Information Theory

Lecture 13
Discrete Input Gaussian channel

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M-PAM

Pulse Amplitude Modulation with M levels

- Amplitudes $A_i = M - 1 - 2i, \quad i = 0, 1, \dots, M-1$
- Signal alternative

$$s_i(t) = A_i g(t)$$

where $g(t)$ has duration $0 \le t \le T_s$ and T_s signalling time.

- The total signal is

$$s(t) = \sum_{\ell=-\infty}^{\infty} A_{i_\ell} g(t - \ell T_s)$$

- Assume equally likely signal alternatives, $P_i = \frac{1}{M}$
- Energy per signal alternative

$$E_s = \frac{M^2 - 1}{3} E_g, \quad E_g = \int g^2(t) dt$$

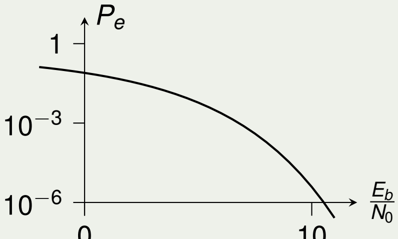
2-PAM vs BSC

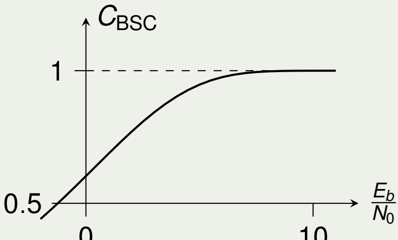
Example

In 2-PAM signalling over AWGN the error is

$$P_e = Q\left(\sqrt{2\frac{E_b}{N_0}}\right), \text{ where } Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

P_e error probability in a BSC. The capacity is $C = 1 - h(P_e)$.

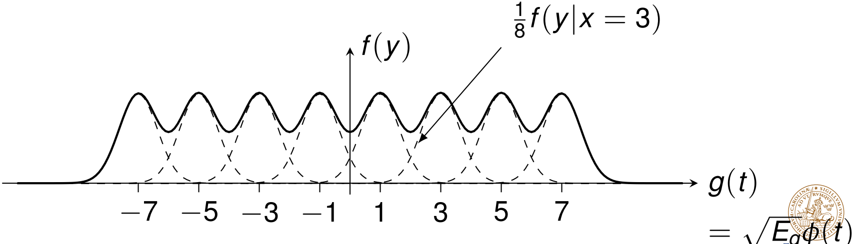




M-PAM signals and capacity

With equally likely signals, $p(x) = \frac{1}{M}$, how much information can be transmited over an AWGN channel?

$$I(X;Y) = H(Y) - H(Y|X)$$
$$H(Y|X) = \frac{1}{2}\log \pi e N_0$$
$$f(y) = \sum_x \frac{1}{M} f(y|x) = \frac{1}{M}\sum_x \frac{1}{\sqrt{\pi N_0}} e^{(y-x)^2/N_0}$$



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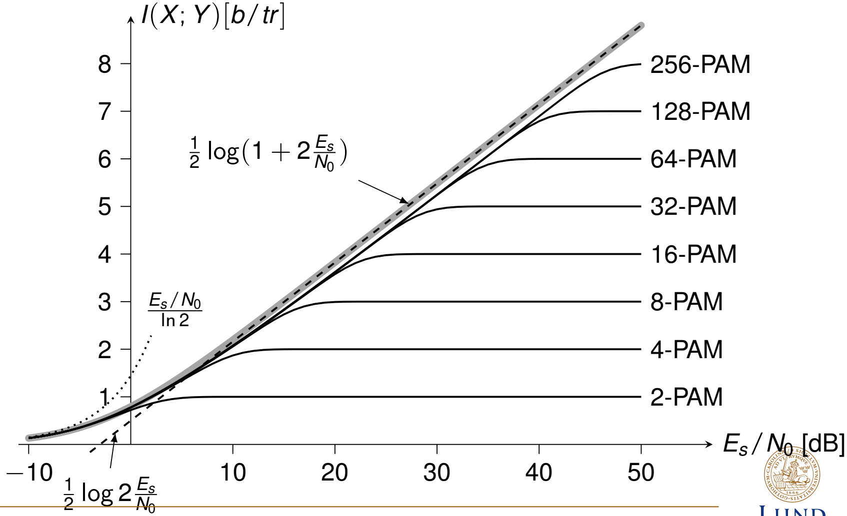
M-PAM constraint capacity



