LUND
UNIVERSITY
Electrical and Information Technology

Home Exam in
Information Theory, EITN45

May 29 – June 5, 2017

Name: ______________________________

Id Number: ______________________________

Programme: ______________

Nbr of sheets: ______________

Mark the problems you solved with a cross.

1	2	3	4	5	6

Signature: ______________________________

Assessment protocol

1	2	3	4	5	6	Σ	Grade

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- The exam should be solved and handed in individually.
- Write your name on each paper.
- Start a new solution on a new sheet of paper. Use only one side of the paper.
- Solutions should clearly show the line of reasoning.
- Include the cover sheet when handing in the solutions. The preferred way to submit the exam solutions is as scanned copies in pdf at `elearning.eit.lth.se`. Otherwise you can hand in as paper copies directly to Stefan.
- Scripts and program code, e.g. MATLAB/Octave, python, Java or C/C++, used for derivations should be handed in as a complement to the solution. However, the solutions should be fully understandable without the code. Normal command line calculations, e.g. entropy as done in the hand in problem, is not required to hand in.
- There are six problems with 10 points each. The limits for grades are planned as:
  3: 30-39
  4: 40-49
  5: 50-60

Good luck!

Problem 1

The LZW algorithm is used to compress a text. As initialisation the alphabet according to the ASCII table in Table I on page 5 is used. The codewords, written as decimal numbers, from the compression is given in Table 1.1. Ind denote the indices in the decoding. Decompress the received code sequence.

Ind	Codeword	Ind	Codeword	Ind	Codeword	Ind	Codeword
128	73	138	114	148	140	158	139
129	32	139	101	149	101	159	119
130	119	140	129	150	110	160	141
131	105	141	104	151	144	161	143
132	115	142	97	152	130	162	128
133	104	143	116	153	132	163	97
134	32	144	134	154	101	164	109
135	128	145	129	155	100	165	46
136	119	146	97	156	151		
137	101	147	115	157	137		

Table 1.1: Table of codewords from an LZW encoding.

(10p)

Problem 2

Let X be the outcome of a roll with a five sided fair dice and Y the outcome from a roll with an eight sided fair dice.

(a) What is the uncertainty of $Z_a = X + Y$

(b) What is the uncertainty of $Z_b = X - Y$

(c) What is the uncertainty of $Z_c = |X - Y|$

(4+3+3=10p)

Problem 3

Consider a dice with memory. It can not give the same number twice in row, while the other outcomes are equally likely. That is, if the previous outcome was 3, the next will be 1, 2, 4, 5 or 6, with the equal probabilities.

(a) After a substantial number of consecutive rolls, what is the uncertainty of the next outcome if you do not know the previous sequence of outcomes?

(b) After a substantial number of consecutive rolls, what is the uncertainty of the next outcome if you know the previous sequence of outcomes?

(c) Let $X_0 = j$ be the number shown when the dice roll is started. Then let X_n be the outcome at time n. Draw a plot showing the uncertainty of $\{X_n | X_0 = j\}$ for the first ten rolls, i.e. for $n = 1, 2, \dots, 10$.
Hint: It is not required to give a closed expression of $P(X_n = i | X_0 = j)$.

(3+4+3=10p)

Problem 4

Consider a 4-PAM communication system with the signal alternatives

$$s_x(t) = A_x \sqrt{E_g}\phi(t), \quad x = 0, 1, 2, 3$$

where $A_x \in \{-3, -1, 1, 3\}$ are the amplitudes and $\phi(t)$ a normalised basis function. During the transmission the noise Z is added with the distribution $Z \sim N(0, \sqrt{N_0/2})$. At the receiver the signal is decision back to $\{0, 1, 2, 3\}$ according to the decision regions in Figure 4.1.

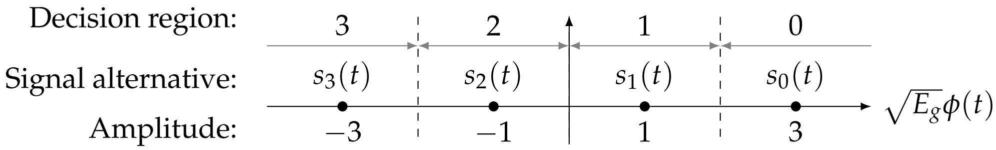

Figure 4.1: 4-PAM modulation and decision regions.

