EITG05 – Digital Communications

Week 4, Lecture 1

Matched Filter Receiver
Performance Binary Signaling

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Chapter 4: Receivers in Digital Communication Systems – Part I

- 4.3 The minimum Euclidean distance receiver
  - 4.3.1 Matched filter implementation
- 4.4 Binary signaling
  - 4.4.1 P_b for minimum Euclidean distance receiver
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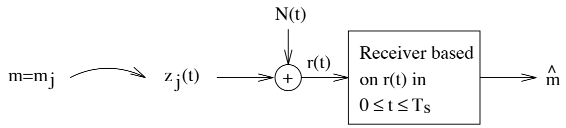
Exercises: 3.11c, Example 3.19 on page 168, 3.23, 4.1, 4.2, 4.6

The Minimum Euclidean Distance Receiver



- The received signal is compared with all noise-free signals $z_i(t)$ in terms of the squared Euclidean distance

$$D_{r,i}^2 = \int_0^{T_s} \left(r(t) - z_i(t)\right)^2 \, dt = E_r - 2\int_0^{T_s} r(t)\, z_i(t) \, dt + E_i$$

- The message is selected according to the following decision rule:

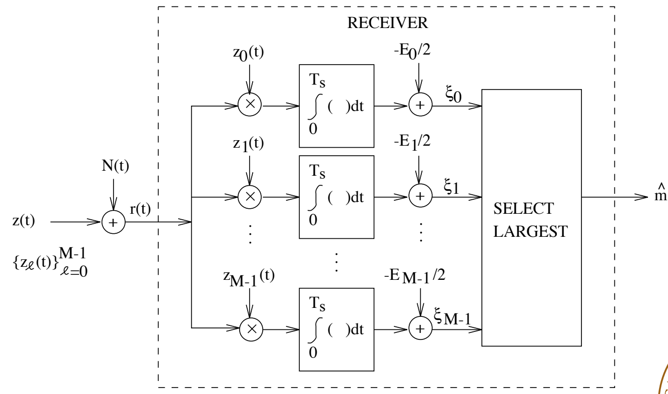
$$\hat{m}(r(t)) = m_\ell \ ,$$

where $\ell = \arg\min_i \ D_{r,i}^2$

Correlation based implementation

Equivalently we can write:

$$\ell = \arg\min_i \ D_{r,i}^2 = \arg\max_i \int_0^{T_s} r(t)\, z_i(t) \, dt - E_i/2$$



Matched filter implementation

- A filter with impulse response $q(t)$ is matched to a signal $z_i(t)$ if

$$q(t) = z_i(-t + T_s) = z_i(-(t - T_s))$$

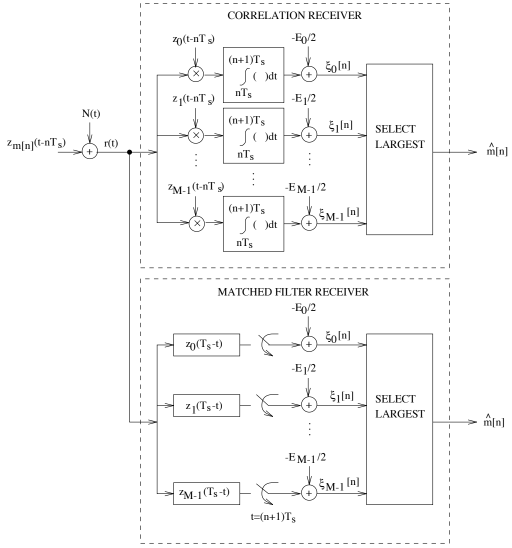
- Let the received signal $r(t)$ enter this matched filter $q(t)$
- The matched filter output, evaluated at time $t = (n+1)T_s$, can be written as

$$r(t) * q(t)\big|_{t=(n+1)T_s} = \int_{nT_s}^{(n+1)T_s} r(\tau)\, z_i(\tau - nT_s)\, d\tau$$

- **Observe:**
  this is exactly the same output value as the correlator produces

$\Rightarrow$ We can replace each correlator with a matched filter which is sampled at times $t = (n+1)T_s$

Matched filter vs correlator implementation



Summary: receiver types

- **Minimum Euclidean distance (MED) receiver:**
  decision is based on the signal alternative $z_i(t)$ closest to $r(t)$
- **Correlation receiver:**
  an implementation of the MED receiver based on correlators
- **Matched filter receiver:**
  an implementation of the MED receiver based on matched filters
- **Maximum likelihood (ML) receiver:**
  equivalent to MED receiver under our assumptions: ML = ED
- **Maximum a-posteriori (MAP) receiver:**
  minimizes symbol error probability P_s
  equivalent to ML if $P_i = 1/M$, $i = 0, \dots, M-1$: ML = ED = MAP

