LUND UNIVERSITY

EITG05 – Digital Communications

Week 4, Lecture 2

Performance M-ary Signaling
Receiver for Multipath Channels

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Chapter 4: Receivers in Digital Communication Systems – Part I

- 4.5 M-ary signaling
- 4.6 Receiver structure for the linear filter channel model

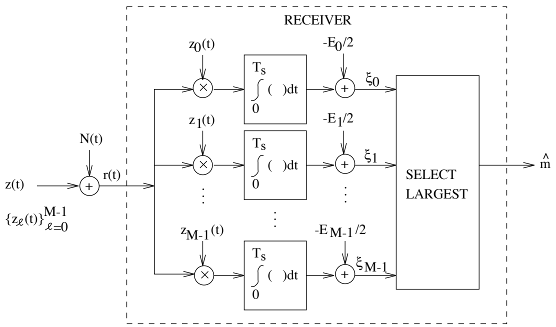
Chapter 5: Receivers in Digital Communication Systems – Part II

- 5.1 The MAP receiver for the AWGN channel
  - 5.1.1 A geometric description
- 5.2 Comparisons
  - 5.2.1 Energy efficiency

Pages 272 – 293, 329 – 331 and 360 – 366

Exercises: 4.7, 4.8, 4.27, 4.10, 4.17c, 4.20, 4.29, Example 4.12 on page 260, 4.32

M-ary Signaling



- The receiver computes M decision variables $\xi_0, \xi_1, \ldots, \xi_{M-1}$
- The selected message $\hat{m}$ is based on the largest value

$$\hat{m} = m_\ell \ , \quad \ell = \arg\max_i \xi_i$$

- **Question:** when do we make a wrong decision?

Probability of a wrong decision

- For $M = 2$ we have considered two error probabilities P_F and P_M
- For a given message $m = m_j$, in general there are $M-1$ ways (events) to make a wrong decision,

$$\{\xi_i > \xi_j \mid m = m_j\} \ , \quad i \neq j$$

- The probability of a wrong decision can be upper bounded by

$$Pr\{\hat{m} \neq m_j | m = m_j\} = Pr\Big\{ \bigcup_{\substack{i=0 \\ i\neq j}}^{M-1} \xi_i > \xi_j \ \Big| \ m = m_j \Big\}$$

$$\leq \sum_{\substack{i=0 \\ i\neq j}}^{M-1} Pr\{\xi_i > \xi_j \mid m = m_j\} \quad \text{(union bound)}$$

- Note: given some events A and B, the union bound states that

$$Pr\{A \cup B\} \leq Pr\{A\} + Pr\{B\} \ ,$$

where equality holds if A and B are independent

Symbol error probability

- The symbol error probability can be upper bounded by

$$P_s \leq \sum_{j=0}^{M-1} P_j \sum_{\substack{i=0\\ i\neq j}}^{M-1} Pr\{\xi_i > \xi_j \mid m = m_j\}$$

- From the binary case $M=2$ we know that (pick $i=0$ and $j=1$)

$$Pr\{\xi_i > \xi_j \mid m = m_j\} = Q\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right)$$

where $D_{i,j}$ is the Euclidean distance between $z_i(t)$ and $z_j(t)$

- We obtain the following main result for M-ary signaling:

$$\max_{\substack{i\\ i\neq j}} Q\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right) \leq P_s \leq \sum_{j=0}^{M-1} P_j \sum_{\substack{i=0\\ i\neq j}}^{M-1} Q\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right)$$

Distances $D_{i,j}$ are important

- P_s is determined by the distances $D_{i,j}$ between the signal pairs
- Let us sort these distances

$$D_{min} < D_1 < D_2 < \cdots < D_{max}$$

- Then the upper bound on P_s can be written as

$$P_s \leq c\, Q\left(\sqrt{\frac{D_{min}^2}{2N_0}}\right) + c_1\, Q\left(\sqrt{\frac{D_1^2}{2N_0}}\right) + \cdots + c_x\, Q\left(\sqrt{\frac{D_{max}^2}{2N_0}}\right)$$

- The coefficients are

$$c_\ell = \sum_{j=1}^{M-1} P_j \cdot n_{j,\ell}\ , \quad \ell = 0,1,2,\ldots,x$$

- $n_{j,\ell}$: number of signals at distance D_ℓ from signal $z_j(t)$

How many distinct terms do exist for 4-PAM?

