(b)

$$
\begin{aligned}
H(Y) &= -\int_0^1 \frac{p}{a}\log\frac{p}{a}dy - \int_1^a \frac{1}{a}\log\frac{1}{a}dy - \int_a^{a+1}\frac{1-p}{a}\log\frac{1-p}{a}dy \\
&= -\frac{p}{a}\log\frac{p}{a} - (a-1)\frac{1}{a}\log\frac{1}{a} - \frac{1-p}{a}\log\frac{1-p}{a} \\
&= \frac{1}{a}h(p) + \log a \\
H(X|Y) &= \sum_x \underbrace{H(Y|X=x)}_{=\log a} P(X=x) = \log a \\
I(X;Y) &= H(Y) - H(Y|X) = \frac{1}{a}h(p)
\end{aligned}
$$

(c) $C = \max_p I(X;Y) = \dfrac{1}{a}$ for $p = \dfrac{1}{2}$.

Chapter 9

9.1. The capacity of this additive white Gaussian noise channel with the output power constraint $E[Y^2] \leq P$ is

$$
\begin{aligned}
C &= \max_{f(X):E[Y^2]\leq P} I(X;Y) = \max_{f(X):E[Y^2]\leq P} (H(Y) - H(Y|X)) \\
&= \max_{f(X):E[Y^2]\leq P} (H(Y) - H(Z))
\end{aligned}
$$

Here the maximum differential entropy is achieved by a normal distribution and the power constraint on Y is satisfied if we choose the distribution of X as $N(0, P-\sigma)$. The capacity is

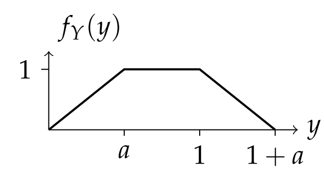
$$
C = \frac{1}{2}\log(2\pi e(P-\sigma+\sigma)) - \frac{1}{2}\log(2\pi e(\sigma)) = \frac{1}{2}\log(2\pi e P) - \frac{1}{2}\log(2\pi e\sigma) = \frac{1}{2}\log(\frac{P}{\sigma})
$$

9.2. From the problem formulation we know that

$$
\begin{aligned}
&X \sim U(1) \\
&Z \sim U(a), \quad 0 < a \leq 1
\end{aligned}
$$

Then the addititve result $Y = X + Z$ has the density function

$$
f_Y(y) = f_X(x) * f_Z(z) = \begin{cases} \frac{y}{a} & 0 \leq y \leq a \\ 1 & a \leq y \leq 1 \\ 1 + \frac{1}{a} - \frac{y}{a} & 1 \leq y \leq 1+a \end{cases}
$$



The mutual information can be derived as

$$
I(X;Y) = H(Y) - H(Y|X) = H(Y) - H(Z) = H(Y) - \log a
$$

To derive the entropy of Y split the derivation in three parts according to the linear slopes in the density function. Use that $\int y \ln y dy = \frac{y^2}{2}\ln y - \int \frac{y^2}{2}\frac{1}{y}dy = \frac{y^2}{2}\ln y - \frac{y^2}{4}$.

$$
\begin{aligned}
H_1 &= -\int_0^a \frac{y}{a}\log\frac{y}{a}dy = \frac{1}{a}\log a\int_0^a ydy - \frac{1}{a}\int_0^a y\log ydy = \frac{a}{2}\log a - \frac{a}{2}\log a + \frac{a}{4\ln 2} = \frac{a}{4\ln 2}\\
H_2 &= -\int_a^1 1\log 1dy = 0\\
H_3 &= -\int_1^{1+a}\Big(1+\frac{1}{a}-\frac{y}{a}\Big)\log\Big(1+\frac{1}{a}-\frac{y}{a}\Big)dy = \begin{bmatrix} s = a+1-y\\ ds = -dy\\ y=1\Rightarrow s=a\\ y=1+a\Rightarrow s=0\end{bmatrix}\\
&= -\int_0^a \frac{s}{a}\log\frac{s}{a}ds = H_1(Y) = \frac{a}{4\ln 2}
\end{aligned}
$$

