Microwave theory 2014: Problems for week 4 and 5

1

Show that the time averages of the stored electric and magnetic energies in a resonance cavity are equal.

2

Estimate the number of resonances with a wavelength larger than 500 nm in a cubic vacuum cavity with volume one cubic meter.

3

a) Determine the fundamental mode in a circular waveguide with radius a with vacuum inside the waveguide.
b) Plot the transverse electric and magnetic field of the fundamental mode in the cross section of the waveguide.
c) Determine the surface current density for the fundamental mode.

4

a) Determine the fundamental mode for a rectangular waveguide.
b) Give the x, y-dependence of the different components of the transverse electric and magnetic fields for the fundamental mode without doing any calculations.
c) Determine the $z-$component of the Poynting vector $\boldsymbol{S}(x, y)$.
d) Determine the attenuation of the fundamental mode in dB/m when $a = 3$ cm, $b = 1$ cm in the frequency interval $[0, f_{c10}]$ where f_{c10} is the cut-off frequency for the fundamental mode.

5

Around 1970 there were attempts to use circular waveguides for communication. Such a system was developed by Bell Telephone Laboratories in USA that managed to send 234 000 two-way telephone channels in one waveguide. One then used the waveguide mode TE_{01} in a waveguide with radius 25 mm in the frequency band 40-110 GHz. Determine the attenuation in dB/m for the fundamental mode TE_{11} and for the mode TE_{01} in a circular waveguide with radius 25 mm as a function of frequency in the interval $[f_c, 40\,\mathrm{GHz}]$, where f_c is the cut-off frequency for the TE_{11} mode. Explain why they used the mode TE_{01} and not the fundamental mode. The waveguide is made out of copper.

6

Determine the three lowest resonance frequencies for a coaxial cable that is terminated by perfectly conducting plates at $z = 0$ and $z = h = 10$ cm. The radius of the inner and outer conductor is $a = 3$ mm and $b = 10$ mm, respectively. The material between the conductors is a plastic material with $\sigma = 0$ and $\varepsilon = 4$.

7

Determine the three lowest resonance frequencies for a rectangular parallelepiped with sides $a = 4$ cm, $b = 3$ cm and height $h = 5$ cm. All of the walls are perfectly conducting and there is vacuum in the cavity.

8

Design a resonator in the shape of a circular cylinder such that it has the two lowest resonance frequencies at $f_1 = 10$ GHz and $f_2 = 15$ GHz.

Solutions

S1

Let $\boldsymbol{E}_n$ be the electric field of a resonance mode in a cavity. Since $\boldsymbol{E}_n$ satisfies Maxwell's equations it follows that

$$\nabla \times (\nabla \times \boldsymbol{E}_n) - k_n^2 \boldsymbol{E}_n = \boldsymbol{0}$$

Take the scalar product with $\boldsymbol{E}_n^*$ and use

$$\boldsymbol{E}_n^* \cdot \nabla \times (\nabla \times \boldsymbol{E}_n) = (\nabla \times \boldsymbol{E}_n^*) \cdot (\nabla \times \boldsymbol{E}_n) - \nabla \cdot (\boldsymbol{E}_n^* \times (\nabla \times \boldsymbol{E}_n) = \frac{1}{\epsilon_0 \epsilon} k_n^2 \boldsymbol{B}_n \cdot \boldsymbol{H}_n^* - \nabla \cdot (\boldsymbol{E}_n^* \times (\nabla \times \boldsymbol{E}_n))$$

Thus

$$\frac{1}{\epsilon_0 \epsilon} k_n^2 \boldsymbol{B}_n \cdot \boldsymbol{H}_n^* - \nabla \cdot (\boldsymbol{E}_n^* \times (\nabla \times \boldsymbol{E}_n)) = k_n^2 |\boldsymbol{E}_n|^2$$

