Microwave theory 2014: Problems week 3

Problems

Problems 5.1, 5.2, 5.4–5.8 and 6.3 in the book.

Problem V3.1

A circular waveguide of length $0 < z < L$ and radius a has a non-reflecting termination at $z = L$. In the region $\frac{L}{4} < z < \frac{L}{2}$ one has inserted a circular metal cylinder with radius $b < a$ and very thin wall.

Determine the radius b such that it is only the TM_{02} mode that can propagate from $z = 0$ to $z = L$ without reflections from the inner cylinder.

Solutions

5.1

We analyze a TM-mode here. The analysis of the TE-mode is done in the same manner.

For a TM-mode in a planar waveguide the electric field reads, cf section 5.5.1:

$$\boldsymbol{E}_n(\boldsymbol{r}) = (\boldsymbol{E}_{Tn}(y) + v_n(y)\hat{z})e^{\mathrm{i}k_{zn}z}$$

where

$$v_n(y) = \sqrt{\frac{2}{b}}\sin k_{tn}y$$

$$\boldsymbol{E}_{Tn}(y) = \mathrm{i}\frac{k_{zn}}{k_{tn}^2}\nabla_T v_n(y) = \mathrm{i}\hat{y}\frac{k_{zn}}{k_{tn}}\sqrt{\frac{2}{b}}\cos k_{tn}y$$

The electric field then reads

$$\begin{aligned}\boldsymbol{E}_n(\boldsymbol{r}) &= \sqrt{\frac{2}{b}}\frac{1}{k_{tn}}\left(\hat{y}\mathrm{i}k_{zn}\cos k_{tn}y + \hat{z}k_{tn}\sin k_{tn}y\right)e^{\mathrm{i}k_{zn}z}\\ &= \sqrt{\frac{2}{b}}\frac{\mathrm{i}}{2k_{tn}}\left((0, k_{zn}, -k_{tn})e^{\mathrm{i}(k_{tn}y+k_{zn}z)} + (0, k_{zn}, k_{tn})e^{\mathrm{i}(-k_{tn}y+k_{zn}z)}\right)\end{aligned}$$

We se that the field consists of two planar transverse waves with direction of propagation $\hat{k}_1 = (0, k_{tn}, k_{zn})/k$ respektive $\hat{k}_2 = (0, -k_{tn}, k_{zn})/k$ and with the electric fields in the direction $\hat{k}_1 \times \hat{x}$ and $\hat{k}_2 \times \hat{x}$, respectively.

We need to determine the corresponding magnetic fields by using the plane wave relation $\boldsymbol{H} = \eta_0^{-1}\hat{k} \times \boldsymbol{E}$ for each of the two plane waves. This gives

$$\begin{aligned}\boldsymbol{H}_n(\boldsymbol{r}) &= -\eta_0^{-1}\sqrt{\frac{2}{b}}\frac{\mathrm{i}}{2k_{tn}}\frac{1}{k}\left((k_{tn}^2 + k_{zn}^2, 0, 0)e^{\mathrm{i}(k_{tn}y+k_{zn}z)} + (k_{tn}^2 + k_{zn}^2, 0, 0)e^{\mathrm{i}(-k_{tn}y+k_{zn}z)}\right)\\ &= -\hat{x}\eta_0^{-1}\sqrt{\frac{2}{b}}\frac{\mathrm{i}}{k_{tn}}k\cos k_{tn}ye^{\mathrm{i}k_{zn}z}\end{aligned}$$

This gives the same function as in equation 5.24 *i.e.*,

$$\begin{aligned}\boldsymbol{H}_n(\boldsymbol{r}) &= \eta_0^{-1}\frac{\mathrm{i}}{k_{tn}^2}k\epsilon\hat{z}\times\nabla_T v_n(\boldsymbol{\rho}) \\ &= -\hat{x}\eta_0^{-1}\sqrt{\frac{2}{b}}\frac{\mathrm{i}}{k_{tn}}k\cos k_{tn}y e^{\mathrm{i}k_{zn}z}\end{aligned}$$

