LUND UNIVERSITY

Information Theory

Lecture 12
Parallel Gaussian channels, OFDM and MIMO

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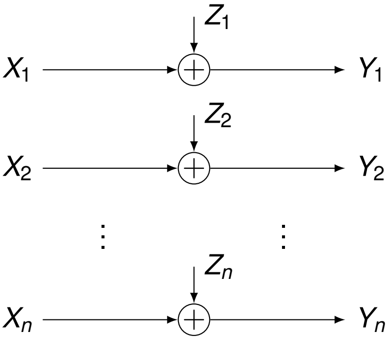
Parallel Gaussian channels

Consider n parallel, independent, Gaussian channels with a total power contraint of $\sum_i P_i \leq P$. The noise on channel i is $Z_i \sim \mathcal{N}(0, \sqrt{N_i})$.



Parallel Gaussian channels
Water filling

Theorem

Given k independent parallel Gaussian channels with noise variance N_i, $i = 1, \ldots, k$, and a restricted total transmitted power, $\sum_i P_i = P$.
The capacity is given by

$$C = \frac{1}{2}\sum_{i=1}^{n} \log(1 + \frac{P_i}{N_i})$$

where

$$P_i = \big(B - N_i\big)^+, \qquad (x)^+ = \begin{cases} x, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

and B is such that $\sum_i P_i = P$.

Frequency division

Wideband channel – OFDM

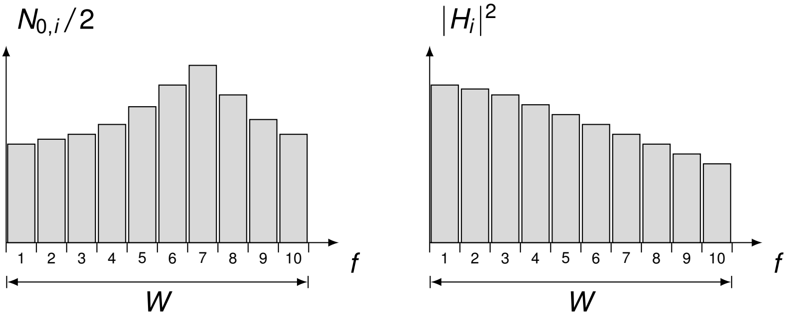
Consider a wideband channel of bandwidth W, where

- Noise and channel attenuation varies over frequency
- Power constraint: $E[x(t)^2] = P$

Split into n independent sub-channels of bandwidth $W_\Delta = \frac{W}{n}$, s.t.

- Noise and channel attenuation constant
- Power constraint: $\sum_i P_i = P$ where $P_i = E[X_i^2]$ power in sub-channel i

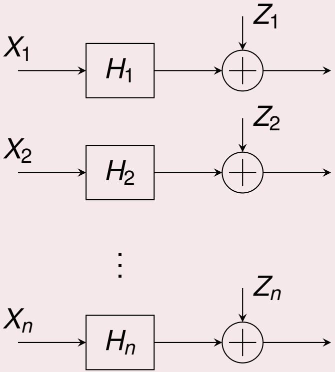
Frequency division



Frequency division

Parallel channels

Equivalent model n parallel channels:



$$Y_1 = X_1 + Z_1, \quad Z_1 \sim N\left(0, \sqrt{\frac{N_{0,1}}{2}}\right)$$

$$Y_2 = X_2 + Z_2, \quad Z_2 \sim N\left(0, \sqrt{\frac{N_{0,2}}{2}}\right)$$

$$Y_n = X_n + Z_n, \quad Z_n \sim N\left(0, \sqrt{\frac{N_{0,n}}{2}}\right)$$

Frequency division

Theorem

Capacity For a wideband channel containing n independent sub-channels with bandwidth W_Δ, attenuation H_i and additive noise $Z_i \sim N(0, \sqrt{N_{0,i}/2})$, the channel capacity is

$$C = \sum_{i=1}^{n} W_\Delta \log\Big(1 + \frac{P_i|H_i|^2}{N_{0,i}W_\Delta}\Big)$$

