Carrier Transmission

The transmitted signal is $y(t) = \sum_k a_k h(t - kT)$. What is the bandwidth? More generally, what is its Fourier transform?

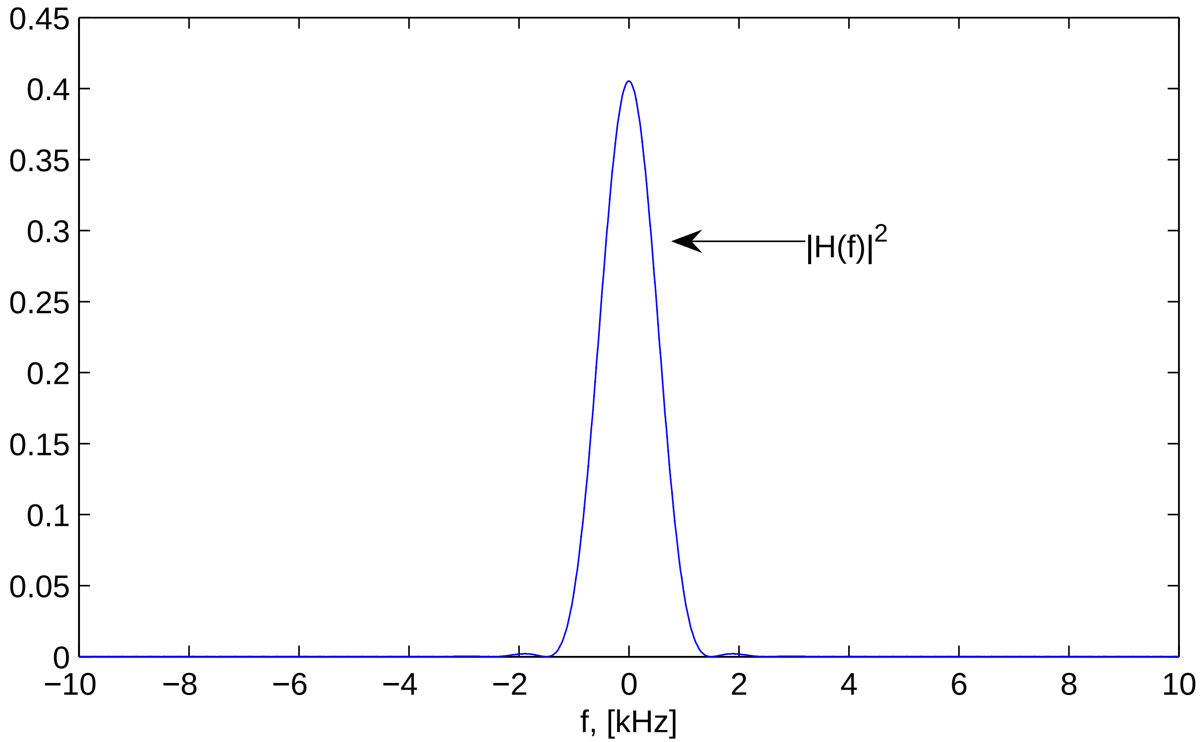
Carrier Transmission

Table 2.3 Properties of the Fourier transform

1.	Linearity	$ax_1(t) + bx_2(t) \leftrightarrow aX_1(f) + bX_2(f)$
2.	Inverse	$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j\omega t}\, df$
3.	Translation (time shift)	$x(t - t_0) \leftrightarrow X(f)\, e^{-j\omega t_0}$
4.	Modulation (frequency shift)	$x(t)\, e^{j\omega_0 t} \leftrightarrow X(f - f_0)$ $x(t)\cos\omega_0 t \leftrightarrow \frac{1}{2}X(f + f_0) + \frac{1}{2}X(f - f_0)$
5.	Time scaling	$x(at) \leftrightarrow \frac{1}{\|a\|}X(f/a)$
6.	Differentiation in time	$\frac{d}{dt}x(t) \leftrightarrow j\omega X(f)$
7.	Differentiation in frequency	$tx(t) \leftrightarrow -\frac{1}{j2\pi}\frac{d}{df}X(f)$
8.	Integration in time	$\int_{-\infty}^{t} x(\tau)\, d\tau \leftrightarrow \frac{1}{j\omega}X(f)$
9.	Duality	$X(t) \leftrightarrow x(-f)$
10.	Conjugate functions	$x^*(t) \leftrightarrow X^*(-f)$
11.	Convolution in time	$x_1(t) * x_2(t) \leftrightarrow X_1(f)X_2(f)$
12.	Multiplication in time	$x_1(t)x_2(t) \leftrightarrow X_1(f) * X_2(f)$
13.	Parseval's formulas	$\int_{-\infty}^{\infty} x_1(t)x_2^*(t)\, dt = \int_{-\infty}^{\infty} X_1(f)X_2^*(f)\, df$ or, when $x_1(t) = x_2(t)$, $\int_{-\infty}^{\infty} \|x(t)\|^2\, dt = \int_{-\infty}^{\infty} \|X(f)\|^2\, df$

Carrier Transmission

The baseband signal is $y(t) = \sum_k a_k h(t - kT)$. The power spectral density of the transmission is $\propto |H(f)|^2$



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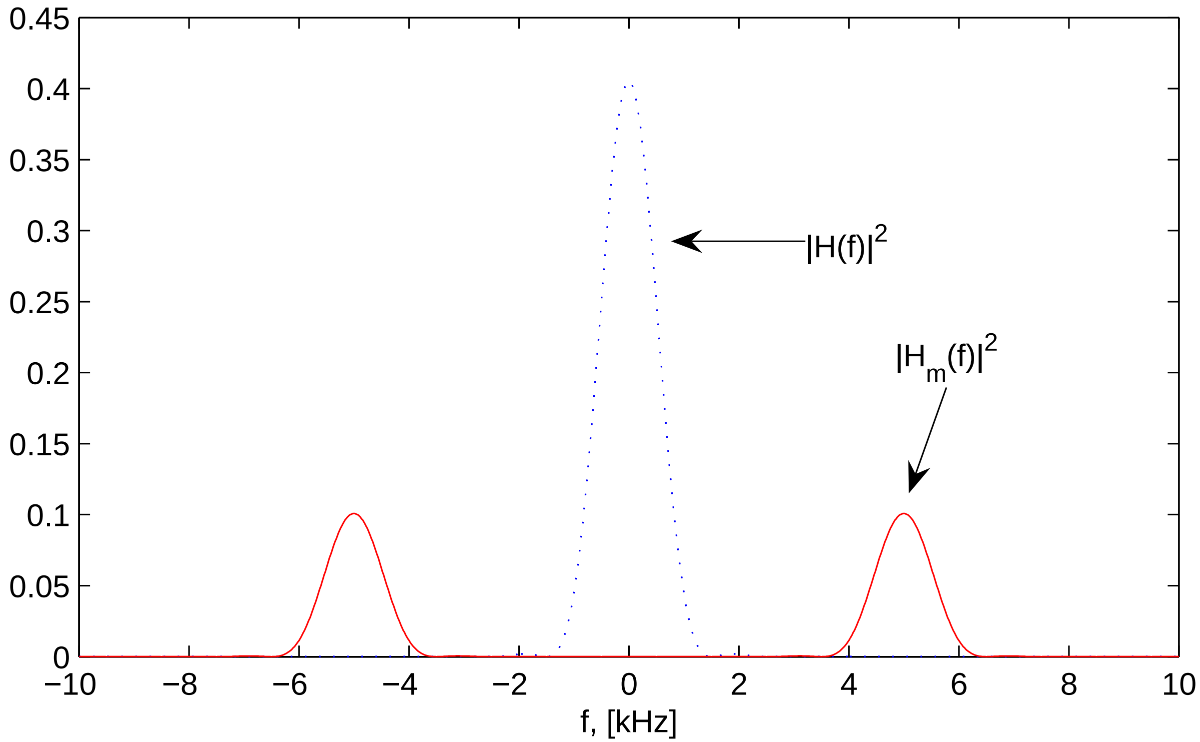

