Problem 1

(a) To start with we need some new probabilities: $p(x) = \sum_y p(x,y)$, $p(y) = \sum_x p(x,y)$, $p(x|y) = \frac{p(x,y)}{p(y)}$ and $p(y|x) = \frac{p(x,y)}{p(x)}$.

x	$p(x)$
0	0.25
1	0.40
2	0.15
3	0.15
4	0.05

y	$p(y)$
0	0.35
1	0.35
2	0.20
3	0.10

$p(x\|y)$	0	1	2	3
0	0.71	0	0	0
1	0.29	0.86	0	0
2	0	0.14	0.50	0
3	0	0	0.25	1
4	0	0	0.25	0

$p(y\|x)$	0	1	2	3
0	1	0	0	0
1	0.25	0.75	0	0
2	0	0.33	0.67	0
3	0	0	0.33	0.67
4	0	0	1	0

Then, the desired functions can be derived as (using $0 \log 0 = 0$)

$$\begin{aligned}
H(X) &= -\sum_x p(x)\log p(x) = 2.07\\
H(Y) &= -\sum_y p(y)\log p(y) = 1.86\\
H(X|Y) &= -\sum_{x,y} p(x,y)\log p(x|y) = 0.81\\
H(Y|X) &= -\sum_{x,y} p(x,y)\log p(y|x) = 0.60\\
H(X,Y) &= -\sum_{x,y} p(x,y)\log p(x,y) = 2.67
\end{aligned}$$

(b) The mutual information between X and Y is $I(X;Y) = 1.26$. The channel probabilities equals $p(y|x)$ in the table above. The capacity is defined as $C = \max_{p(x)} I(X;Y)$, so if we can find another distribution on X that gives a higher mutual information than the given distribution we are ready. Thus, consider a uniform distribution $\tilde{p}(x) = 1/5$, $x = 0,1,2,3,4$. Then the joint distribution is given by $\tilde{p}(x,y) = p(y|x)\tilde{p}(x)$, which gives the mutual information $I(\tilde{X};Y) = 1.36$. Since $I(\tilde{X};Y) > I(X;Y)$ we can directly say that the given distribution is not according to the capacity. It should, however, be noted that we do not know the capacity it self. It is not likely that the newly derived mutual information is the capacity but this we do not know.

Problem 2

Search buffer	Look ahead	Codeword
[There are two minutes differenc]	[e f | rom four to]	(23,2,f)
[re are two minutes difference f]	[ro | m four to two]	(7,1,o)
[are two minutes difference fro]	[m | four to two t]	(22,1,)
[re two minutes difference from]	[fo | ur to two to]	(5,1,o)
[two minutes difference from fo]	[ur | to two to tw]	(23,1,r)
[wo minutes difference from four]	[t | o two to two]	(5,1,t)
[minutes difference from four t]	[o | two to two to]	(5,1,)
[inutes difference from four to]	[tw | o to two to t]	(3,1,w)
[utes difference from four to tw]	[o to | two to two]	(4,3,o)
[difference from four to two to]	[two to two, | an]	(7,11,,)
[from four to two to two to two,]	[a | nd from two t]	(5,1,a)
[om four to two to two to two, a]	[n | d from two to]	(0,0,n)
[m four to two to two to two, an]	[d | from two to t]	(0,0,d)
[four to two to two to two, and]	[fr | om two to tw]	(31,2,r)
[ur to two to two to two, and fr]	[om | two to two t]	(9,1,m)
[to two to two to two, and from]	[two to two to t |]	(28,15,t)
[o two, and from two to two to t]	[wo, t | o]	(28,4,t)
[, and from two to two to two, t]	[oo |]	(4,1,o)

Uncoded (96 letters @ 8 bit each): $L_u = 96 \cdot 8 = 768$
Coded (init 31 letter @ 8 bit, 18 codewords @ 5+4+8=17 bits each): $L_c = 31 \cdot 8 + 18 \cdot 17 = 554$
Rate

$$R = \frac{768}{554} = 1.3863$$

Problem 3

(a) Use the water filling algorithm to derive the capacity. When a sub-channel is deleted ($P_i = 0$) the total number of sub-channel is changed and the power distribution has to be recalculated. We get the following recursion:

