Lund University

# Information Theory

## Lecture 11

## Gaussian channel

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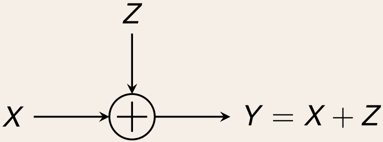
# Gaussian channel

## Definition

A Gaussian channel is a time-discrete channel with input $X$ and output $Y = X + Z$, where $Z$ models the noise and is $\mathcal{N}(0, \sqrt{N})$.




The communication is limited by a power constraint on the transmitter side,

$$E\left[X^2\right] \leq P$$

# Capacity

## Definition

The information capacity for a continuous channel is

$$C = \max_{\substack{E[X^2]=P \\ f(x)}} I(X;Y)$$

## Theorem

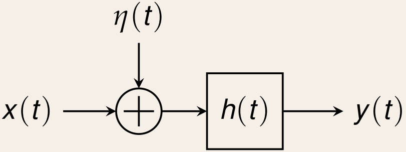
*The mutual information for a Gaussian channel is maximized for* $X \sim \mathcal{N}(0, \sqrt{P})$*, as*

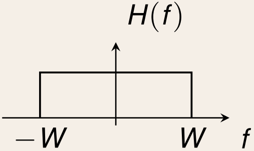
$$C = \max_{\substack{E[X^2]=P \\ f(x)}} I(X;Y) = \max_{\substack{E[X^2]=P \\ f(x)}} H(Y) - H(Z) = \frac{1}{2}\log\Big(1 + \frac{P}{N}\Big)$$

# Band limited signal

## Definition (AWGN)

A band limited Gaussian channel consists of a band limited input $x(t)$, where $f_{max} = W$, additive white Gaussian noise with $R(f) = N_0/2$, and an ideal low-pass filter at the receiver.




The noise at the receiver, after filter and sampling, has the distribution $Z \sim \mathcal{N}(0, \sqrt{N_0/2})$.

# Sampling theorem

## Theorem

*Let $x(t)$ be a band limited signal, $f_{\max} \leq W$. If the signal is sampled with $F_s = 2W$ samples per second to form the sequence $x[k] = x(\frac{k}{2W})$, it can be reconstructed with*

$$x(t) = \sum_{k=-\infty}^{\infty} x\left(\frac{k}{2W}\right) sinc\left(t - \frac{k}{2W}\right)$$

*where*

$$sinc(t) = \frac{\sin(2\pi Wt)}{2\pi Wt}$$

The sinc-function is 1 for $t = 0$ and 0 for $t = k/2W$, $k \neq 0$.

# Capacity for band limited signals

**Theorem**

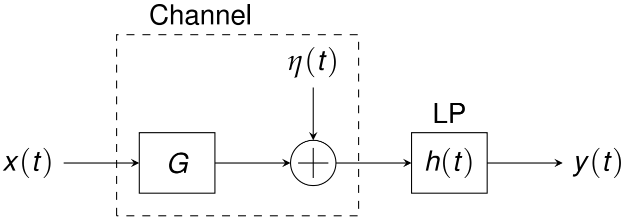
*Let $x(t)$ be a band limited signal, $f_{\max} \le W$, and $z(t)$ noise with power spectra $R(f) = N_0/2$, $|f| \le W$. The channel*

$$y(t) = x(t) + z(t)$$

*has the capacity*

$$C = W \log\left(1 + \frac{P}{N_0 W}\right) \quad \text{[b/s]}$$

# Attenuated channel




where $G$ is the attenuation on the channel ($|G| < 1$) which reduces the received power. The capacity is:

$$C = W \log\left(1 + \frac{P|G|^2}{N_0 W}\right)$$



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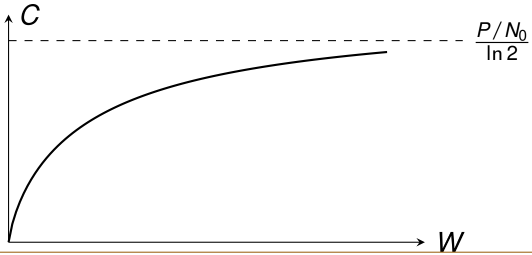
# Fundamental limit

## Limit of capacity

As the bandwidth $W$ grows the capacity approaches a limit,

$$C_\infty = \lim_{W\to\infty} W \log\Big(1 + \frac{P}{N_0 W}\Big) = \frac{P/N_0}{\ln 2}$$



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# Fundamental limit

**Theorem (Shannon limit)**

*For reliable communication over white Gaussian noise it is required that*

$$\frac{E_b}{N_0} > \ln 2 = 0.69 = -1.59 \text{ dB}$$

I.e. reliable communication is possible *if and only if* $\frac{E_b}{N_0} > -1.6\text{dB}$.
But what does it require?