The carrier modulated signal is $y_m(t) = y(t)\cos(2\pi t f_c)$

Carrier Transmission

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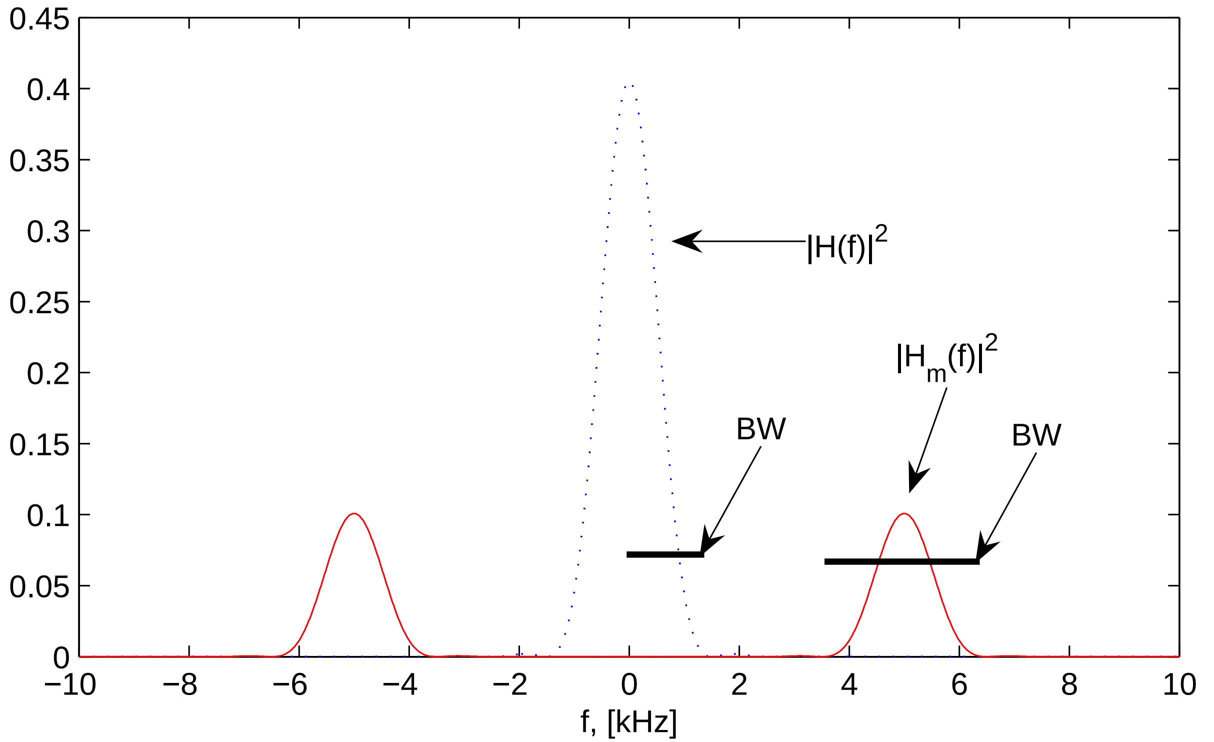

The carrier modulated signal is $y_m(t) = y(t)\cos(2\pi t f_c)$

But bandwidth gets twice as large!

Carrier Transmission

Where did the energy go?

Basic Fourier relations:

$$\cos(2\pi f_c t)h(t) \longleftrightarrow \frac{1}{2}H(f-f_c)+\frac{1}{2}H(f+f_c)$$

$$\sin(2\pi f_c t)h(t) \longleftrightarrow \frac{i}{2}H(f-f_c)-\frac{i}{2}H(f+f_c)$$

Carrier Transmission

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The $1/2$ factor corresponds to a $1/4$ of the energy. Since there are two terms, $1/2$ of the energy is preserved.

Carrier Transmission

Where did the energy go?

Basic Fourier relations:

$$\cos(2\pi f_c t)h(t) \longleftrightarrow \frac{1}{2}H(f-f_c) + \frac{1}{2}H(f+f_c)$$

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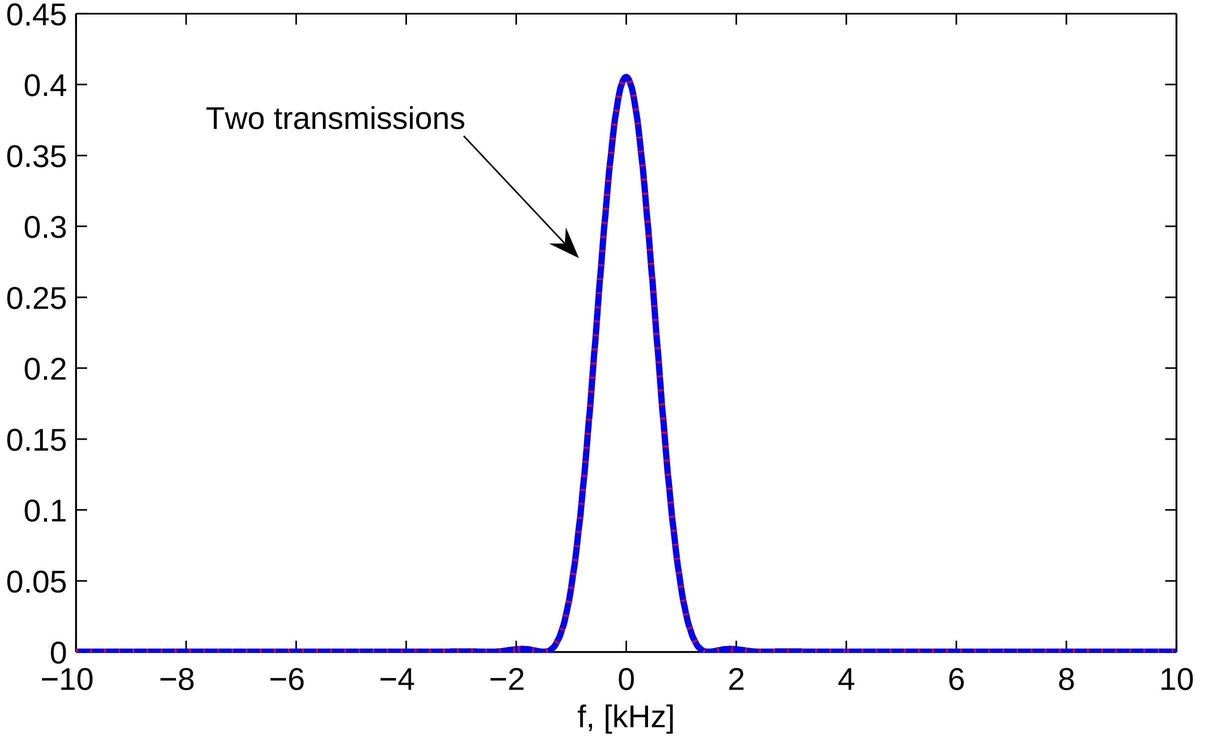
What about the increased bandwidth?