Example: orthogonal signaling

- Consider M orthogonal signals with equal energy E
- Examples: FSK, PPM
- For each pair $z_i(t)$ and $z_j(t)$ we get

$$D_{i,j}^2 = E + E = 2E$$

- From the union bound we obtain

$$P_s \leq \sum_{j=0}^{M-1} P_j \sum_{\substack{i=0\\ i\neq j}}^{M-1} Q\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right)$$

$$= (M-1)\, Q\left(\sqrt{\frac{2E}{2N_0}}\right) = (M-1)\, Q\left(\sqrt{\frac{E}{N_0}}\right)$$

- This generalizes the binary case from the previous lecture

A useful approximation of P_s

- The union bound is easy to compute if we know all distances D_ℓ
- At large signal-to-noise ratio (small N_0), i.e., when P_s is small, the first term provides a good approximation

$$P_s \approx c\, Q\left(\sqrt{\frac{D_{min}^2}{2N_0}}\right)$$

- We see that the minimum distance D_{min}^2 and the average number of closest signals c dominate the performance in this case
- **Explanation:**
  the function $Q(x)$ decreases very fast as x increases (faster than exponentially). The other terms become negligible at some point.

$\Rightarrow$ at small P_s (small N_0) we can compare different signal constellations by means of D_{min}^2, similarly to the binary case

Energy efficiency and normalized distances

- Consider the case $P_\ell = 1/M$, $\ell = 0, 1, \ldots, M-1$
- The average received energy per bit is given by

$$\mathcal{E}_b = \frac{1}{k} \sum_{i=0}^{M-1} \frac{1}{M} \int_0^{T_s} z_i^2(t)\, dt = \frac{1}{k} \frac{E_0 + E_1 + \cdots E_{M-1}}{M}$$

- Using the normalized squared Euclidean distances

$$d_\ell^2 = \frac{D_\ell^2}{2\,\mathcal{E}_b} ,$$

  the union bound can be written as

$$P_s \le c\, Q\left(\sqrt{d_{min}^2 \frac{\mathcal{E}_b}{N_0}}\right) + c_1\, Q\left(\sqrt{d_1^2 \frac{\mathcal{E}_b}{N_0}}\right) + \cdots + c_x\, Q\left(\sqrt{d_{max}^2 \frac{\mathcal{E}_b}{N_0}}\right)$$

- The parameters d_ℓ^2 determine the energy efficiency

Example 4.19

Assume two signal constellations, denoted A and B respectively, with corresponding parameters $d^2_{\min,A}$ and $d^2_{\min,B}$. From the equality (see e.g. the dominating term in the union bound),

$$d^2_{\min,A}\mathcal{E}_{b,A}/N_0 = d^2_{\min,B}\mathcal{E}_{b,B}/N_0$$

we find that the difference (in dB) in received energy per information bit is (compare with (2.13) on page 16),

$$10\log_{10}(\mathcal{E}_{b,B}) - 10\log_{10}(\mathcal{E}_{b,A}) = 10\log_{10}\left(\frac{d^2_{\min,A}}{d^2_{\min,B}}\right)$$

Calculate the value $10\log_{10}\left(\frac{d^2_{\min,A}}{d^2_{\min,B}}\right)$ if "A" is binary antipodal PAM, and if "B" is 4-ary PAM. Assume, that the conditions leading to (2.50) are satiesfied.

- For M-ary PAM we have (Table 4.1 or Table 5.1)

$$d^2_{min} = 6\log_2(M)/(M^2-1) \quad \Rightarrow \; d^2_{min,A} = 2,\; d^2_{min,B} = 4/5$$

- $10\log_{10} d^2_{min,A}/d^2_{min,B} = 10\log_{10} 5/2 = 3.98$ dB

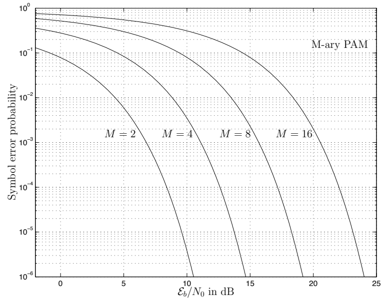
Binary PAM is 3.98 dB more energy efficient than 4-ary PAM!