The variable substitution is found from $\frac{s}{a} = 1+\frac{1}{a}-\frac{y}{a}$. Summing up gives the entropy $H(Y) = H_1+H_2+H_3 = \frac{a}{2\ln 2}$, and the mutual information becomes

$$I(X;Y) = \frac{a}{2\ln 2} - \log a$$

9.3. (a) The received power is

$$P_Z = |H_2|^2 P_Y = |H_1|^2|H_2|^2 P_X$$

and the received noise is Gaussian with variance

$$N = N_1|H_1|^2 + N_2$$

Hence, an equivalent channel model from X to Z has the attenuation H_1H_2 and additive noise with distribution $n \sim N(0, \sqrt{N_1|H_1|^2+N_2})$. That means the capacity becomes

$$C = W\log\Big(1+\frac{|H_1|^2|H_2|^2P_X}{W(N_1|H_1|^2+N_2)}\Big)$$

(b) From the problem we get the SNRs for the two channels

$$
\begin{aligned}
\mathrm{SNR}_1 &= \frac{|H_1|^2P_X}{WN_1} = \frac{P_Y}{WN_1}\\
\mathrm{SNR}_1 &= \frac{|H_2|^2P_Y}{WN_2} = \frac{P_Z}{WN_2}
\end{aligned}
$$

Then the total SNR can be expressed as

$$
\begin{aligned}
\mathrm{SNR} &= \frac{|H_1|^2|H_2|^2P_X}{W(N_1|H_1|^2+N_2)} = \frac{\frac{|H_1|^2P_X|H_2|^2P_Y}{WN_1WN_2}}{\frac{P_Y}{W^2N_1N_2}W(N_2+N_1|H_1|^2)}\\
&= \frac{\frac{|H_1|^2P_X}{WN_1}\frac{|H_2|^2P_Y}{WN_2}}{\frac{P_Y}{WN_1}+\frac{|H_1|^2P_Y}{WN_2}} = \frac{\mathrm{SNR}_1\cdot\mathrm{SNR}_2}{\mathrm{SNR}_1+\mathrm{SNR}_2}
\end{aligned}
$$

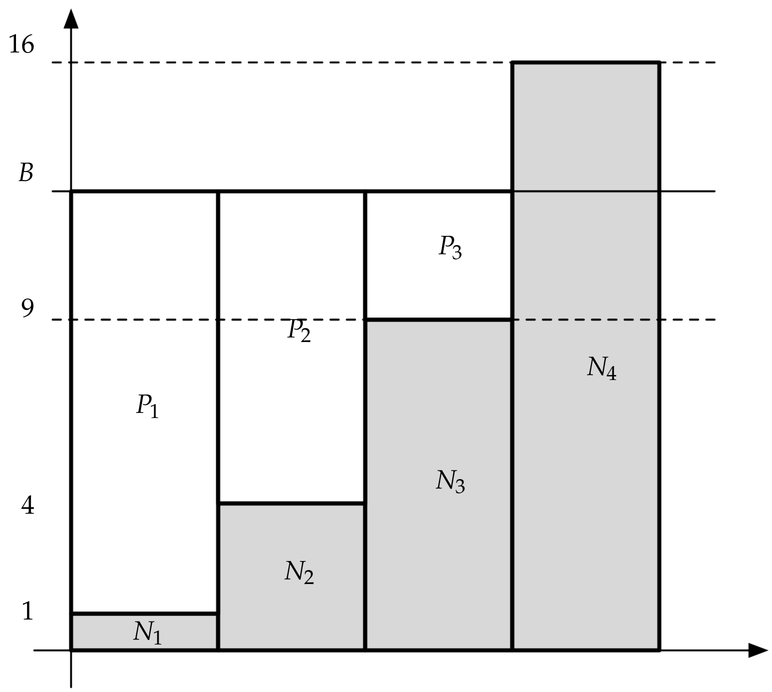
Notice, that by considering the invers of the SNR, the noise to signal ratio, the derivations can be considerably simplified,

$$\frac{1}{\mathrm{SNR}} = \frac{W(N_1|H_1|^2+N_2)}{|H_1|^2|H_2|^2P_X} = \frac{WN_1}{|H_2|^2P_X} + \frac{WN_2}{|H_1|^2|H_2|^2P_X} = \frac{1}{\mathrm{SNR}_1}+\frac{1}{\mathrm{SNR}_2}$$

which is equivalent to the desired result.