The volume integral of this relation and the use of Gauss theorem give

$$k_n^2 \iiint_V \boldsymbol{B}_n \cdot \boldsymbol{H}_n^* \, \mathrm{d}V + \epsilon_0 \epsilon \oint_S \hat{\boldsymbol{n}} \cdot (\boldsymbol{E}_n^* \times (\nabla \times \boldsymbol{E}_n)) \, \mathrm{d}S = k_n^2 \epsilon_0 \epsilon \iiint |\boldsymbol{E}_n|^2 \, \mathrm{d}V$$

Due to the boundary condition $\hat{\boldsymbol{n}} \times \boldsymbol{E}_n = \boldsymbol{0}$ we get $\hat{\boldsymbol{n}} \cdot (\boldsymbol{E}_n^* \times (\nabla \times \boldsymbol{E}_n)) = \nabla \times \boldsymbol{E}_n \cdot (\hat{\boldsymbol{n}} \times \boldsymbol{E}_n^*) = 0$. Since $\boldsymbol{D} = \epsilon_0 \epsilon$ we get

$$\frac{1}{4} \iiint_V \boldsymbol{B}_n \cdot \boldsymbol{H}_n^* \, \mathrm{d}V = \frac{1}{4} \iiint_V \boldsymbol{E}_n \cdot \boldsymbol{D}_n^* \, \mathrm{d}V$$

The volume integral to the left is he time average of the magnetic energy in the cavity and the volume integral to the right is the time average of the electric energy.

S2

The resonance wavenumbers (TE or TM) are given by $k_{nml}^2 = \left(\frac{2\pi}{\lambda_{nml}}\right)^2 = (n^2 + m^2 + l^2)\left(\frac{\pi}{a}\right)^2$ where $a = 1$ m. Now $\lambda_{nml} > \lambda = 500$ nm. This means that the n, m, l have to satisfy

$$n^2 + m^2 + l^2 \leq \left(\frac{2a}{\lambda}\right)^2 = 16 \cdot 10^{12}$$

Only positive n, m, l are allowed. The total number of combinations n, m, l that satisfies this relation equals the volume of an eight of a sphere with radius $4 \cdot 10^6$ units. This means that there are $\frac{1}{8} \cdot \frac{4}{3}\pi(4 \cdot 10^6)^3$ resonances. Since there are both TE and TM modes this should be multiplied by two. The total number is then $6.7 \cdot 10^{19}$.

S3

a) For the TE modes $E_z(\boldsymbol{r}) = 0$ and $H_z(\boldsymbol{r}) = w(\rho, \phi)e^{ik_z z}$ where

$$w_{mn}(\rho, \phi) = A_{mn} J_m(\eta_{tmn}\rho/a) \cos m\phi$$

where A_{mn} denotes the normalization constant and where η_{mn} is the n :th zero of the derivative $J_m'(k_t a)$. The cut-off frequency for the mn-mode is given by

$$f_{cmn} = \frac{c_0 \eta_{mn}}{2\pi a}$$

The lowest zero is $\eta_{11} = 1.841$ which gives the cut-off frequency

$$f_{c11} = \frac{3 \cdot 10^8 \cdot 1.841}{2\pi a} = \frac{8.79 \cdot 10^7}{a} \text{ Hz}$$

where a is in meter.
For the TM-modes $H_z(\boldsymbol{r}) = 0$ and $E_z(\boldsymbol{r}) = v(\boldsymbol{\rho})e^{ik_z z}$ where

$$v_{mn}(\rho, \phi) = B_{mn} J_m(\xi_{mn}\rho/a) \cos m\phi$$

where B_{mn} denotes the normalization constant and ξ_{mn} is the nth zero of $J_m(k_t a)$. The lowest cut-off frequency is the TM_{01} mode which has the cut-off frequency

$$f_c = \frac{3 \cdot 10^8 \cdot 2.405}{2\pi a} = \frac{1.14 \cdot 10^8}{a} \text{ Hz}$$