Comparisons

$M=2$	P_b	$Q\left(\sqrt{d^2_{\min}\frac{\mathcal{E}_b}{N_0}}\right)$, (4.55)
	$d^2_{\min}$	$0 \le d^2_{\min} \le 2$, (4.57)
	ρ	ρ_{bin} , (2.21)
M-ary PAM	P_s	$2\left(1-\frac{1}{M}\right) Q\left(\sqrt{d^2_{\min}\frac{\mathcal{E}_b}{N_0}}\right)$, (5.35)
	$d^2_{\min}$	$\frac{6\log_2(M)}{M^2-1}$, Table 4.1 on page 281, (2.50)
	ρ	$\rho_{2-PAM}\cdot\log_2(M)$, (2.220)
M-ary PSK	P_s	$< 2Q\left(\sqrt{d^2_{\min}\frac{\mathcal{E}_b}{N_0}}\right)$, (5.43)
	$d^2_{\min}$	$2\sin^2(\pi/M)\log_2(M)$, Table 4.1, Fig. 5.11
	ρ	$\rho_{BPSK}\cdot\log_2(M)$, (2.229)
M-ary QAM (rect., k even)	P_s	$4\left(1-\frac{1}{\sqrt{M}}\right) Q\left(\sqrt{d^2_{\min}\frac{\mathcal{E}_b}{N_0}}\right) - -4\left(1-\frac{1}{\sqrt{M}}\right)^2 Q^2\left(\sqrt{d^2_{\min}\frac{\mathcal{E}_b}{N_0}}\right)$, (5.50)
(QPSK with $M=4$)	$d^2_{\min}$	$\frac{3\log_2(M)}{M-1}$, Table 4.1, Subsection 2.4.5.1
	ρ	$\rho_{BPSK}\cdot\log_2(M)$, (2.229)
M-ary FSK (orthogonal FSK)	P_s	$\le (M-1)Q\left(\sqrt{d^2_{\min}\frac{\mathcal{E}_b}{N_0}}\right)$, Example 4.18c, Table 4.1
	$d^2_{\min}$	$\log_2(M)$, Table 4.1 on page 281
	ρ	See (2.245)

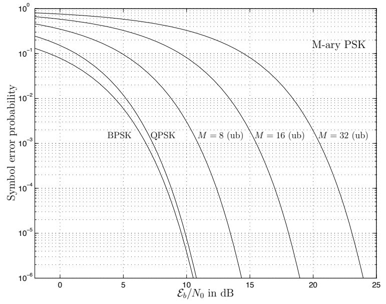
Table 5.1, p. 361

Symbol error probability comparison



M-ary PAM, $M = 2, 4, 8, 16$

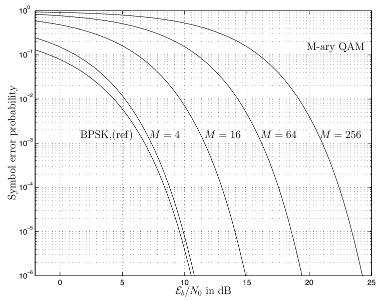
$$d^2_{min} = 6 \cdot \frac{\log_2 M}{M^2 - 1}$$

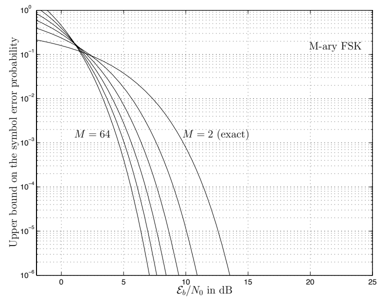


M-ary PSK, $M = 2, 4, 8, 16, 32$

$$d^2_{min} = 2\sin^2(\pi/M)\; \log_2 M$$

Symbol error probability comparison





M-ary QAM, $M = 4, 16, 64, 256$

$$d^2_{min} = 3 \cdot \frac{\log_2 M}{M-1}$$

M-ary FSK, $M = 2, 4, 8, 16, 32, 64$

$$d^2_{min} = \log_2 M$$

Gain in d^2_{min} compared with binary antipodal

Antipodal	$M=2$	0[dB]
Orthogonal	$M=2$	-3.01
M-ary PAM	$M=2$	0
	$M=4$	-3.98
	$M=8$	-8.45
	$M=16$	-13.27
	$M=32$	-18.34
	$M=64$	-23.57
M-ary PSK	$M=2$	0
	$M=4$	0
	$M=8$	-3.57
	$M=16$	-8.17
	$M=32$	-13.18
	$M=64$	-18.40
M-ary QAM	$M=4$	0
	$M=16$	-3.98
	$M=64$	-8.45
	$M=256$	-13.27
	$M=1024$	-18.34
	$M=4096$	-23.57

M-ary FSK	$M=2$	-3.01
	$M=4$	0
	$M=8$	1.76
	$M=16$	3.01
	$M=32$	3.98
	$M=64$	4.77
M -ary bi-orthogonal	$M=2$	0
	$M=4$	0
	$M=8$	1.76
	$M=16$	3.01
	$M=32$	3.98
	$M=64$	4.77