Binary Signaling

- Binary signaling ($M = 2$, $T_s = T_b$) simplifies the general receiver
- Consider the two decision variables

$$\xi_i[n] = \int_{nT_s}^{(n+1)T_s} r(t)\, z_i(t - nT_s)\, dt - E_i/2\,, \quad i = 0, 1$$

- The decision $\hat{m}[n]$ is made according to the larger value, i.e.,

$$\xi_1[n] \underset{\hat{m}[n]=m_0}{\overset{\hat{m}[n]=m_1}{\gtrless}} \xi_0[n]$$

- This can be reduced to a single decision variable only

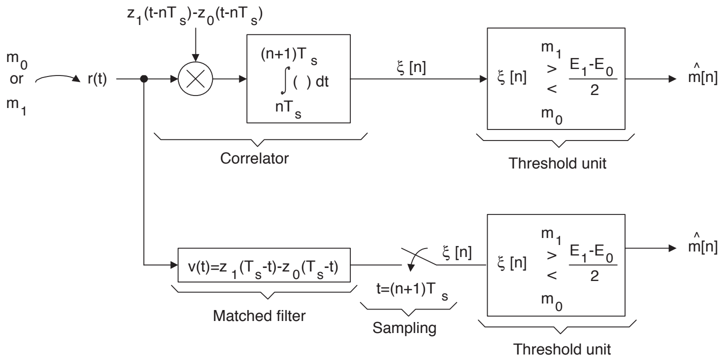
$$\xi[n] = \int_{nT_s}^{(n+1)T_s} r(t)\, \big(z_1(t - nT_s) - z_0(t - nT_s)\big)\, dt$$

which is compared to a threshold value

$$\xi[n] \underset{\hat{m}[n]=m_0}{\overset{\hat{m}[n]=m_1}{\gtrless}} \frac{E_1 - E_0}{2}$$

Receiver for Binary Signaling

- Only one correlator or one matched filter is now required:



- Matched filter output needs be sampled at correct time

When do we make a wrong decision?

- Assuming $m = m_0$ is sent, the decision variable becomes

$$\xi[n] = \int_0^{T_s} r(t)\big(z_1(t)-z_0(t)\big)\, dt = \int_0^{T_s} \big(z_0(t)+N(t)\big)\cdot\big(z_1(t)-z_0(t)\big)\, dt$$

- We can divide this into a signal component β_0 and a noise component $\mathcal{N}$

$$\xi[n] = \beta_0 + \mathcal{N}$$

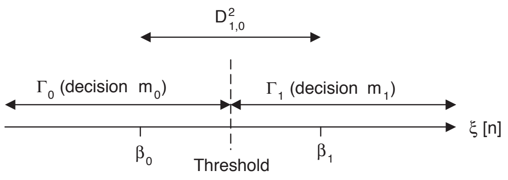
$$\beta_0 = \int_0^{T_s} z_0(t)\,\big(z_1(t)-z_0(t)\big)\, dt\ , \quad \mathcal{N} = \int_0^{T_s} N(t)\,\big(z_1(t)-z_0(t)\big)\, dt$$

- **Wrong decision:** if $\xi[n] > (E_1 - E_0)/2$ then $\hat{m} = m_1 \neq m_0 = m$
- Analogously, when $m = m_1$ is sent we get

$$\xi[n] = \beta_1 + \mathcal{N}$$

$$\beta_1 = \int_0^{T_s} z_1(t)\,\big(z_1(t)-z_0(t)\big)\, dt$$

Decision regions



- With

$$\beta_0 + \beta_1 = -\int_0^{T_s} z_0^2(t)\, dt + \int_0^{T_s} z_1^2(t)\, dt = E_1 - E_0$$

the decision threshold lies in the center between β_0 and β_1:

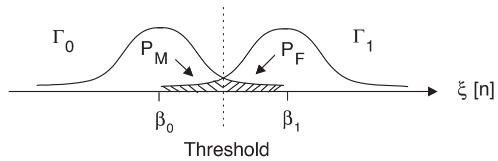
$$\frac{E_1 - E_0}{2} = \frac{\beta_0 + \beta_1}{2}$$