where a is in meter. The fundamental mode is the mode with the lowest cut-off frequency and in this case it is the TE_{11} mode.
b) The transverse fields for the TE-modes are given by Eqs. (6.23) and (6.24)

$$\boldsymbol{E}_T = -\frac{\mathrm{i}}{k_{tmn}^2}\frac{\omega}{c_0}\hat{\boldsymbol{z}} \times \nabla_T w_{mn} = -\frac{\mathrm{i}A_{mn}\omega}{\eta_{mn}^2 c_0}(\hat{\boldsymbol{\phi}}\eta_{mn} J_m'(\eta_{mn}\rho/a)\cos m\phi + \hat{\boldsymbol{\rho}}\frac{ma}{\rho} J_m(\eta_{mn}\rho/a)\sin m\phi)$$

$$\boldsymbol{H}_T = Z_{mn}^{-1}\hat{\boldsymbol{z}} \times \boldsymbol{E}_T = Z_{mn}^{-1}\frac{\mathrm{i}A_{mn}\omega}{\eta_{mn}^2 c_0}(\hat{\boldsymbol{\rho}}\eta_{mn} J_m'(\eta_{mn}\rho/a)\cos m\phi - \hat{\boldsymbol{\phi}}\frac{ma}{\rho} J_m(\eta_{mn}\rho/a)\sin m\phi)$$

where $Z_{mn} = \dfrac{\omega\mu_0}{k_{zmn}}$ is the TE-mode impedance. For the fundamental mode we get

$$\boldsymbol{E}_T = -\frac{\mathrm{i}}{k_{t11}^2}\frac{\omega}{c_0}\hat{\boldsymbol{z}} \times \nabla_T w_{11} = -\frac{\mathrm{i}A_{11}\omega}{\eta_{11}^2 c_0}(\hat{\boldsymbol{\phi}}\eta_{11}J_1'(\eta_{11}\rho/a)\cos\phi + \hat{\boldsymbol{\rho}}\frac{a}{\rho}J_1(\eta_{11}\rho/a)\sin\phi)$$
$$\boldsymbol{H}_T = Z_{11}^{-1}\hat{\boldsymbol{z}} \times \boldsymbol{E}_T = Z_{11}^{-1}\frac{\mathrm{i}A_{11}\omega}{\eta_{11}^2 c_0}(\hat{\boldsymbol{\rho}}\eta_{11}J_1'(\eta_{11}\rho/a)\cos\phi - \hat{\boldsymbol{\phi}}\frac{a}{\rho}J_1(\eta_{11}\rho/a)\sin\phi)$$

When these vectors are plotted in Matlab we get the patterns in figure 1.
c) The surface current density is given by the condition

$$\boldsymbol{J}_s = \hat{\boldsymbol{n}} \times \boldsymbol{H}(\boldsymbol{r}) = -\hat{\boldsymbol{\rho}} \times \boldsymbol{H}(a,\phi,z) = \hat{\boldsymbol{\phi}}H_z(a,\phi,z) - \hat{\boldsymbol{z}}H_\phi(a,\phi,z)$$

This gives

$$\boldsymbol{J}_s(\rho,\phi,z) = \left(\hat{\boldsymbol{\phi}}A_{11}J_1(\eta_{11})\cos\phi - \hat{\boldsymbol{z}}Z_{11}^{-1}\frac{\mathrm{i}A_{11}\omega}{\eta_{11}c_0}J_1(\eta_{11})\sin\phi\right)e^{ik_z z}$$

Notice that the ϕ and $z-$components are 90° out of phase.