Large values M reduce energy efficiency

Example scenario: M-ary QAM

- We want to ensure that $P_s \le P_{s,req}$, where for M-ary QAM

$$P_s \le 4\, Q\left(\sqrt{d^2_{min}\frac{\mathcal{E}_b}{N_0}}\right) = 4\, Q\left(\sqrt{\mathcal{X}}\right) \;, \quad d^2_{min} = 3\, \log_2 \frac{M}{M-1}$$

- The pulse shape $g(t)$ is chosen such that

$$\rho = \log_2(M)\, \rho_{BPSK} \;, \quad \text{where } \rho = \frac{R_b}{W} \le \frac{d^2_{min}}{\mathcal{X}} \cdot \frac{\mathcal{P}_z}{N_0\, W}$$

- Combining these requirements we obtain

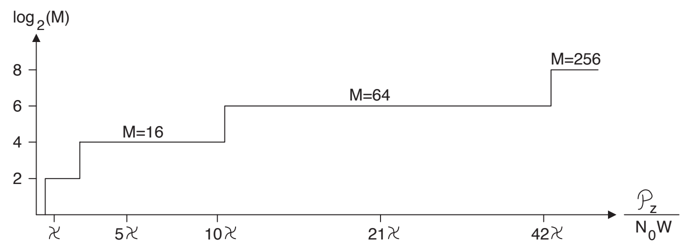
$$M \le 1 + \frac{3}{\mathcal{X}\, \rho_{BPSK}} \cdot \frac{\mathcal{P}_z}{N_0\, W} = 1 + \frac{3}{\mathcal{X}} \cdot \frac{\mathcal{P}_z T_s}{N_0}$$

- Hence we want to choose $M = 2^k$ such that (QAM: k even)

$$2^k \;\le\; 1 + \frac{3}{\mathcal{X}\, \rho_{BPSK}} \cdot \frac{\mathcal{P}_z}{N_0\, W} \;<\; 2^{k+2}$$

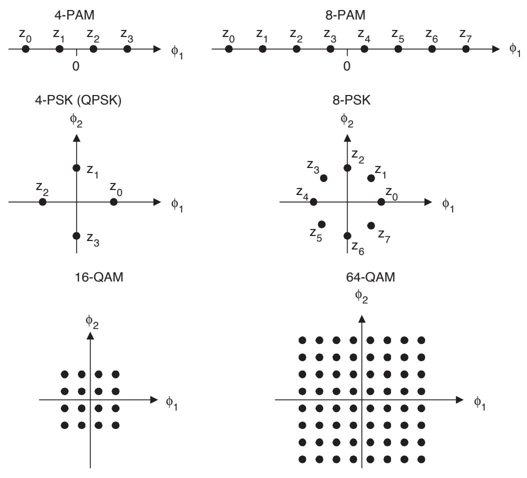
Example 4.22: adapting M to channel quality

Assume that an M-ary QAM system adapts between 4-ary QAM, 16-ary QAM, 64-ary QAM and 256-ary QAM. Show when a new M is chosen by plotting M (or $\log_2(M)$*) versus* $\mathcal{P}_z/N_0W$*. How large is the bit rate in each case? Assume that* $\rho_{BPSK} = 1/2$ [bps/Hz].



Depending on the channel quality we can achieve different bit rates $R_b = W$, $2W$, $3W$, or $4W$[bps]

Signal Space Representation



$$\phi_1(t) = \frac{g(t)}{\sqrt{E_g}}$$

$$\phi_1(t) = \frac{g(t)\cos(2\pi f_c t)}{\sqrt{E_g/2}}$$

$$\phi_2(t) = \frac{g(t)\sin(2\pi f_c t)}{\sqrt{E_g/2}}$$

A geometric description

- As we have seen in Chapter 2 we can represent our signal alternatives $z_j(t)$ as vectors (points) in signal space

$$\mathbf{z}_j = \left(z_{j,1}\right) = \left(A_j \sqrt{E_g}\right) \qquad \text{PAM}$$

$$\mathbf{z}_j = \left(z_{j,1} \quad z_{j,2}\right) = \left(A_j \sqrt{\frac{E_g}{2}} \quad B_j \sqrt{\frac{E_g}{2}}\right) \qquad \text{QAM, PSK}$$

- The signal energy can be written as

$$E_j = \int_0^{T_s} z_j^2(t)\, dt = z_{j,1}^2 + z_{j,2}^2$$

- Likewise, the squared Euclidean distance becomes

$$D_{i,j}^2 = \int_0^{T_s} \left(z_i(t) - z_j(t)\right)^2 dt = (z_{i,1} - z_{j,1})^2 + (z_{i,2} - z_{j,2})^2$$