- Furthermore we see that

$$\beta_1 - \beta_0 = \int_0^{T_s} \big(z_1(t)-z_0(t)\big)^2\, dt = D^2_{1,0} = D^2_{0,1}$$

Probability of a wrong decision

- There exist two ways to make an error:



P_F: false alarm probability P_M: missed detection probability

- The two probabilities of error can be determined as

$$P_F = Pr\{\hat{m}[n] = m_1 | m = m_0\} = Pr\{\beta_0 + \mathcal{N} > (\beta_0+\beta_1)/2\}$$
$$P_M = Pr\{\hat{m}[n] = m_0 | m = m_1\} = Pr\{\beta_1 + \mathcal{N} < (\beta_0+\beta_1)/2\}$$

- We can express these in terms of the $Q(x)$-function:

$$P_F = P_M = Q\left(\frac{\beta_1 - \beta_0}{2\sigma}\right)$$

Bit error probability

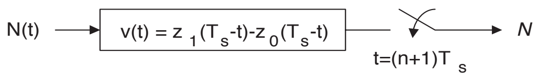
- The bit error probability can be written as

$$P_b = P_0 P_F + P_1 P_M = (P_0 + P_1) P_F = P_F = P_M$$

- With $\beta_1 - \beta_0 = D^2_{0,1}$ and $\sigma^2 = N_0/2 \cdot D^2_{0,1}$ we obtain

$$P_b = Q\left(\frac{\beta_1 - \beta_0}{2\sigma}\right) = Q\left(\frac{D^2_{0,1}}{2\sigma}\right) = Q\left(\sqrt{\frac{D^2_{0,1}}{2N_0}}\right)$$

- This fundamental result provides the bit error probability P_b of an ML receiver for binary transmission over an AWGN channel
- The additive noise $\mathcal{N}$ is sampled from a filtered noise process



$$\sigma^2 = N_0/2 \cdot E_v = N_0/2 \int_0^{T_s} \left(z_1(t) - z_0(t)\right)^2 \, dt$$

Example

- Let $z_0(t) = 0$ and $z_1(t)$ rectangular with amplitude A and $T = T_b$
- The information bit rate is $R_b = 400$ kbps
- Regarding the noise we know that $A^2/N_0 = 70$ dB

Task: determine the bit error probability P_b

Solution:

- First we find that $D^2_{0,1} = A^2/R_b$
- Then

$$\frac{D^2_{0,1}}{2N_0} = \frac{A^2}{N_0} \cdot \frac{1}{2R_b} = 12.5$$

- $P_b = Q\left(\sqrt{12.5}\right) = Q(3.536) = 2.3 \cdot 10^{-4}$
- Last step: check Table 3.1 on page 182

An energy efficiency perspective

- Consider the case $P_0 = P_1 = 1/2$
- The average received energy per bit is then

$$\mathcal{E}_b = \frac{1}{2}\int_0^{T_b} z_0^2(t)\, dt + \frac{1}{2}\int_0^{T_b} z_1^2(t)\, dt = \frac{E_0 + E_1}{2}$$

- We can then introduce the normalized squared Euclidean distance

$$d^2_{0,1} = \frac{D^2_{0,1}}{2\mathcal{E}_b} = \frac{1}{2\mathcal{E}_b}\int_0^{T_b} \left(z_1(t) - z_0(t)\right)^2 \, dt$$

- With this the bit error probability becomes

$$P_b = Q\left(\sqrt{\frac{D^2_{0,1}}{2N_0}}\right) = Q\left(\sqrt{d^2_{0,1}\frac{\mathcal{E}_b}{N_0}}\right)$$

- The parameter $d^2_{0,1}$ is a measure of energy efficiency

Special case 1: antipodal signals

- In case of antipodal signals we have $z_1(t) = -z_0(t)$ and

$$D^2_{0,1} = \int_0^{T_b} \left(z_1(t) - z_0(t)\right)^2 \, dt = 4\int_0^{T_b} z_1^2(t)\, dt = 4E$$

- From $E_0 = E_1 = E$ follows

$$\mathcal{E}_b = \frac{E + E}{2} = E$$

and

$$d^2_{0,1} = \frac{D^2_{0,1}}{2\mathcal{E}_b} = \frac{4E}{2E} = 2$$

- The bit error probability for any pair of antipodal signals becomes

$$P_b = Q\left(\sqrt{2\frac{\mathcal{E}_b}{N_0}}\right)$$

Special case 2: orthogonal signals

- In case of orthogonal signals we have

$$\int_0^{T_b} z_0(t)\, z_1(t)\; dt = 0$$

and hence (compare page 28)

$$D_{0,1}^2 = \int_0^{T_b} \big(z_1(t) - z_0(t)\big)^2\; dt = E_0 + E_1$$

- This gives

$$\mathcal{E}_b = \frac{E_0 + E_1}{2}$$

and

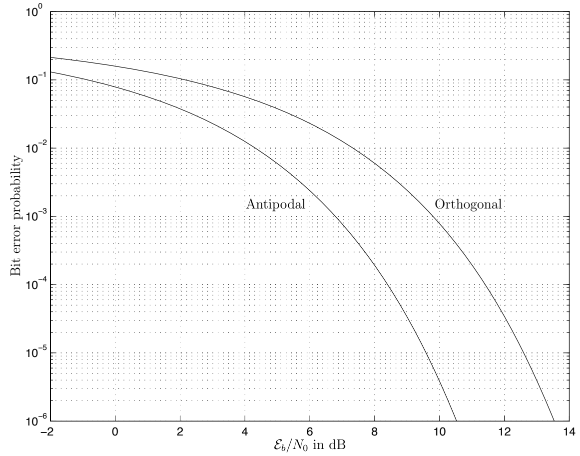
$$d_{0,1}^2 = \frac{D_{0,1}^2}{2\,\mathcal{E}_b} = \frac{E_0 + E_1}{E_0 + E_1} = 1$$

- The bit error probability for any pair of orthogonal signals is

$$P_b = Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right)$$

Comparison

Antipodal vs orthogonal signaling:



Larger values of $d_{0,1}^2$ give better energy efficiency

Antipodal vs orthogonal signaling

- There is a constant gap between the two curves
- We can measure the difference in energy efficiency by the ratio

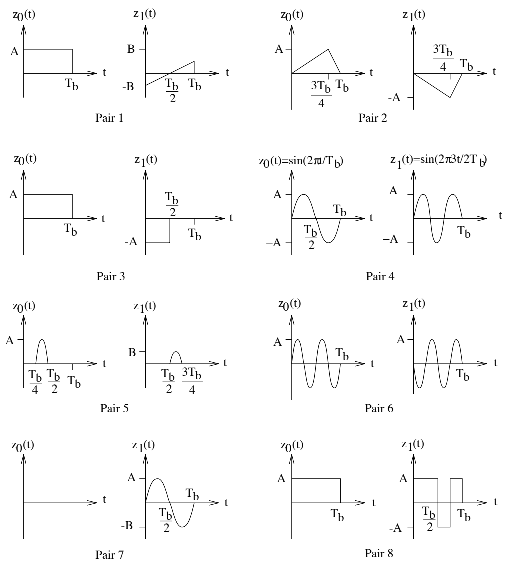
$$\frac{\mathcal{E}_{b,atp}}{\mathcal{E}_{b,ort}} = \frac{d_{0,1,ort}^2}{d_{0,1,atp}^2} = \frac{1}{2}$$

- In terms of dB this corresponds to

$$10\log_{10}\left(\frac{\mathcal{E}_{b,atp}}{\mathcal{E}_{b,ort}}\right) = 10\log_{10}\left(\frac{d_{0,1,ort}^2}{d_{0,1,atp}^2}\right) = -3\ [\text{dB}]$$

$\Rightarrow$ antipodal signaling requires 3 dB less energy for equal P_b

Example 4.11: rank pairs with respect to $d_{0,1}^2$



Can we do better?

- It is possible to show that for two equally likely signal alternatives we always have

$$d_{0,1}^2 \leq 2$$

- Antipodal signaling is hence optimal for binary signaling ($M=2$)

Remark:

- Channel coding can be used to further increase $d_{0,1}^2$
- Sequences of binary pulses with large separation are designed
- This does not contradict the result from above: coded binary signals correspond to uncoded signals with $M>2$

Channel coding can be used for improving energy efficiency
Cost: complexity, latency, (bandwidth)

Relationship between parameters

- The bit error probability can be expressed in different ways

$$P_b = Q\left(\sqrt{\frac{D_{0,1}^2}{2N_0}}\right) = Q\left(\sqrt{d_{0,1}^2\,\frac{\mathcal{E}_b}{N_0}}\right) = Q\left(\sqrt{d_{0,1}^2\,\frac{P_z}{R_b N_0}}\right)$$