5.2

Assume a waveguide mode that is propagating in the positive z-direction in a conducting material with $\mathrm{Im}\epsilon\mu > 0$. The time average of the power transported by the modes is, *cf.*, page 117

$$P = \mathrm{Re}P_{n\nu}^E |a_{n\nu}^+|^2 e^{-2\mathrm{Im}k_{zn}z}$$

where $P_{n\nu}^E$ is given by equation 5.36

$$P_{n\nu}^E = \frac{\omega}{k_{tn}^2(\omega)}\begin{cases} k_{zn}\epsilon_0\epsilon^* & \nu = \mathrm{TM} \\ k_{zn}^*\mu_0\mu & \nu = \mathrm{TE}\end{cases}$$

For a passive material $\mathrm{Re}\epsilon\mu > 0$, $\mathrm{Im}\epsilon\mu > 0$ and

$$k_{zn} = \sqrt{k^2 - k_{tn}^2} = \sqrt{\omega^2(\mathrm{Re}\{\epsilon\mu\} + \mathrm{i}\mathrm{Im}\{\epsilon\mu\})/c_0^2 - k_{tn}^2} = \alpha + \mathrm{i}\beta$$

where $\alpha > 0$ and $\beta > 0$. This implies $\mathrm{Re}k_{zn}\epsilon^* = \alpha\mathrm{Re}\epsilon + \beta\mathrm{Im}\epsilon > 0$ and $\mathrm{Re}P_{nTM}^E > 0$ and hence $P > 0$ for TM-modes. In the same manner we can show that $\mathrm{Re}Y_{nTE}^E > 0$.

The power transported through a cross section $z = z_0$ of the waveguide for frequencies below the cut-off frequency is transferred to heat in the region $z > z_0$.

5.4

We only consider TE-modes. For a TE-mode $H_z(\boldsymbol{r}) = w(\rho,\phi)e^{ik_z z}$ where

$$\begin{aligned}&\nabla_T^2 w + k_t^2 w = 0 \\ &\frac{\partial w}{\partial\rho}(a,\phi) = \frac{\partial w}{\partial\phi}(\rho,0) = \frac{\partial w}{\partial\phi}(a,2\pi) = 0 \\ &w(\rho,\phi) \text{ finite}\end{aligned}$$

Separation of variables $w = f(\rho)g(\phi)$ inserted in the Helmholtz equation $\nabla_T^2 w + k_t^2 w = 0$ gives the eigenvalue problem

$$g''(\phi) + \gamma g(\phi) = 0 \tag{0.1}$$

$$g'(0) = g'(2\pi) = 0 \tag{0.2}$$

Equation (0.1) gives $g(\phi) = A\sin\sqrt{\gamma}\phi + B\cos\sqrt{\gamma}\phi$ where equation (0.2) gives $A = 0$ and $\gamma = (m/2)^2$, $m = 0, 1, \ldots$. In the ρ-direction we get the Bessel differential equation

$$\begin{aligned}&\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial f(\rho)}{\partial\rho} + \left(k_t^2 - \left(\frac{m}{2\rho}\right)^2\right)f(\rho) = 0 \\ &f'(a) = 0\end{aligned}$$

with the finite solutions

$$f_{mn}(\rho) = J_{m/2}(k_t\rho) = J_{m/2}(\eta_{m/2,n}\rho/a)$$

where $J'_{m/2}(\eta_{m/2,n}) = 0$. The normalized modes are given by

$$w_{mn} = BJ_{m/2}(k_{tmn}\rho)\cos m\phi/2$$

where $k_{tmn} = \eta_{m/2,n}/a$ and

$$B = \sqrt{\frac{\epsilon_m}{2\pi}}\left(\int_0^a \left(J_{m/2}(k_{tmn}\rho)\right)^2 \rho d\rho\right)^{-1/2}$$

with $\epsilon_m = 2 - \delta_{m,0}$.