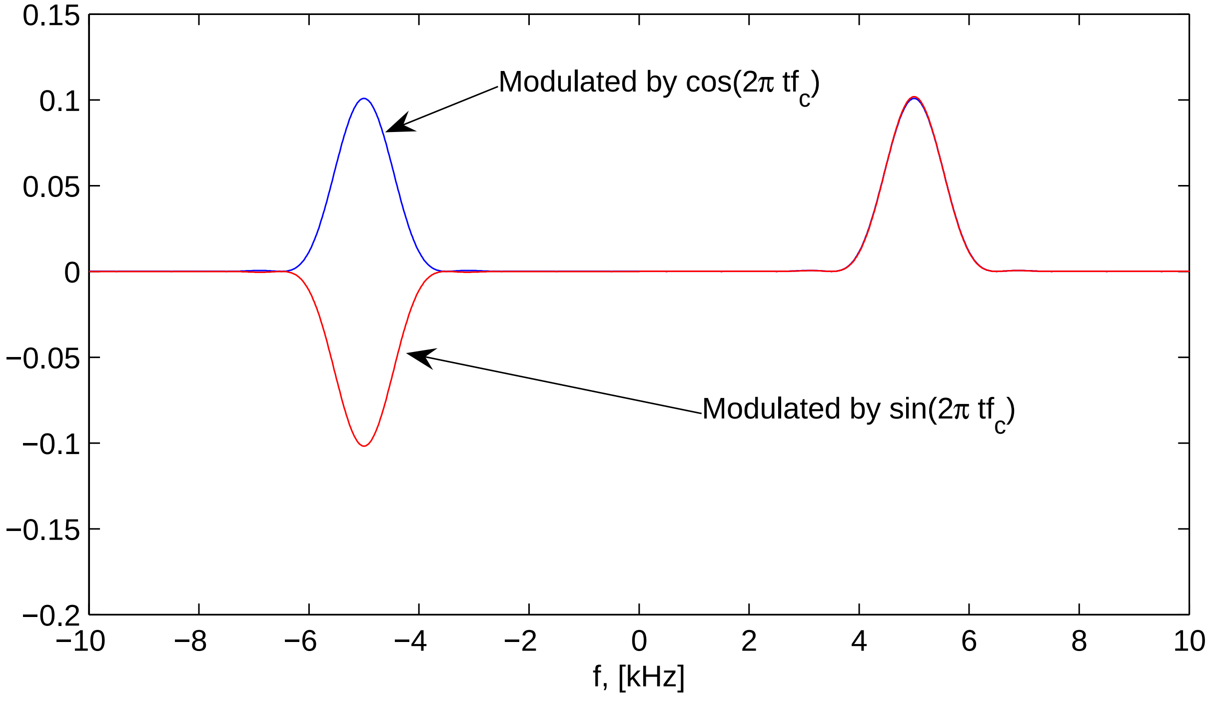
Carrier Transmission

Assume two independent baseband transmissions



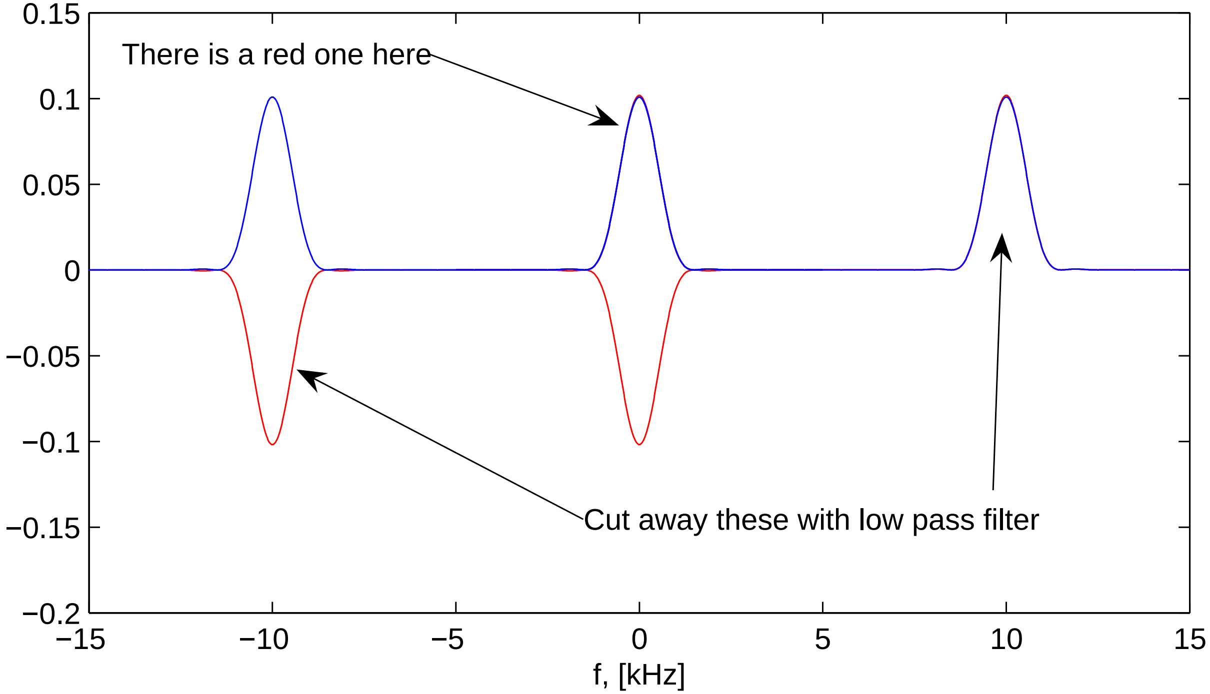
Carrier Transmission

Assume two independent baseband transmissions
After modulation with $\cos(2\pi t f_c)$ and $\sin(2\pi t f_c)$ we get



Carrier Transmission

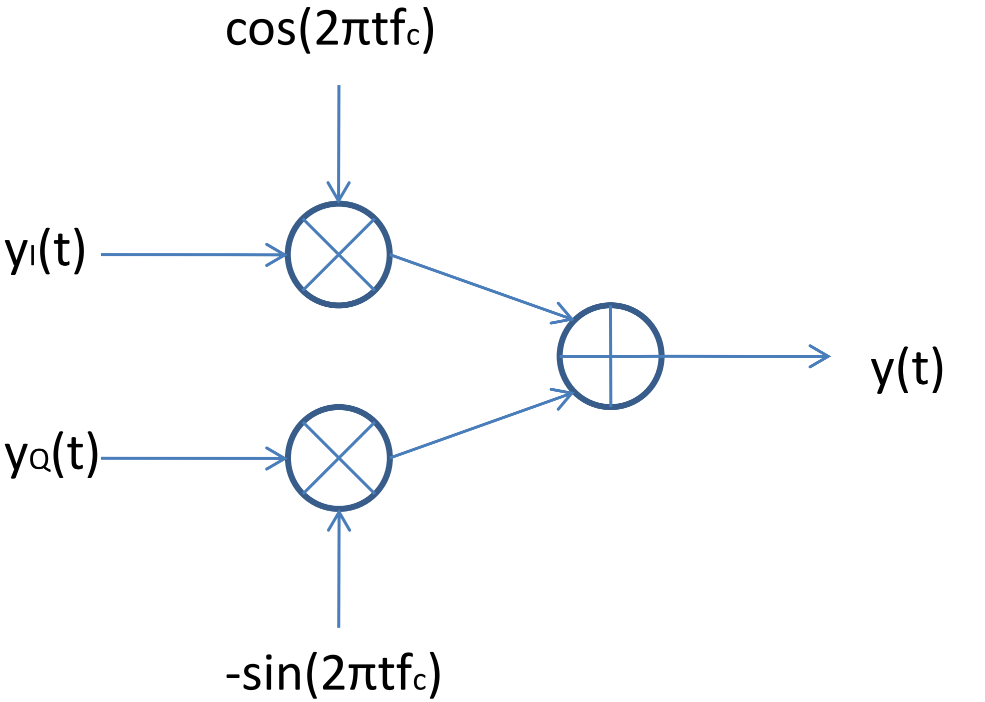
Assume two independent baseband transmissions
After demodulation with $\cos(2\pi t f_c)$ we get



The red spectras cancel out, thus, we can detect the blue independently from the red
Similar for demodulation with $\sin(2\pi t f_c)$

Carrier Transmission

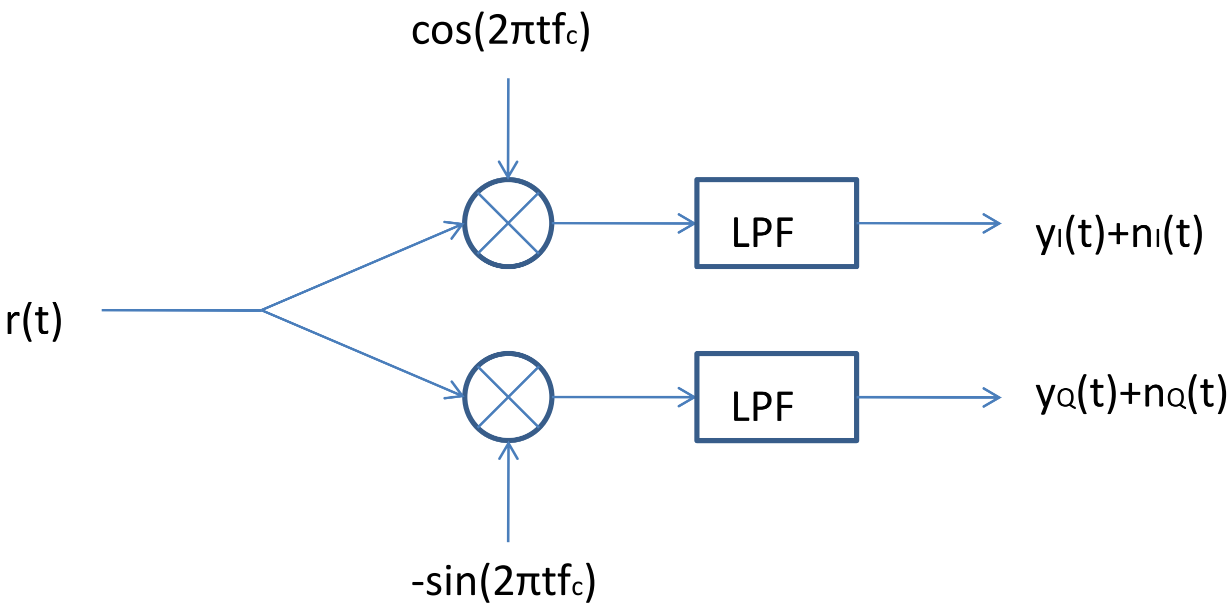
The block diagram of the transmitter is



$$y(t) = y_I(t)\cos(2\pi f_c t) - y_Q(t)\sin(2\pi f_c t)$$

Carrier Transmission

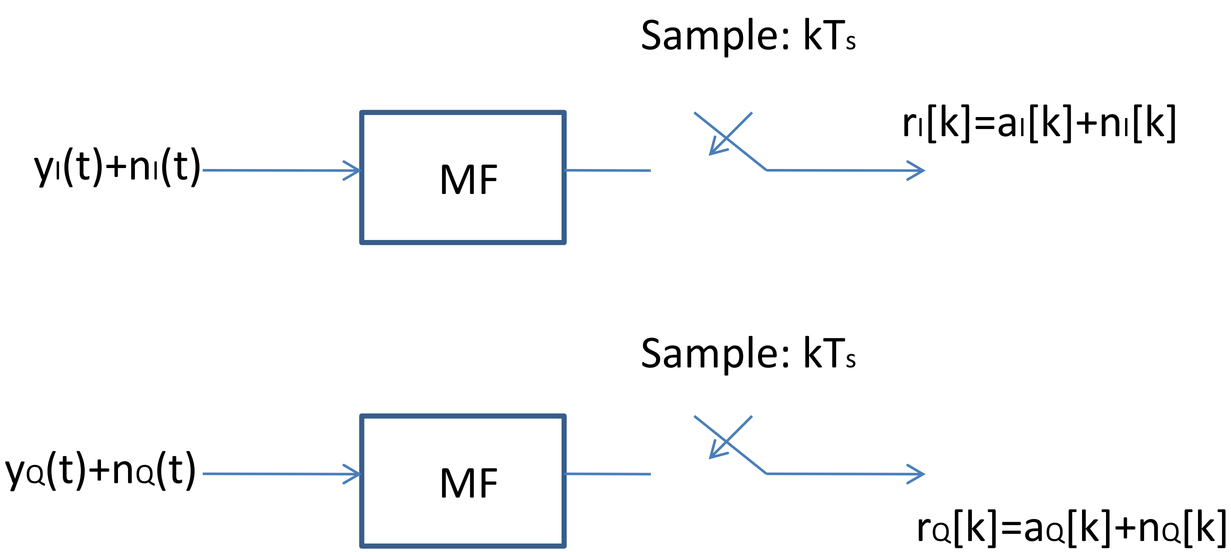
The block diagram of the receiver is



The in-phase and the quadrature components can be independently detected!
The LPF (low pass filters) can be taken as a matched filter to $h(t)$

Carrier Transmission

The signals at both rails are baseband signals, and conventional processing follows: matched filter $\rightarrow$ sampling every T_s second $\rightarrow$ decision unit



What is a complex-valued symbol $1+i$?

In QPSK, we transmit complex valued symbols. In one symbol interval, we have

$$y(t) = \underbrace{h(t)}_{y_I(t)} \cos(2\pi f_c t) - \underbrace{h(t)}_{y_Q(t)} \sin(2\pi f_c t)$$

Carrier Transmission

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Real part goes here and imaginary here

Carrier Transmission

We can alternatively express the signal $y(t)$ as

$$\begin{aligned} y(t) &= y_I(t)\cos(2\pi f_c t) - y_Q(t)\sin(2\pi f_c t) \\ &= e(t)\cos(2\pi f_c t + \theta(t)) \end{aligned}$$

where $e(t)$ is the envelope and $\theta(t)$ is the phase

For QPSK, $e(t) = \sqrt{2}h(t)$ and $\theta(t) \in \{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\}$

We can further manipulate $y(t)$ into

$$\begin{aligned} y(t) &= \mathrm{Re}\{(y_I(t) + iy_Q(t))\mathrm{e}^{2\pi f_c t}\} \\ &= \mathrm{Re}\{\tilde{y}(t)\mathrm{e}^{i2\pi f_c t}\} \end{aligned}$$

where

$$\tilde{y}(t) = y_I(t) + iy_Q(t)$$

Carrier Transmission

Example

Assume that we have two bits to transmit, say +1 and -1.