9.4. We can use the total power $P_1 + P_2 + P_3 + P_4 = 17$ and for the four channels the noise power is $N_1 = 1$, $N_2 = 4$, $N_3 = 9$, $N_4 = 16$. Let $B = P_i + N_i$ for the used channels. Since $(16-1) + (16-4) + (16-9) > 17$ we should not use channel four when reaching capacity. Similarly, since $(9-1) + (9-4) < 17$ we should use the rest of the three channels. These tests are marked as dashed lines in the figure below. Hence, $B = P_1 + 1 = P_2 + 4 = P_3 + 9$, which leads to $B = \frac{1}{3}(P_1 + P_2 + P_3 + 14) = \frac{1}{3}(17 + 14) = \frac{31}{3}$. The capacity becomes

$$\begin{aligned} C &= \sum_{i=1}^{3} \frac{1}{2} \log\Big(1 + \frac{P_i}{N_i}\Big) = \sum_{i=1}^{3} \frac{1}{2} \log \frac{B}{N_i} = \frac{1}{2} \log \frac{\frac{31}{3}}{1} + \frac{1}{2} \log \frac{\frac{31}{3}}{4} + \frac{1}{2} \log \frac{\frac{31}{3}}{9} \\ &= \frac{3}{2} \log 31 - \frac{5}{2} \log 3 - 1 \approx 2.4689 \end{aligned}$$



9.5. (a) Use the water filling algorithm to derive the capacity. When a sub-channel is deleted ($P_i = 0$) the total number of sub-channel is changed and the power distribution has to be recalculated. We get the following recursion:

1. Iteration 1
   $B = B - N_i = \frac{1}{6}(\sum_i N_i + P) = 14.17$
   $P_i = (6.17, 2.17, 0.17, 4.17, -1.83, 8.17)$
   Sub-channel 5 should not be used, $P_5 = 0$.
2. Iteration 1
   $B = \frac{1}{5}(\sum_{i \neq 5} N_i + P) = 13.8$
   $P_i = B - N_i = (5.80, 1.80, -0.20, 3.80, 0, 7.80)$
   Sub-channel 3 should not be used, $P_3 = 0$.
3. Iteration 1
   $B = \frac{1}{4}(\sum_{i \neq 3,5} N_i + P) = 13.75$
   $P_i = B - N_i = (5.75, 1.75, 0, 3.75, 0, 7.75)$
   All remaining sub-channels can be used.

The capacities in the sub-channels are

$$C_i = \frac{1}{2}\log\Big(1+\frac{P_i}{N_i}\Big) = (0.39, 0.10, 0, 0.23, 0, 0.60)$$

and the total capacity $C = \sum_i C_i = 1.32$ bit/transmission.

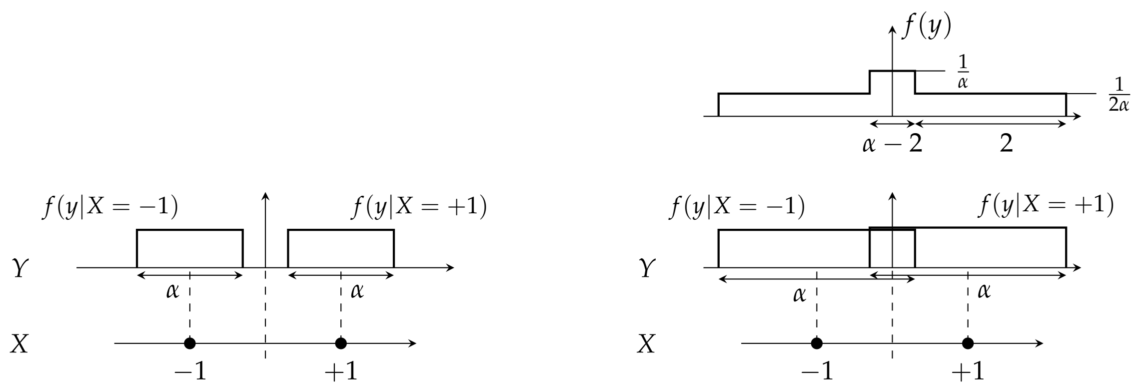
(b) If the power is equally distributed over the sub-channels we get $P_i = 19/6 = 3.17$. That gives the capacities

$$\begin{aligned} C &= \sum_i \frac{1}{2}\log\Big(1+\frac{19/6}{N_i}\Big) \\ &= 0.24 + 0.17 + 0.15 + 0.20 + 0.13 + 0.31 = 1.19 \text{ bit/transmission} \end{aligned}$$

(c) When using only one sub-channel the capacity is maximised if we take the one with least noise, $N = 6$. This gives the capacity $C = \frac{1}{2}\log 2(1 + 19/6) = 1.03$ bit/transmission.