Signal energies and distances have a geometric interpretation

Bit errors vs symbol errors

- Assume that S symbols are transmitted and S_{err} are in error
- If a symbol $\hat{m} \neq m$ is decided, this causes at least 1 bit error and at most $k = \log_2 M$ bit errors

$$S_{err} \leq B_{err} \leq k\, S_{err}$$

- This leads to the following relationship between P_b and P_s:

$$\frac{P_s}{k} = \frac{E\{S_{err}\}}{S \cdot k} \leq P_b \leq \frac{E\{S_{err} \cdot k\}}{S \cdot k} = P_s$$

- P_s depends on the signal constellation only
- The exact P_b depends on the mapping from bits to messages m_ℓ and hence signal alternatives $s_{m_\ell}(t)$

Example: Which mapping is better for 4-PAM? (and why?)

$$(1) \quad m_0 = 00,\ m_1 = 11,\ m_2 = 01,\ m_3 = 10$$

$$(2) \quad m_0 = 00,\ m_1 = 01,\ m_2 = 11,\ m_3 = 10$$

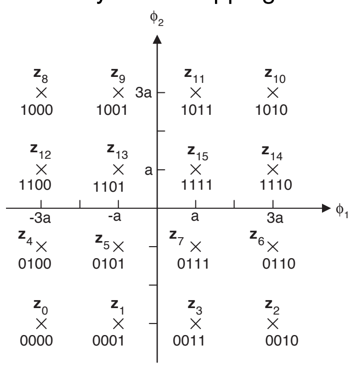
Gray code mappings

- We have seen that for small N_0 we can approximate

$$P_s \approx c\, Q\left(\sqrt{\frac{D_{min}^2}{2N_0}}\right)$$

- This motivates the use of Gray code mappings:

Example:
16-QAM



Receiver for linear filter channel model

- In Chapter 3 we have introduced the model

$$z_\ell(t) = s_\ell(t) * h(t) \ ,$$

  where $h(t)$ denotes the impulse response of the channel filter
- For a simple channel with a direct transmission path only

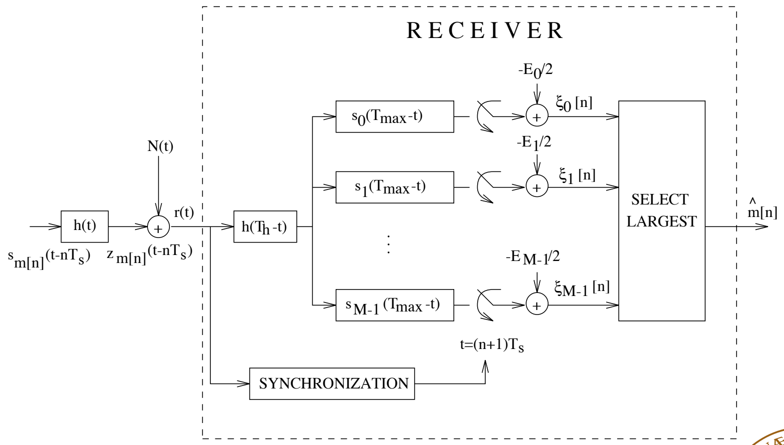
$$h(t) = \alpha\, \delta(t) \quad \Rightarrow \; z_\ell(t) = \alpha\, s_\ell(t)$$

- In case of multipath propagation the channel filter can change the shape and duration of the signals $z_\ell(t)$
- It can be shown that the matched filter of the overall system can be replaced with a cascade of two separate matched filters

$$z_\ell(T_s - t) \quad \Leftrightarrow \quad h(T_h - t)\ ,\ s_\ell(T_{max} - t)\ , \quad T_s = T_{max} + T_h$$

- The channel matching filter $h(T_h - t)$ simplifies the implementation of the receiver

ML receiver with channel matching filter



Example: three-ray channel

- Consider a channel with three signal paths

$$h(t) = \alpha_1\, \delta(t - \tau_1) + \alpha_2\, \delta(t - \tau_2) + \alpha_3\, \delta(t - \tau_3)$$

- Assuming $\tau_1 < \tau_2 < \tau_3$ we have $T_h = \tau_3$
- The channel matching filter becomes

$$\begin{aligned} h(T_h - t) &= h(\tau_3 - t) \\ &= \alpha_3\ \delta(t) + \alpha_2\, \delta(t - (\tau_3 - \tau_2)) + \alpha_1\, \delta(t - (\tau_3 - \tau_1)) \end{aligned}$$

RAKE receiver structure: