Lund University
Electrical and Information Technology

Home Exam in
Information Theory, EITN45

May 30 – June 6, 2016

Name: ______________________________

Id Number: ______________________________

Programme: ______________

Nbr of sheets: ______________

Mark with a cross the problems you solved.				
1	2	3	4	5

Signature: ______________________________

Assessment protocol						
1	2	3	4	5	$\sum$	Grade

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- Problems shall be solved and handed in individually.
- Write your name on each paper.
- Start a new solution on a new sheet of paper. Use only one side of the paper.
- Solutions should clearly show the line of reasoning.
- Include the cover sheet when handing in the solutions.
- Program code, e.g. MATLAB, python or C/C++, used for derivations should be handed in as a complement to the solution. However, the solutions should be fully understandable without the code. Normal command line calculations, e.g. entropy, is not required to hand in.

Good luck!

Problem 1

For the discrete memoryless channel (DMC) in Figure 1.1 show that $H(X|Y) = H(X)$, for any distribution on X.

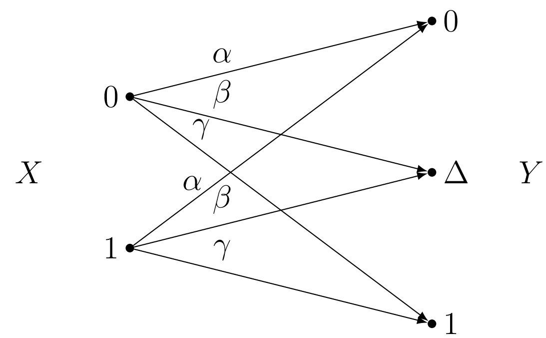

Figure 1.1: A DMC.

(10p)

Problem 2

Consider a fair die and a fair coin. The die is rolled once to determine the number of tosses with the coin. That is, if the die shows 1 the coin is tossed once, if the die shows 2 it is tossed twice, and so on. By observing the number of heads tossed by the coin, how much information do you get about the outcome of the die?

(10p)

Problem 3

Consider a channel with bandwidth $W = 10$ MHz that is split into five frequency divided channels with equal bandwidth. The attenuation in the five sub-bands are

$$|H_i|^2 \in \{-10, -13, -16, -19, -22\} \text{ [dB]}$$

The noise level can be considered to be flat with level $N_0 = -150$ dBm/Hz over the entire bandwidth. The total allowed power consumption is $P = -60$ dBm.

(a) What is the total capacity for the channel?

(b) Give an estimate of the maximum total bit rate when uncoded M-PAM signalling is used in each sub-band. Set the error probability to $P_e = 10^{-6}$.

(5+5=10p)

Problem 4

In Figure 4.1 a three state Markov source is shown. The output from the source, denoted $\boldsymbol{X} = \dots, X_0, X_1, X_2, X_3, \dots$, is a sequence of the letters A, B and C. Assume that the sequence was started a long time ago.

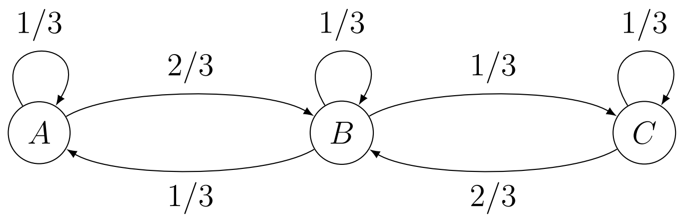

Figure 4.1: A Markov chain.

(a) Derive the entropy rate for the sequence $\boldsymbol{X}$

(b) To form a new sequence, $\boldsymbol{Y}$, every second letter of $\boldsymbol{X}$ is removed. For example the sequence

$$\boldsymbol{X} = \dots, A, B, C, C, B, A, A, B, A, B, C, \dots$$

will result in

$$\boldsymbol{Y} = \dots, A, C, B, A, A, C, \dots$$

Derive the entropy rate for the sequence $\boldsymbol{Y}$.

(5+5=10p)

Problem 5

Let X be a random variable with n outcomes, $x_i, i = 1, \dots, n$, and the non-zero probabilities $p_i, i = 1, \dots, n$. Construct a binary prefix code for it and write the codewords w_i in a tree. Let the $n-1$ inner nodes in the tree, including the root, be denoted $z_i, i = 1, \dots, n-1$ and have the probabilities $q_i, i = 1, \dots, n-1$.

(a) Construct a new random variable $\tilde{X}$ by replacing two sibling leaves in the tree, x_k and x_l, by their parent node, thus reducing the set of outcomes to $n-1$. Show that the entropy of X can be expressed in terms of the entropy of $\tilde{X}$ as

$$H(X) = H(\tilde{X}) + (p_k + p_l)h(\alpha)$$

and determine the probability α.

(b) Find an expression for the entropy $H(X)$ based on the inner node probabilities q_i and the binary entropy function.
Hint: For an inner node z_i, define α_i as the probability for the next bit in the codeword to be a 0, conditioned on z_i, i.e. $\alpha_i = P(\text{next bit } 0 | \text{inner node } z_i)$.

(c) Find a condition for an optimal code to have $L = H(X)$, based on the conditional probabilities α_i defined in the hint of b). What does it mean for the probabilities for the outcomes of X?

(3+4+3=10p)