Lund University
Electrical and Information Technology

Home Exam in Information Theory, EITN45

May 27 – June 2, 2013

- Write your name on each paper.
- Start a new solution on a new sheet of paper. Use only one side of the paper.
- Solutions should clearly show the line of reasoning.
- Program code, e.g. MATLAB, python or C/C++, used for derivations should be handed in as a complement to the solution. However, the solutions should be fully understandable without the code. Normal command line calculations, e.g. entropy, is not required to hand in.

Good luck!

Problem 1

Consider the channels A and B in Figure 1.1.

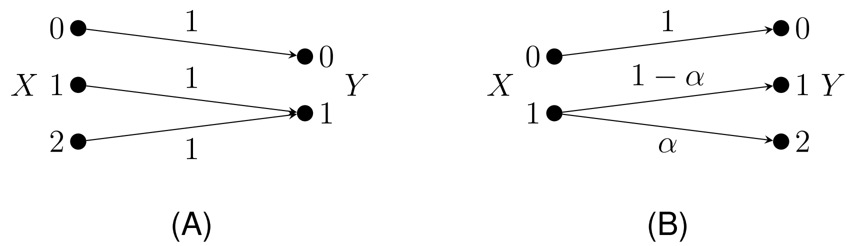

Figure 1.1: Two channels.

(a) For channel A assign $P(X = 1) = P(X = 2) = p$ and give an expression for the mutual information $I(X; Y)$

(b) For channel B assign $P(X = 1) = p$ and give an expression for the mutual information $I(X; Y)$

(c) For each of the channels A and B, show that the capacity is

$$C = \max_p I(X; Y) = 1$$

What is the maximising p? (May be different for the two channels)

(d) The result in c is obvious when you think of it. Explain why.

(3+3+2+2=10p)

Problem 2

Consider the German sentence[1]

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Ich esse essig essig esse ich im salat
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Compress it using the LZ78 algorithm. Give the codewords both in numeric form and binary form, the compression ratio and the dictionary represented in a tree. Assume that characters are encoded with the ASCII table using eight bits each.

(10p)

[1]With a comma between the two *essig*, it means: I eat vinegar, vinegar I eat in salad.

Problem 3

Consider two urns, numbered 1 and 2. Urn 1 has four white balls and three black balls, while Urn 2 has three white balls and seven black. Choose one of the urns with equal probability, and draw one ball from it. Let X be the colour of that ball and Y the number of the chosen urn.

(a) Derive the uncertainty of X.

(b) How much information is obtained about Y when observing X?

(c) Introduce a third urn, Urn 3, with only one white ball (and no black). Redo problems a and b for this case.

(3+3+4=10p)

Problem 4

A *unary* source code is a mapping from the positive integers to binary vectors such that the codeword for the integer n consists of $n-1$ zeros followed by a one, i.e.

n	y
1	1
2	01
3	001
4	0001
5	00001
$\vdots$	$\vdots$

This is clearly a prefix code since there is only one 1, located at the end of each codeword.

On Wikipedia (`https://en.wikipedia.org/wiki/Unary_coding`)[2] it is stated that the unary code is optimal for the probability function

$$P(n) = (k-1)k^{-n}, \quad n = 1, 2, \dots \text{ and } k \geqslant 1.61803\dots$$

The aim of this problem is to clarify if this is correct. However, we will only consider probability functions where k is an integer, i.e. $k = 2, 3, 4, \dots$. Your answers below should be clearly motivated.

(a) Is it true that for any integer $k \geqslant 2$ the function $P(n) = (k-1)k^{-n}$, $n = 1, 2, 3, \dots$ is a probability function?

(b) Is it true that for $k = 2$ the unary code is optimal? What is the average codeword length?

(c) Is it true that for any integer $k = 2, 3, 4, \dots$ the unary code is optimal? What is the average codeword length?

(2+3+5=10p)

[2] You can of course not use any formulas found on Wikipedia, or elsewhere on Internet, without proving it.

Problem 5

Consider the case when two discrete memoryless channels are used in parallel, see Figure 5.1. That is, the transmitted symbol X is transmitted over two channels and the receiver gets two symbols, Y and Z. The two channels work independently in parallel, i.e. $P(y|x)$ and $P(z|x)$ are independent distributions, and hence, $P(y|x,z) = P(y|x)$ and $P(z|x,y) = P(z|x)$). However, it does not mean Y and Z are independent, so in general $p(y|z) \neq p(y)$.

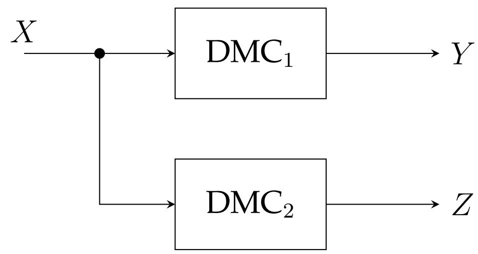

Figure 5.1: Two DMC used in parallel.

(a) Consider the information about X the receiver gets by observing Y and Z, and show that

$$I(X;Y,Z) = I(X;Y) + I(X;Z) - I(Y;Z)$$

(b) Consider the case when both the channels are BSC with error probability p, and let $P(X=0) = P(X=1) = \frac{1}{2}$. Use the result in a and show that

$$\begin{aligned} I(X;Y,Z) &= H(Y,Z) - 2h(p) \\ &= p^2 \log \frac{2p^2}{p^2 + (1-p)^2} + (1-p)^2 \log \frac{2(1-p)^2}{p^2 + (1-p)^2} \\ &= \Big(p^2 + (1-p)^2\Big)\left(1 - h\left(\frac{p^2}{p^2 + (1-p)^2}\right)\right) \end{aligned}$$

Hint: Consider the distribution $P(y,z|x)$ to get $P(y,z)$.

One interpretation of this channel is that the transmitted symbol is sent twice over the BSC.

(4+6=10p)