Carrier Transmission

Example

Assume that we have two bits to transmit, say +1 and -1.

We can either do this as

$$y(t) = h(t)\cos(2\pi f_c t) - (-h(t))\sin(2\pi f_c t)$$

Example

Assume that we have two bits to transmit, say +1 and -1.

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$$y(t) = h(t)\cos(2\pi f_c t) - (-h(t))\sin(2\pi f_c t)$$

or as

$$y(t) = \sqrt{2}h(t)\cos(2\pi f_c t + 7\pi/4)$$

Carrier Transmission

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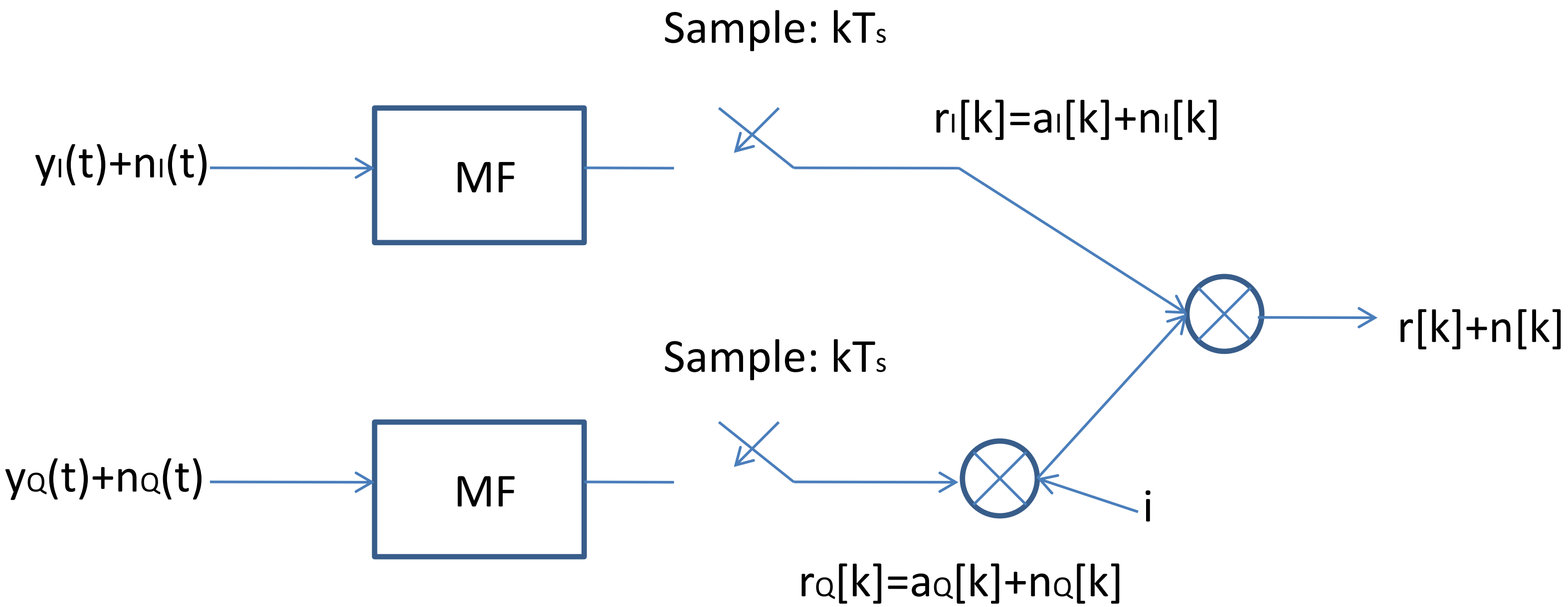
or as

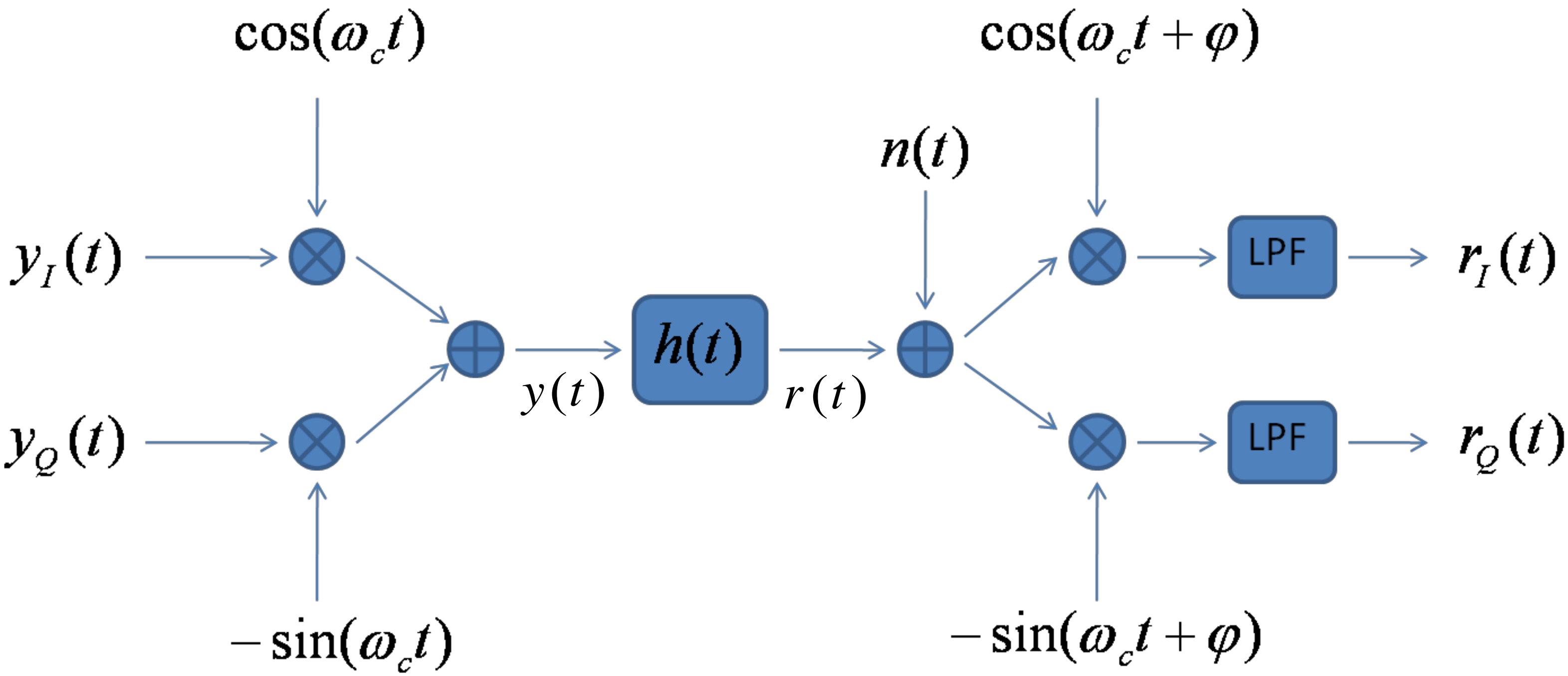
$$y(t) = \mathrm{Re}\{(1-i)h(t)\mathrm{e}^{2\pi f_c t}\}$$

Carrier Transmission

In the last representation, we can change the receiver processing into



System model



- **We want to represent the outputs as functions of the inputs**
- **Note that the receiver and transmitter are not synchronous**

Models of input and channel

The transmitted signal $y(t)$ equals

$$y(t) = y_I(t)\cos(\omega_c t) - y_Q(t)\sin(\omega_c t).$$

Similarily, the channel impulse response can be expressed as

$$h(t) = h_I(t)\cos(\omega_c t) - h_Q(t)\sin(\omega_c t).$$

Channel output in the Fourier domain

To evaluate $r(t) = y(t) * h(t)$, we consider the signals in the Fourier domain:

$$\begin{aligned} R(f) &= Y(f)H(f) \\ &= \frac{1}{4}\left[Y_I(f+f_c) + Y_I(f-f_c) + \jmath Y_Q(f+f_c) - \jmath Y_Q(f-f_c)\right] \\ &\quad \times \left[H_I(f+f_c) + H_I(f-f_c) + \jmath H_Q(f+f_c) - \jmath H_Q(f-f_c)\right] \end{aligned}$$

Now observe that a product of the type $Y_{I/Q}(f \pm f_c)H_{I/Q}(f \mp f_c) = 0$

Channel output in the Fourier domain

$$
\begin{aligned}
R(f) \quad = \quad & \frac{1}{4}\left[Y_I(f+f_c)H_I(f+f_c) + \jmath Y_I(f+f_c)H_Q(f+f_c) + Y_I(f-f_c)H_I(f-f_c)\right. \\
& -\jmath Y_I(f-f_c)H_Q(f-f_c) + \jmath Y_Q(f+f_c)H_I(f+f_c) - Y_Q(f+f_c)H_Q(f+f_c) \\
& \left. -\jmath Y_Q(f-f_c)H_I(f+f_c) - Y_Q(f-f_c)H_Q(f-f_c)\right]
\end{aligned}
$$

Channel output in the Fourier domain

$$
\begin{aligned}
R(f) \quad = \quad & \frac{1}{4}\big[\underbrace{Y_I(f+f_c)H_I(f+f_c)} + \jmath Y_I(f+f_c)H_Q(f+f_c) + \underbrace{Y_I(f-f_c)H_I(f-f_c)} \\
& -\jmath Y_I(f-f_c)H_Q(f-f_c) + \jmath Y_Q(f+f_c)H_I(f+f_c) - Y_Q(f+f_c)H_Q(f+f_c) \\
& -\jmath Y_Q(f-f_c)H_I(f+f_c) - Y_Q(f-f_c)H_Q(f-f_c)\big]
\end{aligned}
$$

By identifying terms, we get that

$$r(t) = \tilde{r}_I(t)\cos(\omega_c t) - \tilde{r}_Q(t)\sin(\omega_c t),$$

with

$$\tilde{r}_I(t) = \frac{1}{2}[y_I(t) * h_I(t) - y_Q(t) * h_Q(t)]$$

and

$$\tilde{r}_Q(t) = \frac{1}{2}[y_I(t) * h_Q(t) + y_Q(t) * h_I(t)].$$

Channel output in the Fourier domain

$$R(f) = \frac{1}{4}[Y_I(f+f_c)H_I(f+f_c) + \jmath Y_I(f+f_c)H_Q(f+f_c) + Y_I(f-f_c)H_I(f-f_c)$$
$$-\jmath Y_I(f-f_c)H_Q(f-f_c) + \jmath Y_Q(f+f_c)H_I(f+f_c) - Y_Q(f+f_c)H_Q(f+f_c)$$
$$-\jmath Y_Q(f-f_c)H_I(f+f_c) - Y_Q(f-f_c)H_Q(f-f_c)]$$

By identifying terms, we get that

$$r(t) = \tilde{r}_I(t)\cos(\omega_c t) - \tilde{r}_Q(t)\sin(\omega_c t),$$

with

$$\tilde{r}_I(t) = \frac{1}{2}[y_I(t) * h_I(t) - y_Q(t) * h_Q(t)]$$

and

$$\tilde{r}_Q(t) = \frac{1}{2}[y_I(t) * h_Q(t) + y_Q(t) * h_I(t)].$$

Channel output in the Fourier domain

$$R(f) = \frac{1}{4}\Big[Y_I(f+f_c)H_I(f+f_c) + \jmath Y_I(f+f_c)H_Q(f+f_c) + Y_I(f-f_c)H_I(f-f_c)$$
$$-\jmath Y_I(f-f_c)H_Q(f-f_c) + \jmath Y_Q(f+f_c)H_I(f+f_c) - Y_Q(f+f_c)H_Q(f+f_c)$$
$$-\jmath Y_Q(f-f_c)H_I(f+f_c) - Y_Q(f-f_c)H_Q(f-f_c)\Big]$$

By identifying terms, we get that

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with

$$\tilde{r}_I(t) = \frac{1}{2}[y_I(t) * h_I(t) - y_Q(t) * h_Q(t)]$$

and

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Channel output in the Fourier domain

$$\begin{aligned} R(f) &= \frac{1}{4}\big[Y_I(f+f_c)H_I(f+f_c) + \jmath Y_I(f+f_c)H_Q(f+f_c) + Y_I(f-f_c)H_I(f-f_c) \\ &\quad -\jmath Y_I(f-f_c)H_Q(f-f_c) + \jmath Y_Q(f+f_c)H_I(f+f_c) - Y_Q(f+f_c)H_Q(f+f_c) \\ &\quad -\jmath Y_Q(f-f_c)H_I(f+f_c) - Y_Q(f-f_c)H_Q(f-f_c)\big] \end{aligned}$$

