|^2$$

It is given that $a=4cm$, $b=3cm$ and $\epsilon=2$. The fundamental TM-mode is TM_{11}. The frequency is chosen such that it is the same as the cut-off frequency for the second TM-mode, *i.e.*, TM_{21} that has $k^2_{t21}=(2\pi/a)^2+(\pi/b)^2$. For $z<0$ this corresponds to the frequency

$$f_{21} = k_{t21}\frac{c_0}{2\pi} = 9\text{GHz}$$

This gives

$$\frac{P_r}{P_i} = 6.1\cdot 10^{-3}$$

b) $P_r=0$ when $r_{11}=0$ *i.e.*, when $\tilde{k}_{zn}=\epsilon k_{zn}$. This gives

$$f = \frac{1}{2\pi}\sqrt{\frac{1+\epsilon}{\epsilon}}c_0k_{t11}$$

The numerical value is $f=7.65$ GHz.

5.6

a) We first determine the modes that can propagate when $a = 3$ cm and $f = 5$ GHz. The lowest cut-off frequencies are obtained from the tables of zeros for $J_m(x)$ (för TM) and $J'_m(x)$ (för TE) in appendix A

$$f_{11}^{TE} = \frac{c_0}{2\pi}\frac{1.841}{3}10^2 = 2.93\,\mathrm{GHz} < 5\,\mathrm{GHz} \tag{0.3}$$

$$f_{21}^{TE} = \frac{c_0}{2\pi}\frac{3.053}{3}10^2 = 4.86\,\mathrm{GHz} < 5\,\mathrm{GHz} \tag{0.4}$$

$$f_{01}^{TM} = \frac{c_0}{2\pi}\frac{2.405}{3}10^2 = 3.83\,\mathrm{GHz} < 5\,\mathrm{GHz} \tag{0.5}$$

The next modes are f_{01}^{TE} and f_{11}^{TM} which both are non-propagating modes since they have cut-off frequency 6.1 GHz.

b) The waveguide is filled with a plastic material with $\sigma = 10^{-11}$ S and $\varepsilon = 3$. The z-dependence of the fundamental mode TE_{11} is given by $e^{\mathrm{i}k_z z}$ where $k_z = \sqrt{k^2 - k_{t11}^2}$. The wave number k is given by

$$k^2 = \left(\frac{\omega}{c_0}\right)^2 \epsilon_{ny} = \left(\frac{\omega}{c_0}\right)^2 \left(\epsilon + \mathrm{i}\frac{\sigma}{\omega\epsilon_0}\right)$$

It is seen that $\sigma/(\epsilon\epsilon_0) \approx 10^{-11}/(3\cdot 8.854\,10^{-12})$ In the microwave region $\sigma/(\omega\epsilon\epsilon_0) \ll 1$ and the following approximations are valid

$$\begin{aligned} k_z = (k^2 - k_{t11}^2)^{1/2} &= \left(\left(\frac{\omega}{c_0}\right)^2 \epsilon - k_{t11}^2\right)^{1/2} \left(1 + \mathrm{i}\frac{\sigma\omega\mu_0}{((\omega/c_0)^2\epsilon - k_{t11}^2)}\right)^{1/2} \\ &\approx \left(\left(\frac{\omega}{c_0}\right)^2 \epsilon - k_{t11}^2\right)^{1/2} \left(1 + \mathrm{i}\frac{\sigma\omega\mu_0}{2((\omega/c_0)^2\epsilon - k_{t11}^2)}\right) \end{aligned}$$

and hence $k_z = \mathrm{Re}(k_z) + \mathrm{i}\,\mathrm{Im}(k_z)$ where

$$\mathrm{Im}(k_z) = \frac{\sigma\omega\mu_0}{2}\left((\omega/c_0)^2\epsilon - k_{t11}^2\right)^{-1/2} = \frac{\sigma\eta}{2}\left(1 - (f_c/f)^2\right)^{-1/2}$$

where $f_c = c_0\xi_{11}/(2\pi a\sqrt{\epsilon})$ och $\eta = \sqrt{\mu_0/(\epsilon\epsilon_0)}$ =wave impedance. The numerical value is $f_c = 1.7$ GHz.

5.7

For the TE$_{10}$-mode the electric field in the region $z < 0$ is

$$\boldsymbol{E}(\boldsymbol{r}) = \hat{y}E_0 \sin\frac{\pi x}{a} e^{ik_z z}$$

This is the fundamental mode with cut-off frequency $f_c = 0.5c_0/b = 2.5$ GHz. When this mode hits the plate it couples to the TE$_{m0}$-modes in $z > 0$ and to the reflected TE$_{m0}$-modes in $z < 0$.

Assume that $x_0 > a/2$. The fundamental mode in $z > 0$, $x < x_0$ is TE$_{10}$. This mode has the cut-off frequency $f_c = 0.5c_0/x_0$.

a) According to the text, power propagates in $z > 0$ for frequencies above 3.75 GHz. This means that 3.75 GHz is the cut-off frequency for the fundamental mode TE_{10} in $x < x_0$. Hence the plate is placed at $x_0 = 0.5c_0/f_c = 0.5 \cdot 3 \cdot 10^8/3.75 \cdot 10^9 = 4$ cm.

b) The electric for the TE_{03}-mode in $z < 0$

$$\boldsymbol{E}(\boldsymbol{r}) = \hat{x}E_0 \sin\frac{3\pi y}{b} e^{\mathrm{i}k_z z}$$

and then the boundary condition at $x = x_0$ is already satisfied since the tangential component is zero. Thus $P_r/P_i = 0$. The corresponding cut-off frequency is at $f_c = 3 \cdot 0.5 \cdot c_0/b = 15$ GHz and hence the mode propagates at 20 GHz.