Cut-off frequencies

For even m we get the zeros of $J'_{m/2}$ from appendix A. For odd m we can utilize that $J'_\nu(x) = \frac{\nu}{x}J_\nu(x) - J_{\nu+1}(x)$ and determine the zeros numerically in *e.g.*, Matlab. This gives the following values of $\eta_{m/2,n}$

	n=1	n=2	n=3	n=4
m=0	3.832	7.016	10.17	13.32
m=1	1.1656	4.60	7.79	19.95
m=2	1.841	5.331	8.536	11.71
m=3	2.46	6.03	9.26	12.44

We see that $TE_{1/2,1}$ has the lowest cut-off frequency $f_{1/2,1} = c_0 1.1656/(2\pi a)$. Without the metal plate, according to table 3.4 in the book, the TE_{11} mode has the lowest cut-off frequency $f_{1,1} = c_0 1.841/(2\pi a)$. The cut off frequency for the fundamental mode is then reduced by almost 40% when the metal plate is introduced.

5.5

We utilize the solution to example 5.6. For $z < 0$ there is an incident an a reflected TM-mode and for $z > 0$ a transmitted TM-mode.

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r}) &= \boldsymbol{E}^+_{nTM}(\boldsymbol{r}) + r_n\boldsymbol{E}^-_{nTM}(\boldsymbol{r}) \\ \boldsymbol{H}(\boldsymbol{r}) &= \boldsymbol{H}^+_{nTM}(\boldsymbol{r}) + r_n\boldsymbol{H}^-_{nTM}(\boldsymbol{r}) \end{aligned} \qquad z \le 0$$

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r}) &= t_n\boldsymbol{E}^+_{nTM}(\boldsymbol{r}) \\ \boldsymbol{H}(\boldsymbol{r}) &= t_n\boldsymbol{H}^+_{nTM}(\boldsymbol{r}) \end{aligned} \qquad z \ge 0$$

where r_n is the reflection coefficient and t_n is the transmission coefficient. We let the amplitude of the incident wave be 1 V/m. The boundary conditions at $z = 0$ imply that the transverse components of $\boldsymbol{E}$ and $\boldsymbol{H}$ are continuous. Since $v_n(\boldsymbol{\rho})$ and k_{tn} are independent of z it follows that

$$\begin{aligned} &k_{zn}(1 + r_n) = \tilde{k}_{zn}t_n \\ &1 - r_n = \epsilon t_n \end{aligned}$$

where $k_{zn} = \sqrt{(\omega/c_0)^2 - k_{tn}^2}$ and $\tilde{k}_{zn} = \sqrt{(\omega/c_0)^2\epsilon - k_{tn}^2}$ are the longitudinal wave numbers for $z < 0$ and $z > 0$, respectively. The solution is given by

$$r_n = \frac{\tilde{k}_{zn} - \epsilon k_{zn}}{\tilde{k}_{zn} + \epsilon k_{zn}}$$

If the TM mode number n is above cut-off then k_{zn} and the mode power Y_{nTM}^E are real, see equation 5.36. The power transport in the waveguide in the region $z < 0$ is given by equation (5.36)

$$P_i - P_r = \iint_\Omega \hat{z}\cdot < \boldsymbol{S}(t) > (\boldsymbol{r}, \omega) dxdy = P_{nTM}^E |a_{0TM}^+|^2 (1 - |r_n|^2)$$

where $P_i = P_{nTM}^E |a_{0TM}^+|^2$ is the power of the incident mode and $P_r = P_{nTM}^E |a_{0TM}^+|^2 |r_n|^2$ is the power of the reflected mode. Thus

$$\frac{P_r}{P_i} = |r_n|^2$$

It is given that $a = 4cm$, $b = 3cm$ and $\epsilon = 2$. The fundamental TM-mode is TM_{11}. The frequency is chosen such that it is the same as the cut-off frequency for the second TM-mode, *i.e.*, TM_{21} that has $k_{t21}^2 = (2\pi/a)^2 + (\pi/b)^2$. For $z < 0$ this corresponds to the frequency

$$f_{21} = k_{t21} \frac{c_0}{2\pi} = 9\text{GHz}$$

This gives

$$\frac{P_r}{P_i} = 6.1 \cdot 10^{-3}$$

b) $P_r = 0$ when $r_{11} = 0$ *i.e.*, when $\tilde{k}_{zn} = \epsilon k_{zn}$. This gives

$$f = \frac{1}{2\pi} \sqrt{\frac{1+\epsilon}{\epsilon}} c_0 k_{t11}$$

The numerical value is $f = 7.65$ GHz.