By identifying terms, we get that

$$r(t) = \tilde{r}_I(t)\cos(\omega_c t) - \tilde{r}_Q(t)\sin(\omega_c t),$$

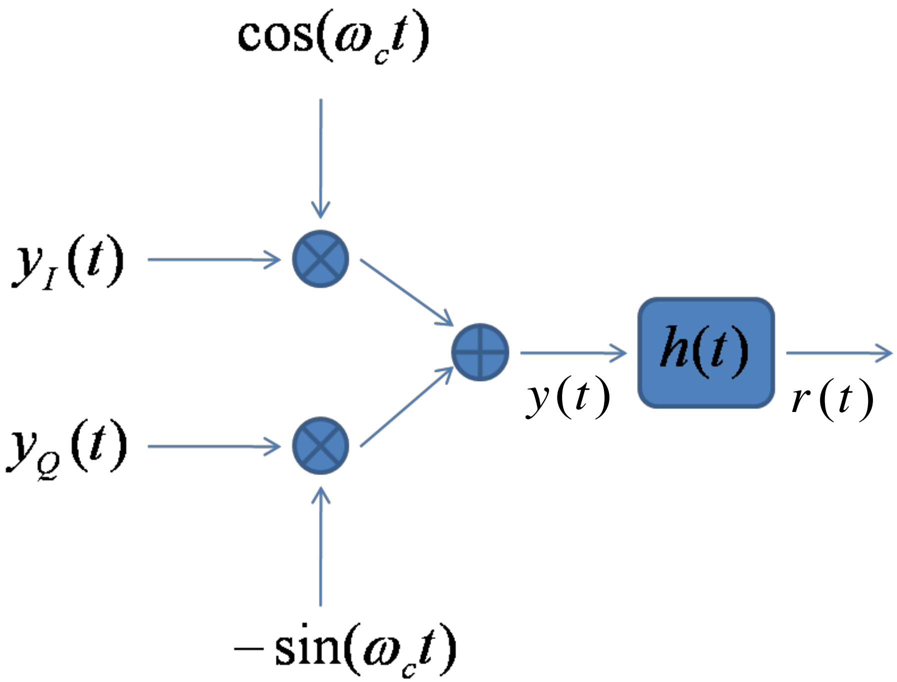
with

$$\tilde{r}_I(t) = \frac{1}{2}[y_I(t) * h_I(t) - y_Q(t) * h_Q(t)]$$

and

$$\tilde{r}_Q(t) = \frac{1}{2}[y_I(t) * h_Q(t) + y_Q(t) * h_I(t)].$$

Interlude

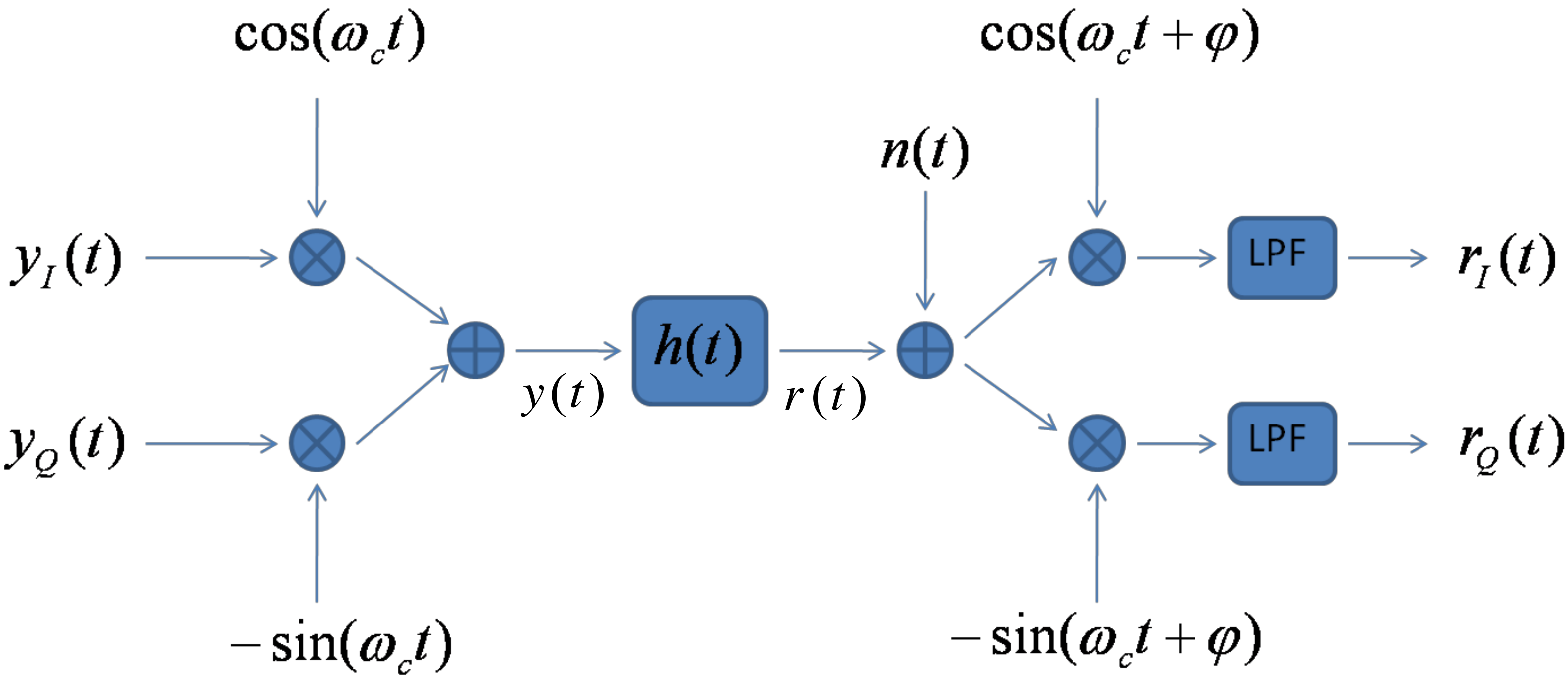


$$r(t) = \tilde{r}_I(t)\cos(\omega_c t) - \tilde{r}_Q(t)\sin(\omega_c t),$$

$$\tilde{r}_I(t) = \frac{1}{2}[y_I(t) * h_I(t) - y_Q(t) * h_Q(t)]$$

$$\tilde{r}_Q(t) = \frac{1}{2}[y_I(t) * h_Q(t) + y_Q(t) * h_I(t)].$$

Interlude



Now, lets multiply by cos(wt + φ)……

Basic trig properties

$$\begin{aligned}\cos(x)\cos(y) &= \tfrac{1}{2}[\cos(x+y)+\cos(x-y)]\\ \sin(x)\cos(y) &= \tfrac{1}{2}[\sin(x+y)+\sin(x-y)]\\ \sin(x)\sin(y) &= \tfrac{1}{2}[\cos(x+y)-\cos(x-y)]\end{aligned}$$

Signal at upper rail equals

$$[\tilde{r}_I(t)\cos(\omega_c t)-\tilde{r}_Q(t)\sin(\omega_c t)]\cos(\omega_c t+\phi)=$$
$$\frac{1}{2}\Big[\,\tilde{r}_I(t)[\cos(2\omega_c t+\phi)+\cos(\phi)]-\ \tilde{r}_Q(t)[\sin(2\omega_c t+\phi)-\sin(\phi)]\Big]$$

Basic trig properties

$$\begin{aligned}\cos(x)\cos(y) &= \tfrac{1}{2}[\cos(x+y)+\cos(x-y)]\\ \sin(x)\cos(y) &= \tfrac{1}{2}[\sin(x+y)+\sin(x-y)]\\ \sin(x)\sin(y) &= \tfrac{1}{2}[\cos(x+y)-\cos(x-y)]\end{aligned}$$

Signal at upper rail equals

$$[\tilde{r}_I(t)\cos(\omega_c t) - \tilde{r}_Q(t)\sin(\omega_c t)]\cos(\omega_c t+\phi) =$$

$$\frac{1}{2}\Big[\,\tilde{r}_I(t)[\cos(2\omega_c t+\phi)+\cos(\phi)] - \tilde{r}_Q(t)[\sin(2\omega_c t+\phi)-\sin(\phi)]\Big]$$

Remove by low pass filtering

Channel output in the Fourier domain

we get that

$$
\begin{aligned}
r_I(t) &= \frac{1}{2}\left[\tilde{r}_I(t)\cos(\phi) + \tilde{r}_Q(t)\sin(\phi)\right] \\
&= \frac{1}{4}\left[(y_I(t) * h_I(t) - y_Q(t) * h_Q(t))\cos(\phi) + (y_I(t) * h_Q(t) + y_Q(t) * h_I(t)))\sin(\phi)\right]
\end{aligned}
$$

and

$$
r_Q(t) = \frac{1}{4}\left[-(y_I(t) * h_I(t) - y_Q(t) * h_Q(t))\sin(\phi) + (y_I(t) * h_Q(t) + y_Q(t) * h_I(t))\cos(\phi)\right]
$$