- Assuming $z_0(t)=\alpha\, s_0(t)$ and $z_1(t)=\alpha\, s_1(t)$ we also get

$$P_b = Q\left(\sqrt{d_{0,1}^2\,\frac{\alpha^2 \bar{P}_{sent}}{R_b N_0}}\right) = Q\left(\sqrt{\frac{d_{0,1}^2}{\rho}\cdot\frac{\alpha^2 \bar{P}_{sent}}{N_0\, W}}\right)$$

- Recall that $\rho = R_b/W$ is the bandwidth efficiency and $N_0\, W$ is the noise power within the bandwidth W

The expression that is most appropriate to use depends on the specific problem to be solved

A "typical" type of problem

- The bit error probability must not exceed a certain level,

$$P_b \leq P_{b,req} = Q(\sqrt{\mathcal{X}})$$

- Example: if $P_{b,req}=10^{-9}$ then $\mathcal{X}\approx 36$
- **Consequences:**

$$d_{0,1}^2\,\frac{\mathcal{E}_b}{N_0} \;\geq\; \mathcal{X}$$

$$R_b \;\leq\; \frac{d_{0,1}^2}{\mathcal{X}}\cdot\frac{P_z}{N_0}$$

$$R_b \;\leq\; \frac{d_{0,1}^2}{\mathcal{X}}\cdot\frac{\alpha^2\bar{P}_{sent}}{N_0}$$

- Note: the received signal power P_z decreases with communication distance

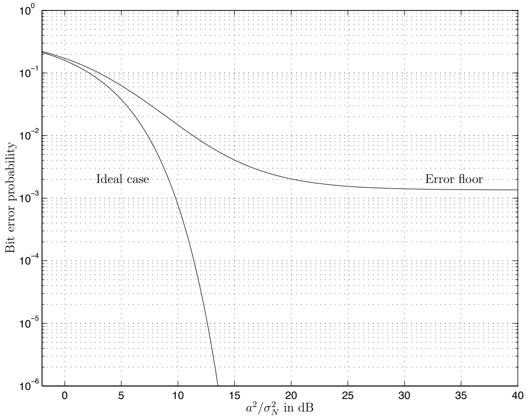
Example 4.12: transmission hidden in noise

In a specific application equally likely binary antipodal signals are used, and the pulse shape is $g_{rc}(t)$ with amplitude A and duration $T \leq T_b$. AWGN with power spectral density $N_0/2$, and the ML receiver is assumed. It is required that the bit error probability must not exceed 10^{-9}. It is also required that the power spectral density satisfies $R(f) \leq N_0/2$ for all frequencies f (the information signal is intentionally "hidden" in the noise). Determine system and signal parameters above such that these two requirements are satisfied.

- $P_b = Q\left(\sqrt{2\,\mathcal{E}_b/N_0}\right) \leq 10^{-9} \Rightarrow \mathcal{E}_b/N_0 \geq 18$
- $R(f) = R_b|G_{rc}(f)|^2$ has maximum at $f=0$
- $R(0) = R_b\, A^2T^2/4 \leq N_0/2$ (check pulse shape)
- $\mathcal{E}_b/N_0 = 3/8A^2T/N_0 \geq 18$
- Hidden in noise: $A^2T/N_0 \leq 2/(R_bT)$
- P_b requirement: $A^2T/N_0 \geq 48$
- Solution: choose $T \leq T_b/24$ and $A^2 = 48N_0/T$

Non-ideal receiver conditions

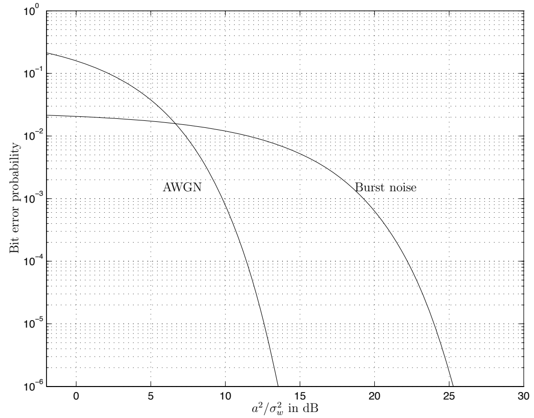
Example 4.15: unexpected additional noise w_x, i.e., $w = w_N + w_x$



Can be analyzed with our methods

Non-ideal receiver conditions

Example 4.16: hostile bursty interference



Observe: at low power an interference in bursts is more severe than continuous interference