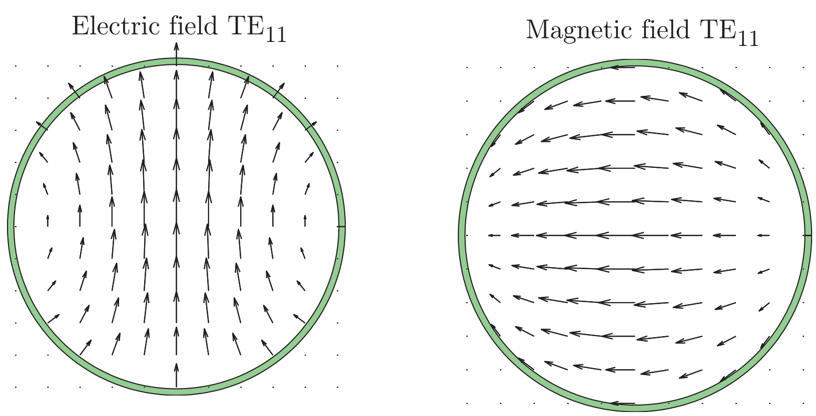

Figure 1: The electric field and the transverse magnetic field for the TE_{11} mode

S4

a) The fundamental mode is the mode with the lowest cut-off frequency. From the book we know that this is the TE_{10}-mode. For this mode

$$H_z(\boldsymbol{r}) = \sqrt{\frac{2}{ab}}\cos\frac{\pi x}{a}e^{ik_z z}$$

where $k_z = \sqrt{(\omega/c)^2 - (\pi/a)^2}$.
b) We do not have any $y-$dependence and the $x-$dependence is either $\sin\dfrac{\pi x}{a}$ or $\cos\dfrac{\pi x}{a}$. The boundary conditions are that the tangential components of the electric field are zero at all walls, that the normal components of $\boldsymbol{H}$ is zero on all walls, and that the normal derivative of the tangential component of the tangential component of $\boldsymbol{H}$ is zero on all walls. Then:

- $E_z = 0$ since it is a TE-mode. It also follows from that E_z must be zero at all walls and hence also for all $y-$values.
- $E_x = 0$ since it must be zero at $y = 0$ and $y = b$ and hence for all y
- $E_y \sim \sin\frac{\pi x}{a}$ since E_y must be zero at $x = 0$ and $x = a$
- $H_x \sim \sin\frac{\pi x}{a}$ since H_x is zero at $x = 0$ and $x = a$
- $H_y = 0$ since it must be zero at $y = 0$ and $y = b$ and hence for all y

c) We now have to calculate the explicit expressions for the transverse electric and magnetic fields. We use Eqs. (6.23) and (6.24) for this

$$\boldsymbol{E}_T(x,y) = -\frac{\mathrm{i}\omega}{k_{t10}^2 c_0}\hat{\boldsymbol{z}} \times \nabla_T\Big(\sqrt{\frac{2}{ab}}\cos\frac{\pi x}{a}\Big) = \hat{\boldsymbol{y}}\frac{\mathrm{i}\omega}{k_{t10}^2 c_0}\frac{\pi}{a}\sqrt{\frac{2}{ab}}\sin\frac{\pi x}{a} = \hat{\boldsymbol{y}}\frac{\mathrm{i}\omega a}{\pi c_0}\sqrt{\frac{2}{ab}}\sin\frac{\pi x}{a}$$
$$\boldsymbol{H}_T(x,y) = Z_{11}^{-1}\hat{\boldsymbol{z}} \times \boldsymbol{E}_T = -\hat{\boldsymbol{x}}Z_{11}^{-1}\frac{\mathrm{i}\omega a}{\pi c_0}\sqrt{\frac{2}{ab}}\sin\frac{\pi x}{a}$$

since $k_{t10} = \dfrac{\pi}{a}$. Here $Z_{11} = \dfrac{\omega\mu_0}{k_{z11}}$ is the TE-mode impedance. The z-component of the complex Poynting vector is

$$S_z = \hat{\boldsymbol{z}}\cdot\frac{1}{2}\boldsymbol{E}_T \times \boldsymbol{H}_T^* = (Z_{11}^*)^{-1}\frac{1}{ab}\left(\frac{\omega a}{\pi c_0}\sin\frac{\pi x}{a}\right)^2$$