Final result

Now construct the two complex valued signals

$$y^c(t) \triangleq y_I(t) + \jmath y_Q(t)$$

and

$$r^c(t) \triangleq r_I(t) + \jmath r_Q(t).$$

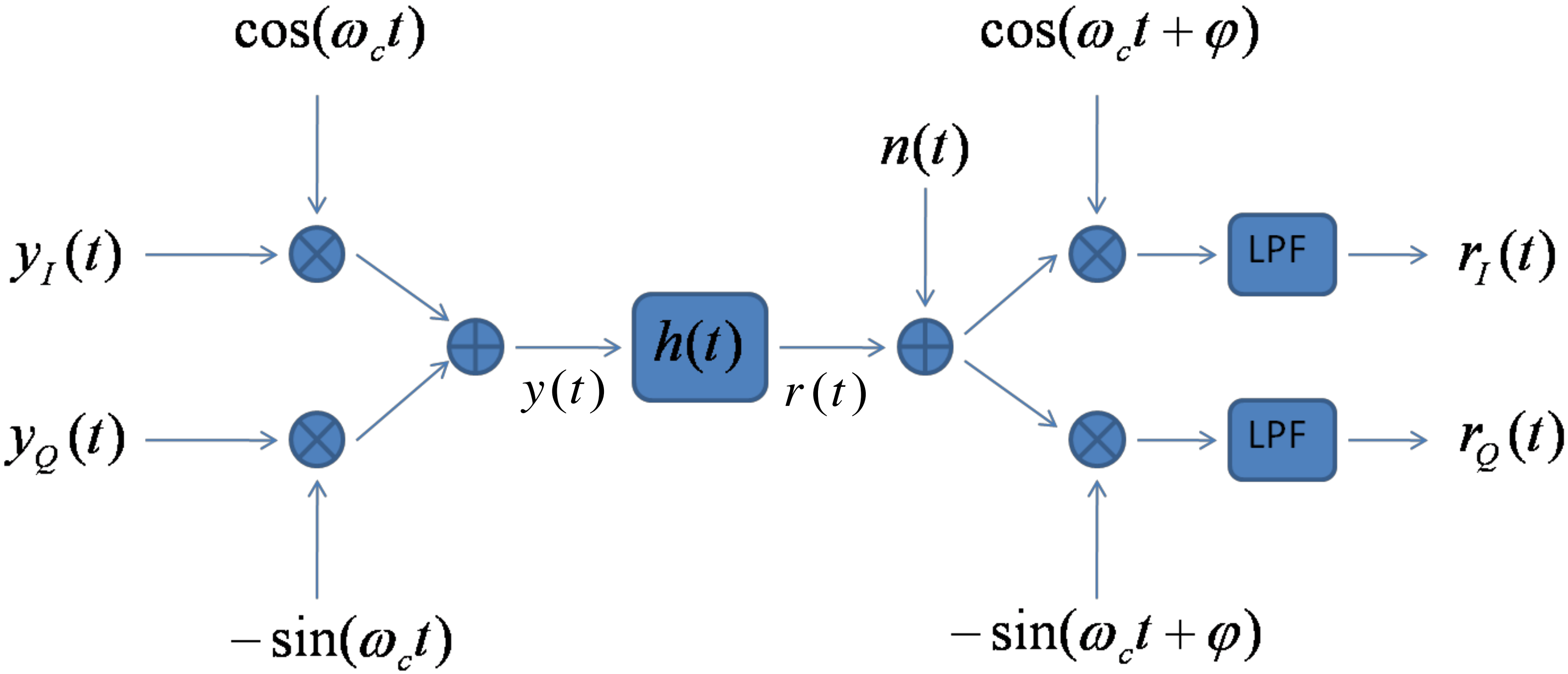
By identyifing some terms we can conclude that

$$r^c(t) = y^c(t) * h^c(t),$$

with

$$h^c(t) \triangleq h_I(t)\cos(\phi) + h_Q(t)\sin(\phi) + \jmath(h_Q(t)\cos(\phi) - h_I(t)\sin(\phi)).$$

Final result



This can be modeled in the complex baseband by………

Final result

$$h^c(t) \triangleq h_I(t)\cos(\phi) + h_Q(t)\sin(\phi) + \jmath(h_Q(t)\cos(\phi) - h_I(t)\sin(\phi)).$$

$$y^c(t) = y_I(t) + \jmath y_Q(t)$$

$$r^c(t) = r_I(t) + \jmath r_Q(t)$$

$$r^c(t) = y^c(t) \star h^c(t)$$

This can be modeled in the complex baseband by………this

No channel case

Let $h(t) = \delta(t)$

This means that $h_I(t) = \delta(t)$ and $h_Q(t) = 0$

We get $h^c(t) = \delta(t)\cos(\phi) - \jmath\delta(t)\sin(\phi)$

Leakage between the In-phase and the qudrature components!

No channel case

Let $h(t) = \delta(t)$

This means that $h_I(t) = \delta(t)$ and $h_Q(t) = 0$

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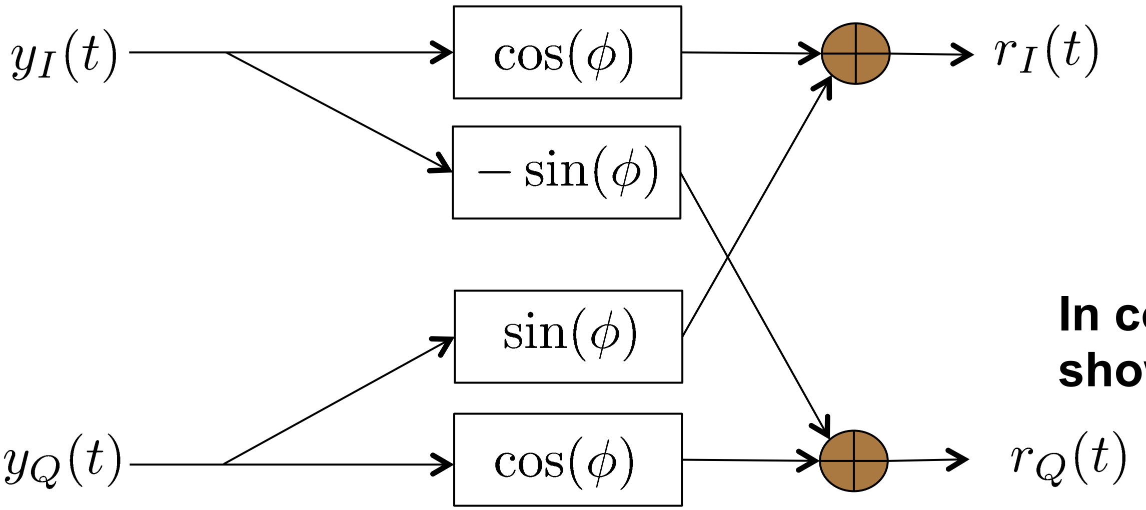
Leakage between the In-phase and the qudrature components!



$$r^c(t) = \exp(-i\phi)y^c(t)$$

In complex baseband, this shows up as a rotation!

Effect of ϕ

What is the effect of ϕ ?

$$h^c(t) \triangleq h_I(t)\cos(\phi) + h_Q(t)\sin(\phi) + \jmath(h_Q(t)\cos(\phi) - h_I(t)\sin(\phi)).$$

Energy of the impulse response = $\int_{-\infty}^{\infty} |h^c(t)|^2 dt = \int_{-\infty}^{\infty} h_I^2(t) + h_Q^2(t) dt$

Energy is independent of ϕ !!

→

Doesn't matter if tx and rx are not synchronous

Summary

We can always work in the complex baseband domain with the input/output relation

$$r^c(t) = y^c(t) \star h^c(t) + n^c(t)$$

And we do not care about ϕ (it must be estimated though)