Chapter 10

10.1. In the following figure the resulting distributions are depicted.



(a) For $\alpha < 2$ the left figure describes the received distribution. Since the density functions $f(y|X = 1)$ and $f(y|X = -1)$ are non-overlapping, the transmitted value can directly be determined from the received Y. Hence, $I(X;Y) = 1$.
The result can also be found from the following derivations:

$$\begin{aligned} H(Y|X=i) &= -\int_{-\alpha/2}^{\alpha/2} \frac{1}{\alpha}\log\frac{1}{\alpha}dx = \log\alpha \\ H(Y|X) &= \sum \frac{1}{2} H(Y|X=i) = \log\alpha \\ H(Y) &= -2\int_{-\alpha/2}^{\alpha/2} \frac{1}{2\alpha}\log\frac{1}{2\alpha}dx = \log 2\alpha = 1 + \log\alpha \\ I(X;Y) &= H(Y) - H(Y|X) = 1 \text{ b/tr} \end{aligned}$$

(b) For $\alpha \geq 2$ there is an overlap between $f(y|X = 1)$ and $f(y|X = -1)$ as shown in the right figure. Still, $H(Y|X) = \log\alpha$, but since the distribution of Y depends on the amount of the overlap, we need to rederive the entropy,

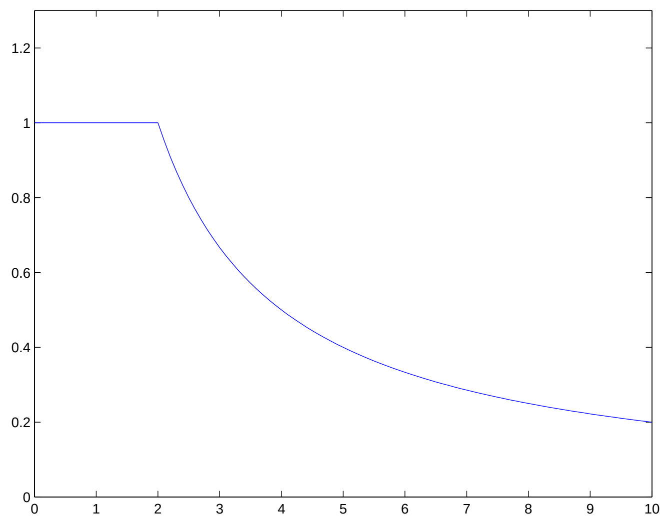
$$
\begin{aligned}
H(Y) &= -\int_{-1-\frac{\alpha}{2}}^{1-\frac{\alpha}{2}} \frac{1}{2\alpha}\log\frac{1}{2\alpha}dx - \int_{1-\frac{\alpha}{2}}^{-1+\frac{\alpha}{2}} \frac{1}{\alpha}\log\frac{1}{\alpha}dx - \int_{-1+\frac{\alpha}{2}}^{1+\frac{\alpha}{2}} \frac{1}{2\alpha}\log\frac{1}{2\alpha}dx \\
&= 2\frac{1}{2\alpha}\log(2\alpha)2 + \frac{1}{\alpha}\log(\alpha)(\alpha-2) \\
&= \frac{2}{\alpha} + \log\alpha
\end{aligned}
$$

Thus, $I(X;Y) = \frac{2}{\alpha}$ b/tr.

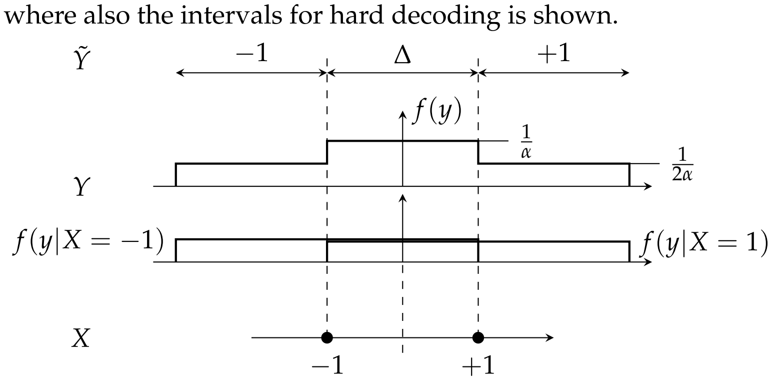
Summarising, the mutual information becomes

$$I(X;Y) = \min\Big\{1, \frac{1}{\alpha}\Big\}, \quad \alpha > 0$$

which is plotted below.



10.2. For $\alpha = 4$ the mutual information is $I(X;Y) = 2/4 = 1/2$. We get the following density functions, where also the intervals for hard decoding is shown.