5.6

a) We first determine the modes that can propagate when $a = 3$ cm and $f = 5$ GHz. The lowest cut off frequencies are obtained from the tables of zeros for $J_m(x)$ (för TM) and $J_m'(x)$ (för TE) in appendix A

$$f_{11}^{TE} = \frac{c_0}{2\pi} \frac{1.841}{3} 10^2 = 2.93\,\text{GHz} < 5\,\text{GHz} \tag{0.3}$$

$$f_{21}^{TE} = \frac{c_0}{2\pi} \frac{3.053}{3} 10^2 = 4.86\,\text{GHz} < 5\,\text{GHz} \tag{0.4}$$

$$f_{01}^{TM} = \frac{c_0}{2\pi} \frac{2.405}{3} 10^2 = 3.83\,\text{GHz} < 5\,\text{GHz} \tag{0.5}$$

The next modes are f_{01}^{TE} and f_{11}^{TM} which both are non-propagating modes since they have cut off frequency 6.1 GHz.

b) The waveguide is filled with a plastic material with $\sigma = 10^{-11}$ S and $\varepsilon = 3$. The z-dependence of the fundamental mode TE_{11} is given by $e^{\mathrm{i}k_z z}$ where $k_z = \sqrt{k^2 - k_{t11}^2}$. The wave number k is given by

$$k^2 = \left(\frac{\omega}{c_0}\right)^2 \epsilon_{ny} = \left(\frac{\omega}{c_0}\right)^2 \left(\epsilon + \mathrm{i}\frac{\sigma}{\omega\epsilon_0}\right)$$

It is seen that $\sigma/(\epsilon\epsilon_0) \approx 10^{-11}/(3{\cdot}8.854\,10^{-12})$ In the microwave region $\sigma/(\omega\epsilon\epsilon_0) \ll 1$ and the following approximations are valid

$$k_z = (k^2 - k_{t11}^2)^{1/2} = \left(\left(\frac{\omega}{c_0}\right)^2 \epsilon - k_{t11}^2\right)^{1/2} \left(1 + \mathrm{i}\frac{\sigma\omega\mu_0}{((\omega/c_0)^2\epsilon - k_{t11}^2)}\right)^{1/2}$$

$$\approx \left(\left(\frac{\omega}{c_0}\right)^2 \epsilon - k_{t11}^2\right)^{1/2} \left(1 + \mathrm{i}\frac{\sigma\omega\mu_0}{2((\omega/c_0)^2\epsilon - k_{t11}^2)}\right)$$

and hence $k_z = \mathrm{Re}(k_z) + \mathrm{i}\,\mathrm{Im}(k_z)$ where

$$\mathrm{Im}(k_z) = \frac{\sigma\omega\mu_0}{2}\left((\omega/c_0)^2\epsilon - k_{t11}^2\right)^{-1/2} = \frac{\sigma\eta}{2}\left(1 - (f_c/f)^2\right)^{-1/2}$$

where $f_c = c_0\xi_{11}/(2\pi a\sqrt{\epsilon})$ och $\eta = \sqrt{\mu_0/(\epsilon\epsilon_0)}$ =wave impedance. The numerical value is $f_c = 1.7$ GHz.

5.7

For the TE$_{10}$-mode the electric field in the region $z < 0$ is

$$\boldsymbol{E}(\boldsymbol{r}) = \hat{y}E_0 \sin\frac{\pi x}{a} e^{ik_z z}$$

This is the fundamental mode with cut-off frequency $f_c = 0.5c_0/b = 2.5$ GHz. When this mode hits the plate it couples to the TE$_{m0}$-modes in $z > 0$ and to the reflected TE$_{m0}$-modes in $z < 0$.