c) The electric field for the TE_{30}-mode in $z < 0$ is

$$\boldsymbol{E}(\boldsymbol{r}) = \hat{y}E_0 \sin\frac{3\pi x}{a} e^{\mathrm{i}k_z z}$$

The corresponding cut-off frequency is at $f_c = 3 \cdot 0.5 \cdot c_0/a = 7.7$ GHz and hence the mode propagates at 10 GHz. At $x = x_0 = 4$ cm we see that $\boldsymbol{E}(x_0, z) = \mathbf{0}$ for the TE_{30}-mode. The electric field satisfies the correct boundary conditions on the plate $x = x_0$. this means that this mode is not affected by the plate and it continues to propagate in $z > 0$, without a reflected wave.Hence $P_r/P_i = 0$.

Comment In $z > 0$ the mode TE_{30} splits up in a TE_{20}-mode in the region $x < x_0$ and one TE_{10}-mode in $x_0 < x < a$.

5.8

A quarter circle

TM-modes:

$E_z(\boldsymbol{r}) = v(\boldsymbol{\rho})e^{ik_z z}$ where v satisfies

$$\begin{cases} \nabla_T^2 v(\boldsymbol{\rho}) + k_t^2 v(\boldsymbol{\rho}) = 0 \\ v(R,\phi) = v(\rho, 0) = v(\rho, \pi/2) = 0 \\ v(\boldsymbol{\rho}) \text{ begränsad} \end{cases}$$

Separation of variables $v(\boldsymbol{\rho}) = f(\rho)g(\phi)$ gives

$$\begin{aligned} &g''(\phi) + \gamma g(\phi) = 0 \\ &g(0) = g(\pi/2) = 0 \end{aligned}$$

$$\Rightarrow g(\phi) = \sin(2m\phi), \quad \gamma = 4m^2$$

In the ρ-direction we get the Bessel differential equation of order $2m$

$$\begin{cases} \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial f(\rho)}{\partial\rho} + \left(k_t^2 - \left(\frac{2m}{\rho}\right)^2\right) = 0 \\ f(R) = 0 \quad |f(0)| < \infty \end{cases}$$

this gives $f(\rho) = J_{2m}(\xi_{2m,n}\rho/R)$ and $k_t^2 = (\xi_{2m,n}/R)^2$, where $J_{2m}(\xi_{2m,n}) = 0$.

The normalized eigenfunctions for the TM-modes are given by

$$v_{2m,n}(\boldsymbol{\rho}) = \sqrt{\frac{2}{\pi}}\frac{J_{2m}(\xi_{2m,n}\rho/R)}{RJ'_{2m}(\xi_{2m,n})}\sin 2m\phi$$

TE-modes:

$H_z(\boldsymbol{r}) = w(\boldsymbol{\rho})e^{ik_z z}$

We get the same problem as in the TE-case except that the boundary conditions are

$$\frac{\partial w(R,\phi)}{\partial\rho} = 0, \ \frac{\partial w(\rho,0)}{\partial\phi} = \frac{\partial w(\rho,\pi/2)}{\partial\phi} = 0$$

This gives the eigenfunctions

$$w_{2m,n}(\boldsymbol{\rho}) = \sqrt{\frac{\epsilon_m}{\pi}}\frac{\eta_{2m,n}J_{2m}(\eta_{2m,n}\rho/R)}{\sqrt{\eta_{2m,n}^2 - 4m^2}RJ_{2m}(\eta_{2m,n})}\cos 2m\phi$$

and the eigenvalues $k_t^2 = (\eta_{2m,n}/R)^2$ where $J'_{2m}(\eta_{2m,n}) = 0$, $m = 0, 1, 2..$, $n = 1, 2, ...$

V3.1

a) and b) Use Eqs 5.24 and 5.25. It is easiest to confirm the statemnets if we, as usual, let the eigenfunctions v and w to be real. The statements follow from the fact that k_z is real for frequencies above the cut-off frequency and imaginary for frequencies below the cut-off frequency.

c) The Poynting vector gives the power flow density. If a component of the vector is imaginary then the power flow in the direction of the component is purely reactive. The time average of the power flow is then zero in that direction. If a component of the poynting vector is real then the time average of the power flow is non-zero in that direction. The time average of the power flow in the transverse directions must be zero when we have perfectly conducting walls since there is nothing that can absorb power in that direction. The time average of the power flow in the $z-$direction must be zero for frequencies below the cut-off frequency since the fields attenuate to zero along the waveguide and there is nothing that can absorb energy along the waveguide. For frequencies above the cut-off frequency the wave propagate without attenuation in the $z-$direction. The time average is then non-zero as long as there is nothing in the waveguide that reflects the wave

V3.2

The TM_{02} mode has $E_z(\boldsymbol{r}) = v_{02}(\rho)e^{\mathrm{i}k_z z}$ where $v_{02}(\rho) = A_{02}J_0\left(\frac{\xi_{02}\rho}{a}\right)$. However $J_0\left(\frac{\xi_{02}\rho}{a}\right)$ is also zero when $\left(\frac{\xi_{02}\rho}{a}\right) = \xi_{01}$. This means that E_z is zero at $\rho = \frac{\xi_{01}}{\xi_{02}}a$. The boundary condition that the tangential component of $\boldsymbol{E}$ is zero at $\rho = b$ is then satisfied if $b = \frac{\xi_{01}}{\xi_{02}}a = 2.405a/5.520 = 0.437a$. The transverse part of the electric field is directed in the radial direction and is not affected by the boundary at $\rho = b$.