The probability for overlap is $P(\Delta|X=i) = 1/2$, and the resulting DMC channel is the binary erasure channel. Hence, the capacity is

$$C_{\text{BEC}} = 1 - \frac{1}{2} = \frac{1}{2}$$

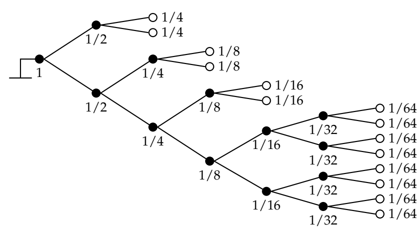
In most cases it is beneficially to use the the soft information, the value of the received symbol instead of the hard decoding, since it should grant some extra information. E.g. in the case of binary transmission and Gaussian noise it is a difference if the received symbol is 3 or 0.5. But in the case here we have uniform noise. Then we get three intervals where for $\tilde{Y} = -1$ it is certain that $X = -1$ and for $\tilde{Y} = 1$ it is certain that $X = 1$. When $\tilde{Y} = \Delta$ the two possible transmitted alternatives are equally likely and we get no information at all. Since the information is either complete or none, there is no difference between the two models.

As a comparison, for $\alpha > 2$ the probability for the overlapped interval is $P(\Delta|X = i) = \frac{\alpha-2}{\alpha} = 1 - \frac{2}{\alpha}$. Thus, the capacity for the BEC is $C_{\text{BEC}} = \frac{2}{\alpha}$, which is the same as for the continuous case.

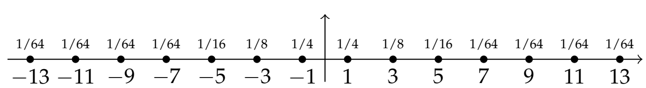
10.3. The mutual information is $I(X;Y) = H(Y) - H(Y|X) = H(Y) - H(X+Z|X) = H(Y) - H(Z)$, where $H(Z) = \log(1-(-1)) = \log 2$. Since Y ranges from -3 to 3 with uniform weights $p_{-2}/2$ for $-3 \leq Y \leq -2$, $(p_{-2}+p_{-1})/2$ for $-2 \leq Y \leq -1$ etc the maximum of $H(Y)$ is obtained for a uniform Y. This can be achieved if the distribution of X is $(\frac{1}{3}, 0, \frac{1}{3}, 0, \frac{1}{3})$. Now $H(Y) = \log(3-(-3)) = \log 6$.

We conclude that $C = \log 6 - \log 2 = \log 3$.

10.4. (a) The original system is 8-PAM with equal probabilities, which has a second order moment $E[X^2] = 21$. Assigning equal probability for zero and one in the tree, the probabilities for the nodes in the tree becoms

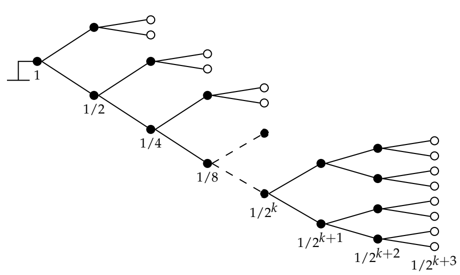


By summing the inner node probabilities, the average length is 3, hence it is comapable in bit rate with the 8-PAM system. If the leaves are mapped to signals in an 14-PAM constellation where the least probable nodes has the highes distance to the center, the following constellation probabilities are obtained



Thus, the second order moment is $E[X^2] = 19$, which gives the shaping gain $\gamma_s = 10\log_{10}\frac{21}{19} = 0.4347$ dB.

(b) Below is a tree with k levels and the probabilities for the levels.



The average lengths for the paths in the tree is, according to the path length lemma,

$$\begin{aligned}
L &= 1 + 2\frac{1}{2} + 2\frac{1}{4} + 2\frac{1}{8} + \cdots + 2\frac{1}{2^{k+1}} + 4\frac{1}{2^{k+2}} \\
&= 1 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^k} + \frac{1}{2^k} \\
&= 1 + \sum_{i=0}^{k} (\tfrac{1}{2})^i + \frac{1}{2^k} = 1 + \frac{1-(\frac{1}{2})^{k+1}}{1-\frac{1}{2}} + \frac{1}{2^k} = 1 + 2 - (\tfrac{1}{2})^k + \frac{1}{2^k} = 3
\end{aligned}$$