The time average of the Power flow density is $\mathrm{Re}\{S_z\}$. This is zero for frequencies below the cut-off frequency since then k_{z11} is imaginary but the rest of the factors are real.

d) For frequencies below the cut-off frequencies then $k_{z10} = i\sqrt{\left(\frac{\pi}{a}\right)^2 - k^2}$ and $e^{ik_{z10}} = e^{-\alpha z}$ where $\alpha = \sqrt{\left(\frac{\pi}{a}\right)^2 - k^2}$. When $a = 3$ cm the cut-off frequency is 5 GHz. The attenuation is given in figure 3.

S5

The power is attenuated as $P(z) = P(0)e^{-\alpha_p z}$ in the waveguide. The attenuation constant is for TE-modes given by the formula on page 124 in the book. Let a denote the radius.

$$\alpha_p = \frac{k_{tmn}^2}{\omega\sigma\delta\mu_0 k_{zmn}}\oint\left(\frac{k_{zmn}^2}{k_{tmn}^4}|\nabla_T w_{mn}|^2 + |w_{mn}|^2\right)dl$$

Here $\sigma = 5.8\cdot 10^7$ is the conductivity of copper and $\delta = \sqrt{2/(\omega\mu_0\sigma}$ is the skin depth. For the TE_{01} mode we have

$$w_{mn}(\rho,\phi) = A_{mn}J_m(k_{tmn}\rho)\cos m\phi$$

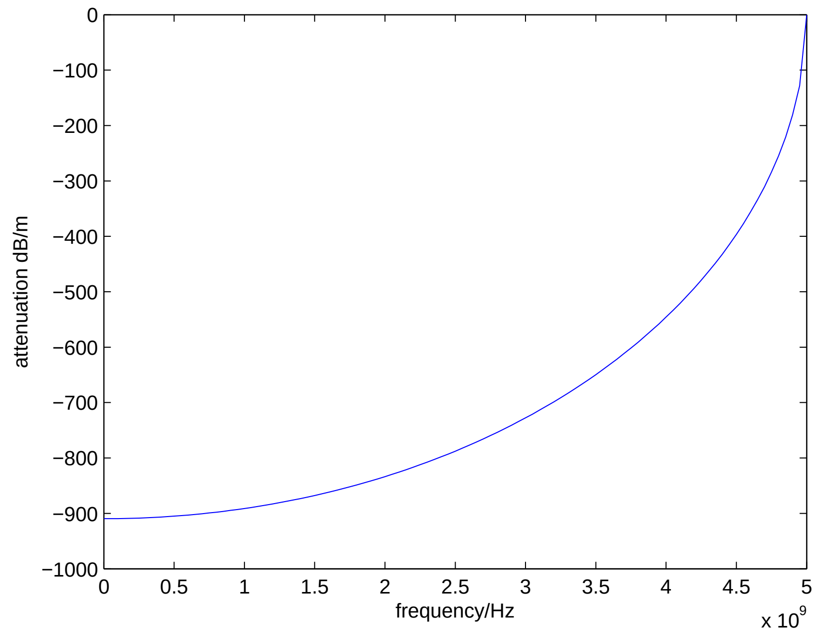

Figure 2: The attenuation of the fundamental mode in a rectangular waveguide. The cut-off frequency is 5 GHz.