where

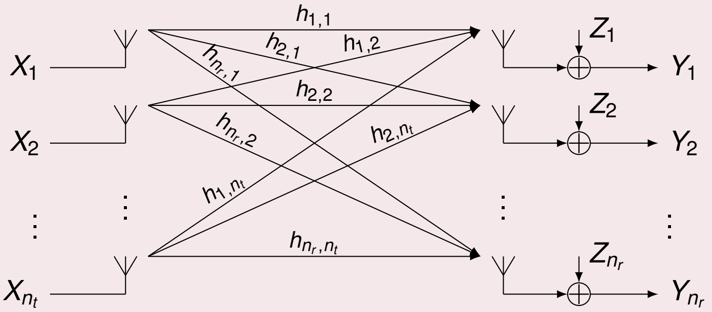
$$\begin{cases} P_i = \Big(B - \frac{N_{0,i}W_\Delta}{|H_i|^2}\Big)^+ \\ \sum_i P_i = P \end{cases}$$

and P is the total power constraint.

MIMO channel

MIMO (Multiple In, Multiple Out)

Consider a radio channel with n_t transmit antennas and n_r receive antennas. The attenuation from antenna i to antenna j is h_{ji}.



MIMO channel

Multi-dimensional Gaussian channel

Let

$$\boldsymbol{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_{n_t} \end{pmatrix} \quad \boldsymbol{Z} = \begin{pmatrix} Z_1 \\ \vdots \\ Z_{n_r} \end{pmatrix} \sim N(\boldsymbol{0}, \Lambda_Z)$$

where $\Lambda_Z = E\big[\boldsymbol{Z}\boldsymbol{Z}^T\big]$ is the covariance matrix for Z. Then

$$\boldsymbol{Y} = H\boldsymbol{X} + \boldsymbol{Z}, \quad \text{where } H = \begin{pmatrix} h_{11} & \dots & h_{1n_t} \\ \vdots & & \vdots \\ h_{n_r1} & \dots & h_{n_rn_t} \end{pmatrix}$$

MIMO channel

Multi-dimensional Gaussian channel

With $\Lambda_X = E\big[\boldsymbol{X}\boldsymbol{X}^T\big]$ as the covariance matrix for X, the power constraint for the channel is

$$tr\Lambda_X = \sum_i E\big[X_i^2\big] = \sum_i P_i \leq P$$

where $tr()$ denotes the trace function (sum of diagonal elements).

n-dim Gaussian distribution

n-dim Gaussian distribution

Let $\boldsymbol{X} \sim N(\boldsymbol{\mu}, \Lambda_X)$ be an n-dim Gaussian column vector with mean $\boldsymbol{\mu} = E[\boldsymbol{X}]$ and covariance matrix $\Lambda_X = E[(\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^T]$. Then

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^n |\Lambda_X|}} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T \Lambda_X^{-1} (\boldsymbol{x}-\boldsymbol{\mu})}$$

This gives the differential entropy

$$H(\boldsymbol{X}) = \frac{1}{2} \log(2\pi e)^n |\Lambda_X| = \frac{1}{2} \log |2\pi e \Lambda_X|$$

The n-dimensional Gaussian distribution maximises the differential entropy for all distributions with mean $\boldsymbol{\mu}$ and covariance matrix Λ_X.

$|A|$ denotes the determinant of the matrix A.

MIMO channel

The mutual information between $\boldsymbol{X}$ and $\boldsymbol{Y}$:

$$\begin{aligned} I(\boldsymbol{X};\boldsymbol{Y}) &= H(\boldsymbol{Y}) - H(\boldsymbol{Y}|\boldsymbol{X}) = H(\boldsymbol{Y}) - H(\boldsymbol{Z}) \\ &= H(\boldsymbol{Y}) - \frac{1}{2}\log(2\pi e)^n|\Lambda_Z| \\ &\le \frac{1}{2}\log(2\pi e)^n|\Lambda_X| - \frac{1}{2}\log(2\pi e)^n|\Lambda_Z| \\ &= \frac{1}{2}\log|\Lambda_Y\Lambda_Z^{-1}| \quad \text{eq. iff } \boldsymbol{X}\sim N(\boldsymbol{0},\Lambda_X) \end{aligned}$$