(c) In the tree there are $2k+4$ leaves, each mapped to a signal point in a PAM constellation. Mapping high probability leaves to short distance fro origin, means that on the positive signal axis, signal $2i-1$ has probability $1/2^{i+1}$, $i = 1, \dots k$. Then therer are also four signals at positions $2k+1$, $2k+3$, $2k+5$ and $2k+7$ with probabilities $1/2^{k+3}$. Deriving the second order moment for the positive half gives

$$\begin{aligned}
\frac{1}{2}E[X^2] &= \sum_{i=1}^{k} (2i-1)^2 \frac{1}{2^{i+1}} + \frac{1}{2^{k+3}} \Big(\underbrace{(2k+1)^2 + (2k+3)^2 + (2k+5)^2 + (2k+7)^2}_{\approx 4\cdot 4k^2 = 2^4 k^2,\ k \text{ large}} \Big) \\
&= \underbrace{2\sum_{i=1}^{k} i^2 \frac{1}{2^i}}_{\to 12} + \underbrace{2\sum_{i=1}^{k} i \frac{1}{2^i}}_{\to 4} + \underbrace{\frac{1}{2}\sum_{i=1}^{k} \frac{1}{2^i}}_{\to \frac{1}{2}} + \underbrace{\frac{2^2}{2^{k-1}}}_{\to 0} \to \frac{17}{2}, \quad k \to \infty
\end{aligned}$$

Hence, $E[X^2] \to 17$ as k grows to infinity. The shaping gain is $\gamma_s = 10 \log_{10} \frac{21}{17} = 0.9177$ dB.

10.5. (a) The radius is derived in the book as $R = \frac{\Gamma(\frac{N}{2}+1)^{1/N}}{\sqrt{\pi}}$.

(b) The integral specifies that one variable (dimension) is fixed and the integral spans over the remaining dimensions. This gives the projection in one dimension. Using the formula

$$\int_{|\boldsymbol{x}|^2 \le R^2} f(|\boldsymbol{x}|) d\boldsymbol{x} = \frac{2\pi^{N/2}}{\Gamma(\frac{N}{2})} \int_0^R x^{N-1} f(x) dx$$

gives

$$\begin{aligned}
f(x) = f_X(x) &= \int_{|\tilde{\boldsymbol{x}}|^2 \le R^2 - x^2} d\tilde{\boldsymbol{x}} = \frac{2\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})} \int_0^{\sqrt{R^2-x^2}} x^{N-2} dx \\
&= \frac{2\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})} \frac{(R^2-x^2)^{\frac{N-1}{2}}}{N-1} \\
&= \frac{\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N+1}{2})} \left(\frac{\Gamma(\frac{N}{2}+1)^{2/N}}{\pi} - x^2 \right)^{\frac{N-1}{2}}
\end{aligned}$$

where in the last equality the radius is inserted and it is used that $s\Gamma(s) = \Gamma(s+1)$.

(c) Using Stirling's approximation gives

$$\begin{aligned}
f(x) &\approx \frac{\pi^{\frac{N-1}{2}}}{\left(\frac{\frac{N-1}{2}}{e}\right)^{\frac{N-1}{2}}} \left(\frac{\left(\frac{\frac{N}{2}}{e}\right)^{\frac{N}{2}\frac{2}{N}}}{\pi} - x^2 \right)^{\frac{N-1}{2}} \\
&= \left(\frac{\pi 2e}{N-1} \left(\frac{N}{2\pi e} - x^2 \right) \right)^{\frac{N-1}{2}} \\
&= \left(1 + \frac{\frac{1}{2} - \pi e x^2}{\frac{N-1}{2}} \right)^{\frac{N-1}{2}}
\end{aligned}$$

(d) Letting $N \to \infty$, and hence $\frac{N-1}{2} \to \infty$, gives

$$\lim_{N\to\infty} f(x) = e^{\frac{1}{2}-\pi e x^2} = \frac{1}{\sqrt{2\pi\frac{1}{2\pi e}}} e^{-\frac{x^2}{2\frac{1}{2\pi e}}} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

where $\sigma^2 = \frac{1}{2\pi e}$. Hence, projecting the infinity-dimensional spherical uniform distribution, to one dimension gives the Normal distribution, $X \sim N(0, \sqrt{\frac{1}{2\pi e}})$.

10.6. (a) There is a constant power level of $P = -70$ dBm/Hz over the whole bandwith. Similarly, the noise level is $N_0 = -140$ dBm/Hz. However the attenuation of the transmitted signal varies over the channel as $|H_i|^2 = 5i + 10$ dB. (In reallity this can resemblance copper cable transmission, where the cable act as a low-pass filter, attenuating higher frequencies stronger than lower. However, the attenuation curve is a bit more complicated than a linerly decreasing function.)