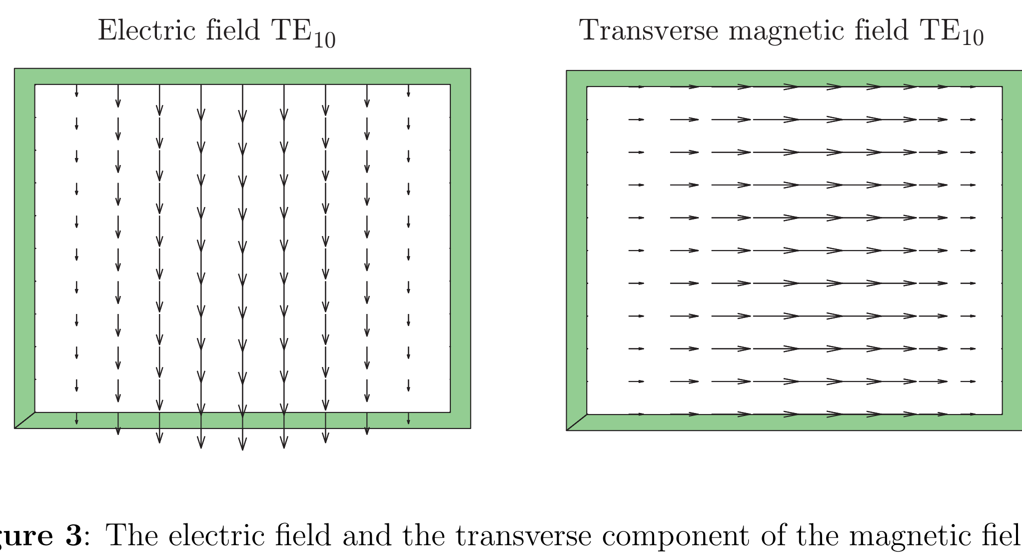

Figure 3: The electric field and the transverse component of the magnetic field for the fundamental mode in a rectangular waveguide.

where the normalization constant is give by the table on page 112

$$A_{mn} = \sqrt{\frac{\epsilon_m}{\pi(\eta_{mn}^2 - m^2)}} \frac{\eta_{mn}}{J_m(\eta_{mn})}$$

where ϵ_m is 1 for $m = 0$ and 2 otherwise and $\eta_{mn} = k_{tmn}a$ is the n :th zero of J'_m. The gradient of w_{mn} is given by

$$\nabla_T w_{mn}(\rho, \phi) = A_{mn} \left(\hat{\rho} k_{tmn} J'_m(k_{tmn}\rho) \cos m\phi - \hat{\phi} \frac{m}{\rho} J_m(k_c\rho) \sin n\phi \right)$$

Since $J'_m(k_{tmn}a) = 0$ it is seen that

$$\begin{aligned} \alpha_p &= \frac{A_{mn}^2 k_{tmn}^2}{\omega\sigma\delta\mu_0 k_{zmn}} \int_0^{2\pi} \left(\frac{m^2 k_{zmn}^2}{k_{tmn}^4 a^2} \sin^2 m\phi + \cos^2 m\phi \right) (J_m(k_c a))^2 \, a d\phi \\ &= \frac{A_{mn}^2 k_{tmn}^2 2\pi a}{\epsilon_m \omega\sigma\delta\mu_0 k_{zmn}} \left(\frac{m^2 k_{zmn}^2}{k_{tmn}^4 a^2} + 1 \right) (J_m(\eta_{mn}))^2 \\ &= \frac{2}{\omega\sigma\delta\mu_0 k_{zmn} a} \left(\frac{m^2 k_{zmn}^2 + k_{tmn}^4 a^2}{k_{tmn}^2 a^2 - m^2} \right) \end{aligned}$$

When the values for the TE_{01} and TE_{11} modes are inserted in this expression figure 5 is obtained. We see that the attenuation of TE_{01} decreases with frequency and gives a very small attenuation at high frequencies.