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Shaping gain γ_s

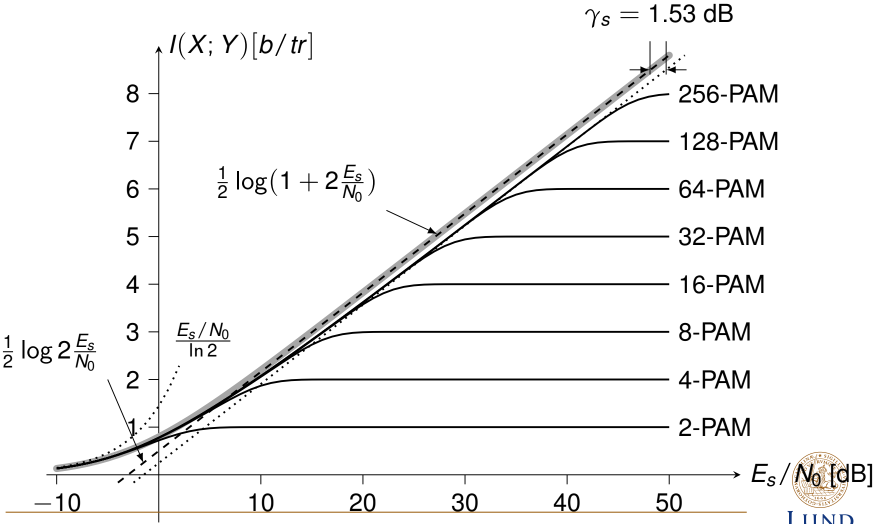
To reach capacity the input distribution is Gaussian, but here we use uniform.
Compare

$$X_g \sim N(0, \sigma)$$
$$X_u \sim U(-a, a)$$

For high SNR, $I(X;Y)$ is dominated by $H(X)$. Compare required powers for equal input entropy,

$$\gamma_s = \frac{P_u}{P_g} = \frac{\pi e}{6} = 1.62 = 1.53 \text{ dB}$$

M-PAM contraint capacity



SNR gap

Error probability for M-PAM

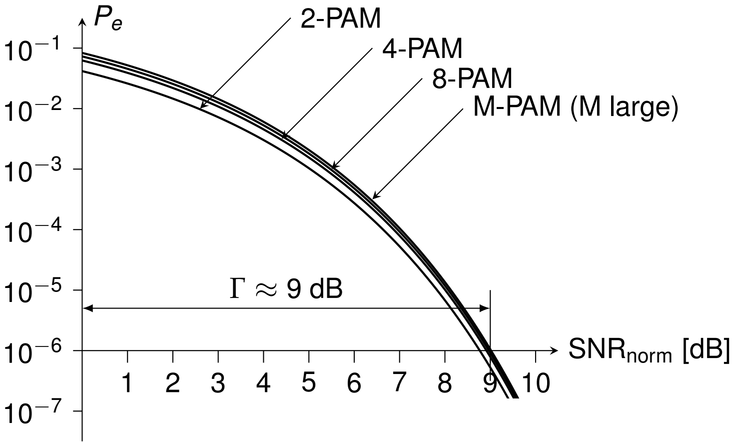
$$\begin{aligned} P_e &= 2\Big(1 - \tfrac{1}{M}\Big) Q\Big(\sqrt{\tfrac{3}{M^2-1}\mathrm{SNR}}\Big) \\ &= 2\Big(1 - \tfrac{1}{M}\Big) Q\Big(\sqrt{3 \cdot \mathrm{SNR}_{\mathrm{norm}}}\Big) \approx 2Q\Big(\sqrt{3 \cdot \mathrm{SNR}_{\mathrm{norm}}}\Big) \end{aligned}$$