Denote the probability for the receiver to make an erroneous decision by

$$\varepsilon = P\big(Z > \sqrt{E_g}\big)$$

You can assume that the probability of making errors to non-neighbouring decision regions are negligible, i.e.

$$P\big(Z > 3\sqrt{E_g}\big) \approx 0$$

(a) What is the corresponding DMC?

(b) Assume that the symbols are transmitted with equal probability. Express the maximum information transmission per channel use for the DMC, in terms of ε and the binary entropy function. Sketch a plot for different ε.

(3+7=10p)

Problem 5

Consider a frequency divided system like OFDM, where the total bandwidth of $W = 10$ MHz, is split in ten equally wide independent subbands. In each of the sub-bands an M-PAM modulation should be used and the total allowed power is $P = 1$ W. On the sub-bands the noise to attenuation ratios are given by the vector

$$\frac{N_{o,i}}{|G_i|^2} = \begin{pmatrix} -53 & -43 & -48 & -49 & -42 & -54 & -45 & -45 & -52 & -49 \end{pmatrix} \; [\text{dBm/Hz}]$$

where $N_{0,i}$ is the noise on sub-band i and G_i the signal attenuation on sub-band i. Notice that it is given in the unit dBm/Hz. The system is supposed to work at an average error probability of $P_e = 10^{-6}$. The aim of this problem is to maximise the total information bit rate R_b for the system.

(a) Show that the total information bit rate for the system can be maximised using the water filling procedure.

(b) Derive the maximum information bit rate for the system.

(c) Assume that you add an error correcting code in each sub-band, with a coding gain of $\gamma_c = 3$ dB. How does that influence the maximum information bit rate?

(3+4+3=10p)

Problem 6

Consider a probability distribution for a random variable with k outcomes, $\{p_1, p_2, \dots, p_k\}$. In this problem the distribution will changed towards a uniform distribution, and the aim is to show how that affects the entropy.

(a) Assume that $p_1 > p_2$ and form a new distribution according to

$$
\begin{aligned}
q_1 &= p_1 - \delta \\
q_2 &= p_2 + \delta \\
q_i &= p_i, \qquad i = 3, \dots, k
\end{aligned}
$$

where $\delta > 0$ and $q_1 \geqslant q_2$. Show that

$$H(q_1, \dots, q_k) \geqslant H(p_1, \dots, p_k)$$

(b) Let $A = [a_{ij}]_{i,j \in \{1,\dots,k\}}$ be a $k \times k$ matrix with non-negative elements, $a_{ij} \geqslant 0$, where $\sum_i a_{ij} = 1$ and $\sum_j a_{ij} = 1$. Form a new distribution from

$$q_i = \sum_j a_{ij} p_j$$

Show that

$$H(q_1, \dots, q_k) \geqslant H(p_1, \dots, p_k)$$

(c) Find a matrix A that shows how sub-problem (a) is a special case of sub-problem (b). **Hint:** Assign $q_1 = \alpha p_1 + \beta p_2$ and $q_2 = \beta p_1 + \alpha p_2$.

(4+4+2=10p)

Ind	Dict	Ind	Dict	Ind	Dict	Ind	Dict
0	NULL	32	Space	64	@	96	‘
1	SOH	33	!	65	A	97	a
2	STX	34	"	66	B	98	b
3	ETX	35	#	67	C	99	c
4	EOT	36	$	68	D	100	d
5	ENQ	37	%	69	E	101	e
6	ACK	38	&	70	F	102	f
7	BEL	39	’	71	G	103	g
8	BS	40	(	72	H	104	h
9	TAB	41	)	73	I	105	i
10	LF	42	*	74	J	106	j
11	VT	43	+	75	K	107	k
12	FF	44	,	76	L	108	l
13	CR	45	-	77	M	109	m
14	SO	46	.	78	N	110	n
15	SI	47	/	79	O	111	o
16	DLE	48	0	80	P	112	p
17	DC1	49	1	81	Q	113	q
18	DC2	50	2	82	R	114	r
19	DC3	51	3	83	S	115	s
20	DC4	52	4	84	T	116	t
21	NAK	53	5	85	U	117	u
22	SYN	54	6	86	V	118	v
23	ETB	55	7	87	W	119	w
24	CAN	56	8	88	X	120	x
25	EM	57	9	89	Y	121	y
26	SUB	58	:	90	Z	122	z
27	ESC	59	;	91	[	123	{
28	FS	60	<	92	\	124	|
29	GS	61	=	93	]	125	}
30	RS	62	>	94	^	126	~
31	US	63	?	95	_	127	-

Table I: The ASCII Table. The first column is only control characters.