The covariance of $\boldsymbol{Y}$:

$$\Lambda_Y = E\Big[(H\boldsymbol{X}+\boldsymbol{Z})(H\boldsymbol{X}+\boldsymbol{Z})^T\Big] = H\Lambda_X H^T + \Lambda_Z$$

This gives

$$I(X;Y) \le \frac{1}{2}\log|I + H\Lambda_X H^T\Lambda_Z^{-1}| \quad \text{eq. iff } \boldsymbol{X}\sim N(\boldsymbol{0},\Lambda_X)$$

MIMO channel

Let $\Lambda_Z = NI_{n_r}$ and $H = USV^T$ the SVD of H, i.e. $S = \text{diag}(s_1, s_2, \dots, s_n)$

$$|U| = |V| = 1 \quad UU^T = I \quad VV^T = I$$

Then,

$$\begin{aligned} C &= \max_{tr\Lambda_X = P} \frac{1}{2} \log \left| I + \frac{1}{N} H \Lambda_X H^T \right| \\ &= \max_{tr\Lambda_X = P} \frac{1}{2} \log \left| I + \frac{1}{N} S V^T \Lambda_X V S^T \right| \end{aligned}$$

With $\widetilde{\boldsymbol{X}} = V^T \boldsymbol{X}$, $\Lambda_{\widetilde{X}} = V^T \Lambda_X V$, and

$$C = \max_{tr\Lambda_{\widetilde{X}} = P} \frac{1}{2} \log \left| I + \frac{1}{N} S \Lambda_{\widetilde{X}} S^T \right|$$

Some Matrix Theory

Theorem

Let A be a matrix with the eigenvalues λ_i. Then

$$trA = \sum_i \lambda_i \quad \text{and} \quad |A| = \prod_i \lambda_i$$

Theorem (Similar)

The matrices A and B are similar if there exists a non-singular matrix M s.t. $B = MAM^{-1}$. Then A and B have the same set of eigenvalues, and consequently the same trace and determinant.

Theorem (Hadamard inequality)

If A is positive semi-definite, then $|A| \leq \prod_i a_{ii}$ with equality if and only if A diagonal.

MIMO channel

That is,

$$C = \max_{tr\Lambda_{\widetilde{X}}=P} \frac{1}{2}\log\left|I + \frac{1}{N}S\Lambda_{\widetilde{X}}S^T\right|$$
$$\leq \max_{tr\Lambda_{\widetilde{X}}=P} \frac{1}{2}\log\prod_i\left(1 + \frac{1}{N}s_i^2\widetilde{P}_i\right)$$

where $\widetilde{P}_i$ are the diagonal elements of $\Lambda_{\widetilde{X}}$.

Maximum occurs for $\Lambda_{\widetilde{X}} = \text{diag}(\widetilde{P}_1, \dots, \widetilde{P}_n)$, which gives

$$C = \sum_i \frac{1}{2}\log\left(1 + \frac{1}{N}s_i^2\widetilde{P}_i\right)$$

MIMO channel

Theorem (Capacity of MIMO channel)

A multi-dimensional Gaussian channel with

$$\boldsymbol{Y} = H\boldsymbol{X} + \boldsymbol{Z}$$

where H is the channel matrix and $\boldsymbol{Z} \sim N(\mathbf{0}, NI_{n_r})$, has the channel capacity

$$C = \sum_{i=1}^{n_t} \frac{1}{2} \log\Big(1 + \frac{s_i^2 \widetilde{P}_i}{N}\Big) \quad \text{where} \begin{cases} \widetilde{P}_i = \Big(B - \frac{N}{s_i^2}\Big)^+ \\ \sum_i P_i = P \end{cases}$$

where the SVD of the channel is $H = USV^T$, and s_i the diagonal elements of S.