1. Iteration 1
   $B = B - N_i = \frac{1}{6}(\sum_i N_i + P) = 14.17$
   $P_i = (6.17, 2.17, 0.17, 4.17, -1.83, 8.17)$
   Sub-channel 5 should not be used, $P_5 = 0$.
2. Iteration 1
   $B = \frac{1}{5}(\sum_{i\neq 5} N_i + P) = 13.8$
   $P_i = B - N_i = (5.80, 1.80, -0.20, 3.80, 0, 7.80)$
   Sub-channel 3 should not be used, $P_3 = 0$.
3. Iteration 1
   $B = \frac{1}{4}(\sum_{i\neq 3,5} N_i + P) = 13.75$
   $P_i = B - N_i = (5.75, 1.75, 0, 3.75, 0, 7.75)$
   All remaining sub-channels can be used.

The capacities in the sub-channels are

$$C_i = \frac{1}{2}\log\Big(1 + \frac{P_i}{N_i}\Big) = (0.39, 0.10, 0, 0.23, 0, 0.60)$$

and the total capacity $C = \sum_i C_i = 1.32$ bit/transmission.

(b) If the power is equally distributed over the sub-channels we get $P_i = 19/6 = 3.17$. That gives the capacities

$$\begin{aligned} C &= \sum_i \frac{1}{2}\log\Big(1 + \frac{19/6}{N_i}\Big) \\ &= 0.24 + 0.17 + 0.15 + 0.20 + 0.13 + 0.31 = 1.19 \text{ bit/transmission} \end{aligned}$$

(c) When using only one sub-channel the capacity is maximised if we take the one with least noise, $N = 6$. This gives the capacity $C = \frac{1}{2}\log 2(1+19/6) = 1.03$ bit/transmission.

Problem 4

If $X = 0$ the received value is $Y = 0$, and if $X = 1$ the received value is $Y = 0$ with probability α and $Y = 1$ with probability $1 - \alpha$. That is, an alternative description of the channel is as in Figure 4.1, which is often called the Z-channel. To derive the capacity first assign a distribution on X, and say that $P(X = 1) = p$ and $P(X = 0) = 1 - p$. Then the distribution on Y yields $P(Y = 0) = (1 - p) + \alpha p = 1 - p(1 - \alpha)$ and $P(Y = 1) = p(1 - \alpha)$.

The capacity can be derived as

$$C = \max_{\alpha} I(X;Y) = \max_{\alpha} H(Y) - H(Y|X)$$

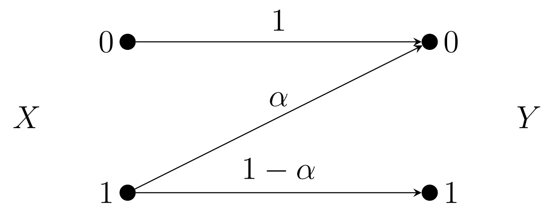


Figure 4.1: The Z-channel.

With $H(Y) = h\big(p(1-\alpha)\big)$ and $H(Y|X) = ph(\alpha)$ the mutual information can be written as

$$I(X;Y) = h\big(p(1-\alpha)\big) - ph(\alpha)$$

Using the derivative of the binary entropy function as $\frac{\partial}{\partial p}h(p) = \log\frac{1-p}{p}$ set the derivative of the mutual information equal to zero,

$$\frac{\partial}{\partial p}I(X;Y) = (1-\alpha)\log\frac{1-p(1-\alpha)}{p(1-\alpha)} - h(\alpha) = 0$$

Solving for p gives

$$p = \frac{1}{(1-\alpha)\big(1+2^{A(\alpha)}\big)}, \qquad \text{where } A(\alpha) = \frac{h(\alpha)}{1-\alpha}$$

Inserting into the mutual information gives

$$C = h\Big(\frac{1}{1+2^{A(\alpha)}}\Big) - \frac{A(\alpha)}{1+2^{A(\alpha)}}$$

As α approaches 1, $A(\alpha)$ will go to infinity. However, since $2^{A(\alpha)}$ is located in the denomitor we get the limits $\lim_{\alpha\to 1}\frac{1}{1+2^{A(\alpha)}} = 0$ and $\lim_{\alpha\to 1}\frac{A(\alpha)}{1+2^{A(\alpha)}} = 0$, and thus $\lim_{\alpha\to 1} C = 0$. This fact can also be argued by viewing the channel for $\alpha = 1$. Then $Y = 0$ independent of what is transmitted, and hence, the capacity is zero.