Assume that $x_0 > a/2$. The fundamental mode in $z > 0$, $x < x_0$ is TE$_{10}$. This mode has the cut-off frequency $f_c = 0.5c_0/x_0$.

a) According to the text, power propagates in $z > 0$ for frequencies above 3.75 GHz. This means that 3.75 GHz is the cut-off frequency for the fundamental mode TE$_{10}$ in $x < x_0$. Hence the plate is placed at $x_0 = 0.5c_0/f_c = 0.5 \cdot 3 \cdot 10^8/3.75 \cdot 10^9 = 4$ cm.

b) The electric for the TE$_{03}$-mode in $z < 0$

$$\boldsymbol{E}(\boldsymbol{r}) = \hat{x}E_0 \sin\frac{3\pi y}{b} e^{\mathrm{i}k_z z}$$

and then the boundary condition at $x = x_0$ is already satisfied since the tangential component is zero. Thus $P_r/P_i = 0$. The corresponding cut-off frequency is at $f_c = 3 \cdot 0.5 \cdot c_0/b = 15$ GHz and hence the mode propagates at 20 GHz.

c) The electric field for the TE_{30}-mode in $z<0$ is

$$\boldsymbol{E}(\boldsymbol{r}) = \hat{y}E_0 \sin\frac{3\pi x}{a} e^{\mathrm{i}k_z z}$$

The corresponding cut-off frequency is at $f_c = 3\cdot 0.5\cdot c_0/a = 7.7$ GHz and hence the mode propagates at 10 GHz. At $x = x_0 = 4$ cm we see that $\boldsymbol{E}(x_0, z) = \boldsymbol{0}$ for the TE_{30}-mode. The electric field satisfies the correct boundary conditions on the plate $x = x_0$. this means that this mode is not affected by the plate and it continues to propagate in $z>0$, without a reflected wave.Hence $P_r/P_i = 0$.

Comment In $z>0$ the mode TE_{30} splits up in a TE_{20}-mode in the region $x<x_0$ and one TE_{10}-mode in $x_0<x<a$.

5.8

A quarter circle

TM-modes:

$E_z(\boldsymbol{r}) = v(\boldsymbol{\rho})e^{ik_z z}$ where v satisfies

$$\begin{cases} \nabla_T^2 v(\boldsymbol{\rho}) + k_t^2 v(\boldsymbol{\rho}) = 0 \\ v(R,\phi) = v(\rho,0) = v(\rho,\pi/2) = 0 \\ v(\boldsymbol{\rho}) \text{ begränsad} \end{cases}$$

Separation of variables $v(\boldsymbol{\rho}) = f(\rho)g(\phi)$ gives

$$\begin{aligned} &g''(\phi) + \gamma g(\phi) = 0 \\ &g(0) = g(\pi/2) = 0 \end{aligned}$$

$$\Rightarrow g(\phi) = \sin(2m\phi), \quad \gamma = 4m^2$$

In the ρ-direction we get the Bessel differential equation of order $2m$

$$\begin{cases} \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\rho\dfrac{\partial f(\rho)}{\partial\rho} + \left(k_t^2 - \left(\dfrac{2m}{\rho}\right)^2\right) = 0 \\ f(R) = 0 \quad |f(0)| < \infty \end{cases}$$

this gives $f(\rho) = J_{2m}(\xi_{2m,n}\rho/R)$ and $k_t^2 = (\xi_{2m,n}/R)^2$, where $J_{2m}(\xi_{2m,n}) = 0$.