The received signal to noise ratio for each sub-channel becomes

$$\text{SNR}_i = -70 - (5i + 10) + 140 = 60 - 5i \text{ dB}$$

and the derived capacity per sub-chanel

$$C_i = \Delta f \log\left(1 + 10^{\text{SNR}_i/10}\right) = 10^4 \log\left(1 + 10^{(60-5i)/10}\right)$$

In the following table the attenuation, SNR and capacity is listed for the sub-channels

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\lvert H_i\rvert^2$[dB]	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85
SNR_i[dB]	60	55	50	45	40	35	30	25	20	15	10	5	0	-5	-10	-15
C_i[kbps]	199	183	166	149	133	116	100	83	66	50	35	21	10	4.0	1.4	0.45

Summing over all sub-chanels gives the total capacity as

$$C = \sum_i C_i = 1317 \text{ [kbps]}$$

(b) Instead of the capacity, we want to derive an estimate of the established bit rate when the system is working with a bit error rate of 10^{-7} and an error correcting code with coding gain $\gamma_c = 3$ dB. The bit error rate gives an SNR gap of $\Gamma = 9$ dB, and the efficient SNR becomes

$$\widetilde{\text{SNR}}_i = \text{SNR}_i - \Gamma + \gamma_c = \text{SNR}_i - 6 \text{ dB}$$

The estimated bit rate is

$$R_i = \Delta f \log\left(1 + 10^{\widetilde{\text{SNR}}_i/10}\right)$$

In the following table the effective SNR and the estimated bit rate is shown.

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\widetilde{\text{SNR}}_i$[dB]	54	49	44	39	34	29	24	19	14	9	4	-1	-6	-11	-16	-21
R_i[kbps]	179	163	146	130	113	96	80	63	47	32	18	8.4	3.2	1.1	0.36	0.11

The total bit rate is

$$R = \sum_i R_i = 1080 \text{ [kbps]}$$

Chapter 11

11.1. From the problem we have $P(X = j) = \frac{1}{k}$, $j = 0, 1, \dots, k-1$, and that te Hamming distortion is used. Assign the probability of distortion as $P(X \neq \hat{X})$. Then the average distortion is $E[d(X, \hat{X})] = \delta$ which is within the minimisation criteria. The mutual information betweeen X and $\hat{X}$ is bounded by

$$I(X; \hat{X}) = H(X) - H(X|\hat{X}) = \log k - H(X|\hat{X}) \geq \log k - \delta \log(k-1) - h(\delta)$$

where the inequality follows from Fano's lemma as

$$H(X|\hat{X}) \leq h\big(P(X \neq \hat{X})\big) + P(X \neq \hat{X}) \log(k-1) = \delta \log(k-1) + h(\delta)$$

To show the rate distortion function we need to find a distribution on $P(X|\hat{X})$ that achieves equality in the bound above. From our asumptions we get $P(X = \hat{X}) = 1 - \delta$. A reasonable attempt is to set uniform distribution for the case when $X \neq \hat{X}$, i.e.

$$P(X = j|\hat{X} = i) = \begin{cases} 1 - \delta, & i = j \\ \frac{\delta}{k-1}, & i \neq j \end{cases}$$

The conditional entropy can be derived as

$$H(X|\hat{X}) = \sum_i P(\hat{X} = i) \sum_i H(X|\hat{X} = i)$$

where

$$\begin{aligned} H(X|\hat{X} = i) &= -\sum_{j \neq j} \frac{\delta}{k-1} \log \frac{\delta}{k-1} - (1-\delta)\log(1-\delta) \\ &= -\delta \log \delta + \delta \log(k-1) - (1-\delta)\log(1-\delta) = \delta \log(k-1) + h(\delta) \end{aligned}$$

Since $iH(X|\hat{X} = i)$ is independent of i, the assumed distribution achieves equality in the bound for the mutual information, and $R(\delta) = \log k - \delta \log(k-1) - h(\delta)$. Finally, we need to find the limits on δ, where the rate distortion function reaches zero. Then, observing that when $\delta = \frac{k-1}{k}$ the conditional distribution $P(X = j|\hat{X} = i) = \frac{1}{k}$, independent of i and j, this gives a point where $H(X|\hat{X}) = \log k$. Hence, at this point $R(\delta) = 0$. Since ther rate distortion function is non-increasing and non-negative, we conclude that

$$R(\delta) = \begin{cases} \log k - \delta \log(k-1) - h(\delta), & 0 \leq \delta \leq \frac{k-1}{k} \\ 0, & \delta \geq 0 \end{cases}$$