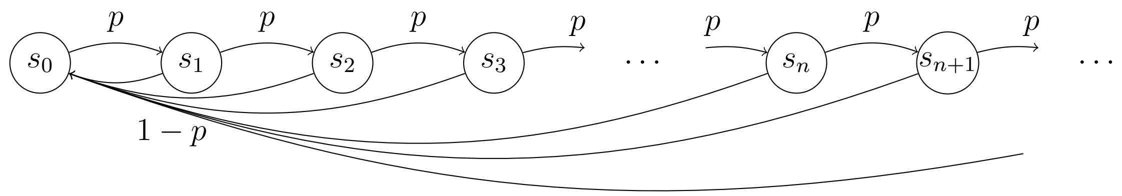
When $\alpha = 0$ the capacity becomes $C = 1$. Similarly, for $\alpha = 1/2$ the capacity becomes $C = 0.322$, which was also seen in one of the problems in the course. In Figure 4.2 the capacity is plotted as a function of α. Such plot can be derived in Matlab as

```
a=[0:0.1:1];
A=Entropy(a)./(1-a);
C=Entropy(1./(1+2.^A))-A./(1+2.^A);
C(end)=0;
plot(a,C)
```

In the plot it is marked the capacity for the end points and for $\alpha = 1/2$ as derived above.

Problem 5

(a) Let the state of the Markov process be the step on ladder. Then the (infinite) state transition graph for the process is



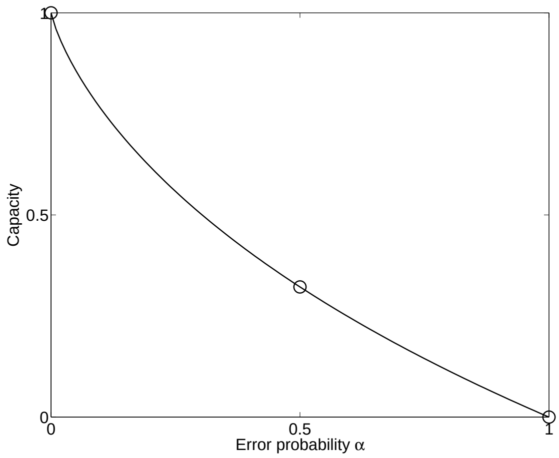


Figure 4.2: The capacity of the Z-channel as a function of the error probability α.

This gives the transition matrix

$$P = \begin{pmatrix} 1-p & p & 0 & 0 & 0 & \cdots \\ 1-p & 0 & p & 0 & 0 & \cdots \\ 1-p & 0 & 0 & p & 0 & \cdots \\ \vdots & & & & \ddots & \ddots \end{pmatrix}$$

(b) Letting S denote the state and S^+ the state at the next time instant. At each state the entropy $H(S^+|S) = h(p)$. With $\boldsymbol{\pi} = \pi_0, \pi_1, \pi_2, \ldots$, denoting the steady state distribution, the entropy rate is

$$H_\infty(S) = \sum_S H(S^+|S)P(S) = \sum_{i=0}^{\infty} h(p)\pi_i = h(p)$$

(c) In this problem we need the staeady state distribution $\boldsymbol{\pi}$. From (a) we get that $\pi_n = \pi_{n-1}p = \pi_{n-2}p^2 = \pi_0 p^n$ for $n = 0, 1, 2, \ldots$. With $1 = \sum_i \pi_i = \pi_0 \sum_i p^i = \pi_0 \frac{1}{1-p}$ we conclude $\pi_n = (1-p)p^n$. The uncertainty that the man is on the ground is then

$$H(S=0) = h(p)$$

To get the uncertainty of the step when the man is not on the ground we first need the corresponding probability as $\nu_n = \frac{\pi_n}{1-\pi_0} = (1-p)p^{n-1}$ for $n = 1, 2, \ldots$. Hence the uncertainty is

$$\begin{aligned} H(N) &= -\sum_{i=1}^{\infty}(1-p)p^{i-1}\log(1-p)p^{i-1} = -\sum_{j=0}^{\infty}(1-p)p^j\log((1-p)p^j) \\ &= -(1-p)\log(1-p)\sum_{j=0}^{\infty}p^j - p(1-p)\log p\sum_{j=0}^{\infty} jp^{j-1} \\ &= -\log(1-p) - \frac{p}{1-p}\log p = \frac{h(p)}{1-p} \end{aligned}$$