9.8. A wide-band Gaussian channel is split in four sub-channels, each with bandwidth $W_\Delta = 1$ kHz. The attenuations and noise parameters are

$$\begin{aligned} |H_i|^2 &= \begin{pmatrix}-36 & -30 & -16 & -21\end{pmatrix} \quad \text{[dB]} \\ N_0 &= \begin{pmatrix}-108 & -129 & -96 & -124\end{pmatrix} \quad \text{[dBm/Hz]} \end{aligned}$$

With the total allowed power on the echannel $P = -50$ dBm, what is the highest possible bit rate on the channel?

Solution:
Use the water-filling procedure and iterate the sub-channel powers.

1. The first itareation gives

$$\begin{aligned} BW_\Delta &= 2.08 \cdot 10^{-5} \text{ mW} \\ P_i &= \begin{pmatrix}-0.42 & 0.21 & 0.11 & 0.21\end{pmatrix} \cdot 10^{-4} \text{ mW} \end{aligned}$$

   Since the first sub-channel has negative power it should be turned of.

2. Second iteartion:

$$\begin{aligned} BW_\Delta &= 6.73 \cdot 10^{-6} \text{ mW} \\ P_i &= \begin{pmatrix}0 & 0.66 & -0.33 & 0.67\end{pmatrix} \cdot 10^{-5} \text{ mW} \end{aligned}$$

   Again, one sub-channel has negative power and sub-channel 3 is turned off.

3. Third iteration

$$\begin{aligned} BW_\Delta &= 5.09 \cdot 10^{-6} \text{ mW} \\ P_i &= \begin{pmatrix}0 & 0.496 & 0 & 0.504\end{pmatrix} \cdot 10^{-5} \text{ mW} \end{aligned}$$

Since all derived powers in the third iteration are positive, they can be used to derive the capacity as,

$$C = 5.34 + 6.67 = 12 \text{ kb/s}$$

9.9. A 3×3 MIMO system with maximum transmit power $P = 10$ and the noise at each receive antenna Gaussian with variance $N = 1$, has the following attenuation matrix

$$H = \begin{pmatrix} 0.05 & 0.22 & 0.73 \\ 0.67 & 0.08 & 0.60 \\ 0.98 & 0.36 & 0.45 \end{pmatrix}$$

Derive the channel capacity for the system. What is the capacity achieving distribution on the transmitted vector $\boldsymbol{X}$?

Hint: The singular value decomposition of H gives

$$U = \begin{pmatrix} -0.36 & 0.88 & 0.30 \\ -0.59 & 0.03 & -0.81 \\ -0.72 & -0.47 & 0.51 \end{pmatrix}, \ S = \begin{pmatrix} 1.51 & 0 & 0 \\ 0 & 0.60 & 0 \\ 0 & 0 & 0.19 \end{pmatrix}, \ V = \begin{pmatrix} -0.74 & -0.65 & -0.16 \\ -0.26 & 0.05 & 0.97 \\ -0.62 & 0.76 & -0.20 \end{pmatrix}$$

Solution:

To start, water-filling should be used to distribute the power over the channels with attenuation according to the singular values. The first iteration gives

$$B = \frac{1}{3}\Big(P + \sum_{i=1}^{3} \frac{N}{s_i^2}\Big) = 13.63$$
$$P_i = B - \frac{N}{s_i^2} = \begin{pmatrix} 13.20 & 10.86 & -14.06 \end{pmatrix}$$

Since channel $i = 3$ has a negative value it should be canceled. The second iteration only contains channels $i = 1, 2$,

$$B = \frac{1}{2}\Big(P + \sum_{i=1}^{2} \frac{N}{s_i^2}\Big) = 6.60$$
$$P_i = B - \frac{N}{s_i^2} = \begin{pmatrix} 6.17 & 3.83 \end{pmatrix}$$

where the power vales are all positive and the iteration can stop. Then the capacity is

$$C = \sum_{i=1}^{2} \frac{1}{2} \log\Big(1 + \frac{s_i^2 P_i}{N}\Big) = 1.96 + 0.63 = 2.59 \text{ bit/channel use}$$

To achieve this information transmission the distribution on the (transformed) transmitter is $\widetilde{\boldsymbol{X}} \sim \mathrm{N}(\boldsymbol{0}, \Lambda_{\widetilde{X}})$, where

$$\Lambda_{\widetilde{X}} = \begin{pmatrix} 6.17 & 0 & 0 \\ 0 & 3.83 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

That gives the distribution $\boldsymbol{X} \sim \mathrm{N}(\boldsymbol{0}, \Lambda_X)$, where

$$\Lambda_X = V \Lambda_{\widetilde{X}} V^T = \begin{pmatrix} 5.01 & 1.05 & 0.95 \\ 1.05 & 0.41 & 1.12 \\ 0.95 & 1.12 & 4.58 \end{pmatrix}$$

9.10. The 5×4 MIMO attenuation matrix

$$H = \begin{pmatrix} 0.76 & 0.40 & 0.61 & 0.90 & 0.42 \\ 0.46 & 0.73 & 0.97 & 0.23 & 0.73 \\ 0.61 & 0.96 & 0.63 & 0.10 & 0.94 \\ 0.36 & 0.88 & 0.78 & 0.83 & 0.30 \end{pmatrix}$$

has the singular value decomposition $H = USV^T$ where

$$\mathrm{diag}(S) = \begin{pmatrix} 2.85 & 0.89 & 0.46 & 0.30 \end{pmatrix}$$

What is the channel capacity for the system if $P = 5$ and $N = 2$?

Solution:

Starting the water-filling to find the power distribution over the channels $i \in \{1, 2, 3, 4\}$ gives

$$B_1 = \frac{1}{4}\Big(P + \sum_{i=1}^{4} \frac{N}{s_i^2}\Big) = 9.90$$
$$P_i = B_1 - \frac{N}{s_i^2} = \begin{pmatrix} 9.65 & 7.38 & 0.45 & -12.48 \end{pmatrix}$$

Removing channel $i = 4$ due to its negative power, the next iteration uses the channels $i \in \{1, 2, 3\}$,

$$B_2 = \frac{1}{3}\Big(P + \sum_{i=1}^{3} \frac{N}{s_i^2}\Big) = 5.74$$
$$P_i = B_2 - \frac{N}{s_i^2} = \begin{pmatrix} 5.49 & 3.22 & -3.71 \end{pmatrix}$$

Again, one of the channels has been assigned negative power so it should be removed. Continuing with the channels $i \in \{1, 2\}$ gives

$$B_3 = \frac{1}{2}\Big(P + \sum_{i=1}^{2} \frac{N}{s_i^2}\Big) = 3.88$$
$$P_i = B_3 - \frac{N}{s_i^2} = \begin{pmatrix} 3.63 & 1.37 \end{pmatrix}$$

All channels have positive powers which gives the capacity

$$C = \sum i = 1^2 \frac{1}{2} \log\Big(1 + \frac{s_i^2 P_i}{N}\Big) = 1.99 + 0.31 = 2.30 \text{ bit/channel use}$$

The distribution on $\widetilde{\boldsymbol{X}}$ is Gaussian with zero mean and covariance matrix

$$\Lambda_{\widetilde{X}} = \begin{pmatrix} 3.63 & 0 & 0 & 0 & 0 \\ 0 & 1.37 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$