The normalized eigenfunctions for the TM-modes are given by

$$v_{2m,n}(\boldsymbol{\rho}) = \sqrt{\frac{2}{\pi}}\frac{J_{2m}(\xi_{2m,n}\rho/R)}{RJ_{2m}'(\xi_{2m,n})}\sin 2m\phi$$

TE-modes:

$H_z(\boldsymbol{r}) = w(\boldsymbol{\rho})e^{ik_z z}$

We get the same problem as in the TE-case except that the boundary conditions are

$$\frac{\partial w(R,\phi)}{\partial \rho} = 0, \ \frac{\partial w(\rho,0)}{\partial \phi} = \frac{\partial w(\rho,\pi/2)}{\partial \phi} = 0$$

This gives the eigenfunctions

$$w_{2m,n}(\boldsymbol{\rho}) = \sqrt{\frac{\epsilon_m}{\pi}} \frac{\eta_{2m,n} J_{2m}(\eta_{2m,n}\rho/R)}{\sqrt{\eta_{2m,n}^2 - 4m^2} R J_{2m}(\eta_{2m,n})} \cos 2m\phi$$

and the eigenvalues $k_t^2 = (\eta_{2m,n}/R)^2$ where $J'_{2m}(\eta_{2m,n}) = 0$, $m = 0, 1, 2..$, $n = 1, 2, ...$

6.3

a) The complex electric field is $\boldsymbol{E} = E(\rho)\hat{z}$ and satisfies the equation

$$\nabla^2 \boldsymbol{E} + k^2 \boldsymbol{E} = 0$$

which leads to the Bessel differential equation of order 0:

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial E(\rho)}{\partial \rho} + k^2 E(\rho) = 0$$

The only condition is that $\boldsymbol{E}$is bounded everywhere. This gives the solution

$$\boldsymbol{E}(\rho) = E_0 J_0(k\rho)\hat{z}$$

where $E_0 = |E_0|e^{i\alpha}$. In the time domain

$$\boldsymbol{E}(\rho, t) = \text{Re}\{\boldsymbol{E}(\rho)e^{-i\omega t}\} = |E_0| J_0(k\rho)\cos(\omega t - \alpha)\hat{z}$$

The corresponding magnetic field is obtained from the induction law

$$\begin{aligned}\boldsymbol{H}(\rho) &= -i\frac{1}{\omega\mu_0}\nabla \times \boldsymbol{E}(\rho) = i\hat{\phi}\frac{1}{\omega\mu_0}\frac{\partial E(\rho)}{\partial \rho} \\ &= i\hat{\phi}\frac{k}{\omega\mu_0}E_0 J'_0(k\rho) = -i\hat{\phi}E_0\frac{1}{\eta_0}J_1(k\rho)\end{aligned}$$

which gives the time domain dependence

$$\boldsymbol{H}(\rho, t) = -\hat{\phi}|E_0|\frac{1}{\eta_0}J_1(k\rho)\sin(\omega t - \alpha)$$

b) The boundary condition is $E(a) = 0$ Which means that only frequencies that

satisfy

$$J_0(ka) = 0$$

are valid. This gives the resonance frequencies

$$f_c = \frac{c}{2\pi}\frac{\xi_{0,n}}{a}, \quad n = 1, 2, 3...$$

where $J_0(\xi_{0,n}) = 0$. The electric field $\boldsymbol{E}(\boldsymbol{r}) = E_0 J_0(k\rho)\hat{z}$ is a field that can exist in a cylindric cavity.

V3.1

The TM_{02} mode has $E_z(\boldsymbol{r}) = v_{02}(\rho)e^{\mathrm{i}k_z z}$ where $v_{02}(\rho) = A_{02}J_0\left(\frac{\xi_{02}\rho}{a}\right)$. However $J_0\left(\frac{\xi_{02}\rho}{a}\right)$ is also zero when $\left(\frac{\xi_{02}\rho}{a}\right) = \xi_{01}$. This means that E_z is zero at $\rho = \frac{\xi_{01}}{\xi_{02}}a$. The boundary condition that the tangential component of $\boldsymbol{E}$ is zero at $\rho = b$ is then satisfied if $b = \frac{\xi_{01}}{\xi_{02}}a = 2.405a/5.520 = 0.437a$. The transverse part of the electric field is directed in the radial direction and is not affected by the boundary at $\rho = b$.