Consider a normalised SNR:

$$\mathrm{SNR}_{\mathrm{norm}} = \frac{\mathrm{SNR}}{2^{2k} - 1} \quad \begin{cases} = 0 \text{ dB}, & k = C \\ > 0 \text{ dB}, & k < C \end{cases}$$

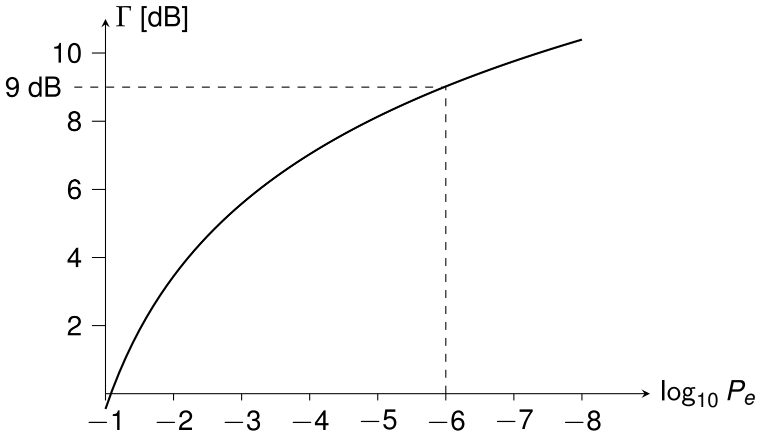
This is a measure of how far from the capacity the system is working.

SNR gap



P_e
2-PAM
4-PAM
8-PAM
M-PAM (M large)
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
$\Gamma \approx 9$ dB
$\mathrm{SNR_{norm}}$ [dB]
1 2 3 4 5 6 7 8 9 10

SNR gap



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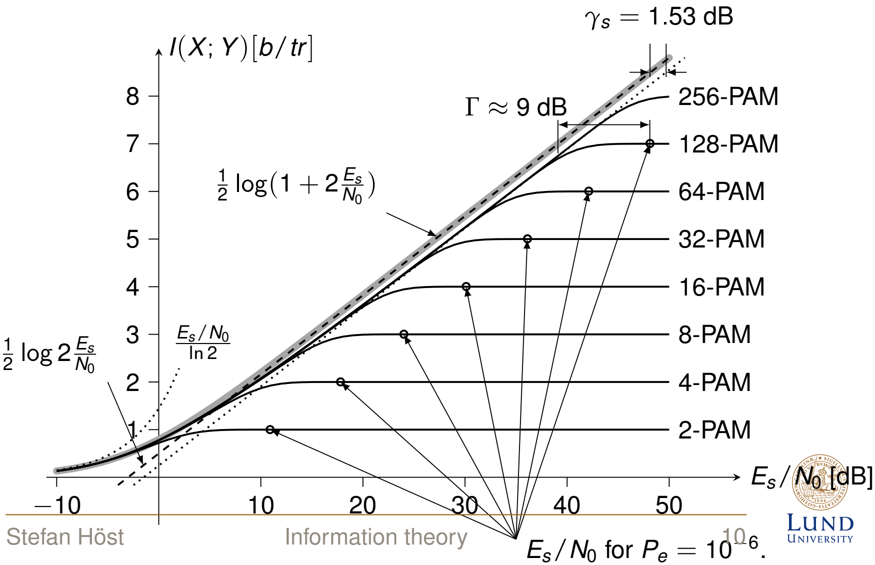
M-PAM contraint capacity



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SNR gap

Bit rate

For an (uncoded) PAM or QAM communication system working at the error probability P_e, the SNR gap is

$$\Gamma = \frac{1}{3}\Big[Q^{-1}\Big(P_e/2\Big)\Big]^2$$

The estimated bit rate is

$$R_b = W\log\Big(1 + \frac{\text{SNR}}{\Gamma}\Big) = W\log\Big(1 + \frac{P}{\Gamma N_0 W}\Big)$$