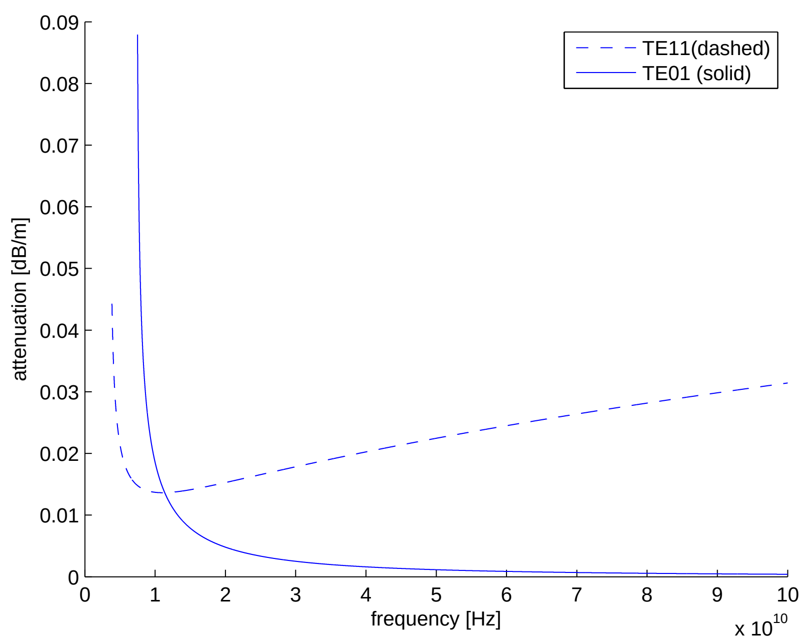

Figure 4: The attenuation of the TE_{01} and TE_{11} modes as a function of frequency

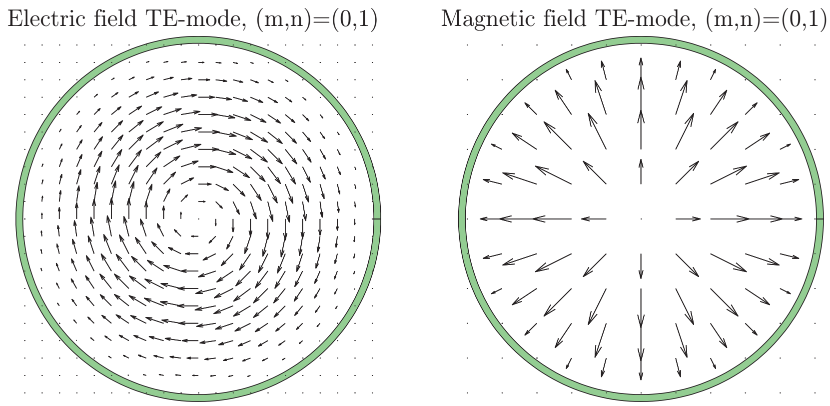

Figure 5: The electric field and the transverse magnetic field for the TE_{01}

S6

In the coaxial cable the TEM-mode propagates at all frequencies. The $z-$dependence is given by

$$\boldsymbol{E}(\boldsymbol{\rho}, z) = \boldsymbol{E}(\boldsymbol{\rho})(a^+ e^{ikz} + a^- e^{-ikz})$$

The boundary conditions says that the electric field is zero at $z = 0$ and $z = h$. Thus

$$\begin{aligned}(a^+ + a^-) &= 0\\ (a^+ e^{ikh} + a^- e^{-ikh}) &= 0\end{aligned}$$

From this we get $a^+ = -a^-$ and $a^+ \sin kh = 0$ and then $kh = \ell\pi$, $\ell = 1, 2, 3 \ldots$. The resonance frequencies are given by $f_r = \dfrac{c\ell}{2h} = \dfrac{c_0 \ell}{2h\sqrt{\varepsilon}}$. That gives the three lowest resonant frequencies

$$\begin{aligned}f_1 &= \frac{c_0}{2h\sqrt{\varepsilon}} = 0.75\,\text{GHz}\\ f_2 &= \frac{2c_0}{2h\sqrt{\varepsilon}} = 1.5\,\text{GHz}\\ f_3 &= \frac{3c_0}{2h\sqrt{\varepsilon}} = 2.25\,\text{GHz}\end{aligned}$$

There are also TE and TM waveguide modes in the coaxial cables that will have resonances. However, their resonances are much higher than 2.25 GHz.