# Fundamental limit

## Theorem

*The maximum achievable bandwidth efficiency is limited by*

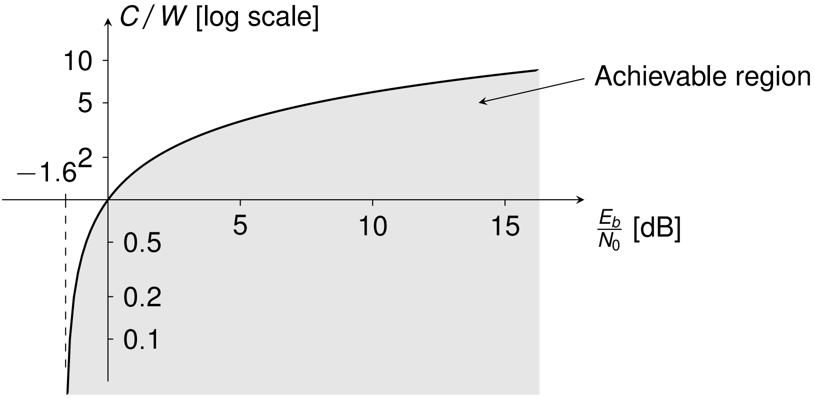
$$\frac{C}{W} < \log\left(1 + \frac{C}{W} \cdot \frac{E_b}{N_0}\right)$$

## Corollary

*For reliable communication over white Gaussian noise it is required that*

$$\frac{E_b}{N_0} > \frac{2^{C/W} - 1}{C/W}$$

# Fundamental limit

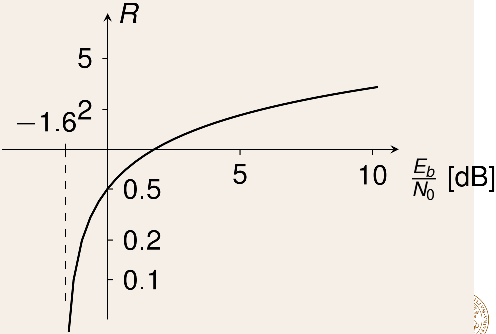


# Fundamental limit

## Theorem

*For reliable communication over white Gaussian noise and a code rate of R it is required that*

$$\frac{E_b}{N_0} > \frac{2^{2R}-1}{2R}$$



# Fundamental limit

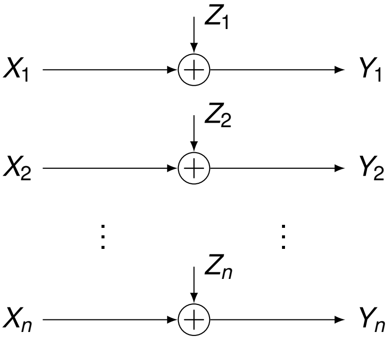
**Limit of capacity**

To reach the reliable communication as $\frac{E_b}{N_0} \to -1.6$ dB the bandwidth efficiency and the code rate approach zero. That indicates a system where

- Codeword length grows to infinity (for finite information word)
- Computational complexity grows to infinity

# Parallel Gaussian channels

Consider $n$ parallel, independent, Gaussian channels with a total power contraint of $\sum_i P_i \leq P$. The noise on channel $i$ is $Z_i \sim \mathcal{N}(0, \sqrt{N_i})$.



# Parallel Gaussian channels
# Water filling

## Theorem

*Given $k$ independent parallel Gaussian channels with noise variance $N_i$, $i = 1, \ldots, k$, and a restricted total transmitted power, $\sum_i P_i = P$.*
*The capacity is given by*

$$C = \frac{1}{2} \sum_{i=1}^{n} \log(1 + \frac{P_i}{N_i})$$

*where*

$$P_i = \big(B - N_i\big)^+, \qquad (x)^+ = \begin{cases} x, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

*and $B$ is such that $\sum_i P_i = P$.*