11.2. —

11.3. Follows directly from Problem 11.2.

11.4. (a) The density function is $f(x) = \frac{1}{2\sqrt{\pi}} e^{x^2/4}$. Since everything is symmetric around $x = 0$, the derivations will be made only for the positive half. The numerical integrations that follows can be erformed in different ways, here a trapetsoid method was used. Assuming a set of x-values, $\boldsymbol{x} = x_1, \dots, x_n$, with constant separation $x_i - x_{i-1} = \Delta$. Let $\boldsymbol{y} = y_1, \dots, y_n$ be the corresponding set of function values. Then the area can be approximated by

$$\int_{x_1}^{x_n} y(x) dx \approx \Delta \Big(\sum_{i=1}^{n} y_i - \frac{y_1 + y_n}{2} \Big)$$

To derive the distortion the intervals $\{[0,1],[1,2],[2,3],[3,\infty]\}$ is used. In the numerical derivations setting ∞ to 10 seems good enough. Then, assign $\delta_i = E[(X - X_{q,i})^2]$ to get

$$\delta_1 = \int_0^1 (x-0.5)^2 f(x)dx \approx 0.0214$$
$$\delta_2 = \int_1^2 (x-1.5)^2 f(x)dx \approx 0.0134$$
$$\delta_3 = \int_2^3 (x-2.5)^2 f(x)dx \approx 0.0053$$
$$\delta_4 = \int_3^\infty (x-3.5)^2 f(x)dx \approx 0.0036$$

To derive the total average distortion we can use $E[(X - X_q)] = \sum_i E[(X - X_{q,i})]$ over both the positive and negative side, which gives the average distortion

$$E[(X - X_q)^2] = 2(\delta_1 + \delta_2 + \delta_3 + \delta_4) \approx 0.0874$$

(b) In general,the distortion in the interval $[a, b]$ when reconstructing to x_q is $\delta = \int_a^b (x - x_q)^2 f(x)dx$. Optimising with respect to the reconstruction value gives

$$\frac{\partial}{\partial x_q}\delta = -\int_a^b 2(x - x_q) f(x)dx = 2x_q \int_a^b f(x)dx - 2\int_a^b x f(x)dx = 0$$

hence, the optimal reqonstruction value is given by

$$x_q^{(opt)} = \frac{\int_a^b x f(x)dx}{\int_a^b f(x)dx} \approx \begin{cases} 0.48, & i=1 \\ 1.44, & i=2 \\ 2.40, & i=3 \\ 3.51, & i=4 \end{cases}$$

The corresponding distortion measures are given by

$$\delta_i^{(opt)} \approx \begin{cases} 0.0213, & i=1 \\ 0.0129, & i=2 \\ 0.0047, & i=3 \\ 0.0036, & i=4 \end{cases}$$

and the total distortion $E[(X - x_q^{(opt)})^2] \approx 0.0850$.

(c) If the quantiser is followed by a compression algorithm, and the samples can be viewed as independent, a limit on the number of bits per symbol is given by the entropy,

$$L \geq H(P) = 2.55 \text{ bit/sample}$$

Note: If the minimum length is estimetaed by a Huffamncode instead, it becomes 2.6 bit/sample.

11.5. The distortion is

$$\begin{aligned} E\Big[\big(X - X_Q\big)^2\Big] &= \int_{-\infty}^0 \Big(x + \sqrt{\frac{2}{\pi}}\sigma\Big)^2 f(x) + \int_0^\infty \Big(x - \sqrt{\frac{2}{\pi}}\sigma\Big)^2 f(x) \\ &= \int_{-\infty}^\infty x^2 f(x)dx + 2\sqrt{\frac{2}{\pi}}\sigma\Big(\int_{-\infty}^0 x f(x)dx - \int_0^\infty x f(x)dx\Big) + \frac{2}{\pi}\sigma^2 \int_{-\infty}^\infty f(x)dx \\ &= \sigma^2 - 4\sqrt{\frac{2}{\pi}}\sigma \int_0^\infty x f(x)dx + \frac{2}{\pi}\sigma^2 \\ &= \sigma^2 - 4\sqrt{\frac{2}{\pi}}\sigma \frac{\sigma}{\sqrt{2\pi}} + \frac{2}{\pi}\sigma^2 = \sigma^2\Big(1 - \frac{2}{\pi}\Big) = \frac{\sigma^2}{\pi}(\pi - 2) \end{aligned}$$

where it is used that $\int_{-\infty}^{0} xf(x)dx = -\int_{0}^{\infty} xf(x)dx$ and that

$$\int_0^\infty xf(x)dx = \frac{1}{\sqrt{2\pi}\sigma}\int_0^\infty xe^{-x^2/2\sigma^2}dx = \frac{1}{\sqrt{2\pi}\sigma}\left[-\sigma^2 e^{-x^2/2\sigma^2}\right]_0^\infty = \frac{\sigma}{\sqrt{2\pi}}$$

Chapter 12