S7

We view the parallelpiped as a waveguide in the $z-$direction with cross section $0 < x < a$, $0 < y < b$. The general theory for finite waveguide cavities gives the resonance frequencies

$$f_{mn\ell} = \frac{c}{2\pi}\sqrt{k_{tmn}^2 + \left(\frac{\ell\pi}{h}\right)^2} = \frac{c}{2\pi}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{\ell\pi}{h}\right)^2}$$

The $mn\ell$ values are

$$\begin{cases} m=1,2,3\ldots,\ n=1,2,3\ldots,\ \ell=0,1,2,3\ldots, & \text{TM-modes} \\ m=0,1,2,3\ldots,\ n=0,1,2,3\ldots,\ (m,n)\neq(0,0),\ \ell=1,2,3\ldots, & \text{TE-modes} \end{cases}$$

Only one of $mn\ell$ can be zero. Hence the three lowest resonance frequencies are

$$f_{TE101} = \frac{c}{2\pi}\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{h}\right)^2} = 4.80\,\text{GHz}$$
$$f_{TE011} = \frac{c}{2\pi}\sqrt{\left(\frac{\pi}{b}\right)^2 + \left(\frac{\pi}{h}\right)^2} = 5.83\,\text{GHz}$$
$$f_{TM110} = \frac{c}{2\pi}\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2} = 6.25\,\text{GHz}$$

S8

We let the waveguide be directed in the $z-$direction with perfectly conducting plates at $z=0$ and $z=h$. The radius of the waveguide is a. The general theory for finite waveguide cavities gives the resonance frequencies

$$f_{mn\ell} = \frac{c}{2\pi}\sqrt{k_{tmn}^2 + \left(\frac{\ell\pi}{h}\right)^2}$$

The transverse wavenumbers and the ℓ-values are given by

$$\begin{cases} k_{tmn} = \xi_{mn}/a, & \ell = 0,1,2,3\ldots\ \text{TM-modes} \\ k_{tmn} = \eta_{mn}/a, & \ell = 1,2,3\ldots\ \text{TM-modes} \end{cases}$$

where ξ_{mn} is the n:th zero of the Bessel function $J_m(k_t a)$ and η_{mn} is the n:th zero of the derivative $J_m'(k_t a)$ The lowest resonance frequencies depend on the height h and the radius a. The candidates are

1. TM_{010} with $f_c = \dfrac{c\xi_{01}}{2\pi a} = \dfrac{3\cdot 10^8\cdot 2.405}{2\pi a}$
2. TM_{110} with $f_c = \dfrac{c\xi_{11}}{2\pi a} = \dfrac{3\cdot 10^8\cdot 3.832}{2\pi a}$
3. TE_{111} with $f_c = \dfrac{c}{2\pi}\sqrt{\left(\dfrac{\eta_{11}}{a}\right)^2 + \left(\dfrac{\pi}{h}\right)^2} = \dfrac{c}{2\pi}\sqrt{\left(\dfrac{1.841}{a}\right)^2 + \left(\dfrac{\pi}{h}\right)^2}$
4. TE_{112} with $f_c = \dfrac{c}{2\pi}\sqrt{\left(\dfrac{\eta_{11}}{a}\right)^2 + \left(\dfrac{2\pi}{h}\right)^2} = \dfrac{c}{2\pi}\sqrt{\left(\dfrac{1.841}{a}\right)^2 + \left(\dfrac{2\pi}{h}\right)^2}$

We have two parameters to determine, namely a and h. There are several options. One is to have $f_{cTM010} = 10$ GHz and $f_{cTE111} = 15$ GHz. This gives $a = 1.15$ cm and $h = 1.16$ cm. The next resonance frequency is $f_{cTM110} = 15.93$ GHz.