EITG05 – Digital Communications

Week 2, Lecture 1

Bandwidth of Transmitted Signals

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Chapter 2: Model of a Digital Communication System

- 2.5 The bandwidth of the transmitted signal
  - 2.5.1 Basic Fourier transform concepts
  - 2.5.2 $R(f)$: M-ary transmission
  - 2.5.3 $R(f)$: binary signaling
  - 2.5.4 Some definitions of bandwidth

Pages 61 – 72 (excluding 2.5.1.2) and 77 – 88

Exercises: 2.18, 2.16, 2.17a, 2.19a, Example 2.17 on page 64

What did we do last week?

Concepts of M-ary digital signaling:

- Modulation of amplitude, phase or both: PAM, PSK, QAM
- Orthogonal signaling: FSK, OFDM
- Pulse position and width: PPM, PWM

We have paid special attention to:

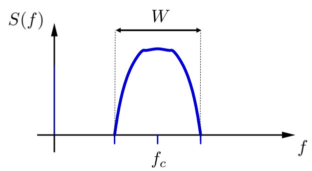
- Average symbol energy $\overline{E}_s$
- Euclidean distance $D_{i,j}$
- Both values could be related to the energy E_g of the pulse $g(t)$

What about the bandwidth of the signal?

How is it related to $g(t)$?

Bandwidth of Transmitted Signal

- The bandwidth W of a signal is the width of the frequency range where most of the signal energy or power is located



- W is measured on the positive frequency axis
- The bandwidth is a limited and precious resource
- We must have control of the bandwidth and use it efficiently

Questions:

What is the relationship between information bit rate and required bandwidth?

How does the bandwidth depend on the signaling method?

Energy Spectrum

- We have seen last week that the energy of a signal $x(t)$ can be determined as
$$E_x = \int_{-\infty}^{\infty} x^2(t)\ dt$$
- The function $x^2(t)$ shows how the energy E_x is distributed along the time axis
- According to Parseval's relation we can alternatively express the energy as
$$E_x = \int_{-\infty}^{\infty} |X(f)|^2\ df\ ,$$
where $X(f)$ denotes the Fourier transform of the signal $x(t)$
- The function $|X(f)|^2$ shows how the energy E_x is distributed in the frequency domain

$\Rightarrow$ We need the Fourier transform as a tool for finding the bandwidth of our signals

Fourier Transform

- The Fourier transform of a signal $x(t)$ is given by
$$X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\ e^{-j2\pi f t}\ dt = X_{Re}(f) + j\ X_{Im}(f)\ ,$$
where $j = \sqrt{-1}$, i.e., the solution to $j^2 = -1$
- We can also express $X(f)$ in terms of magnitude $|X(f)|$ and phase $\varphi(f) = \arg X(f)$ (argument)
$$X(f) = |X(f)|\ e^{j\,\varphi(f)}$$
- Then
$$|X(f)| = \sqrt{X_{Re}^2(f) + X_{Im}^2(f)}$$
$$X_{Re}(f) = |X(f)|\ \cos(\varphi(f))$$
$$X_{Im}(f) = |X(f)|\ \sin(\varphi(f))$$
- Assuming $x(t)$ is a real-valued signal, it can be shown that
$$|X(f)| = |X(-f)|\ ,\ \text{(even)} \qquad \varphi(f) = -\varphi(-f)\ ,\ \text{(odd)}$$

Fourier Transform

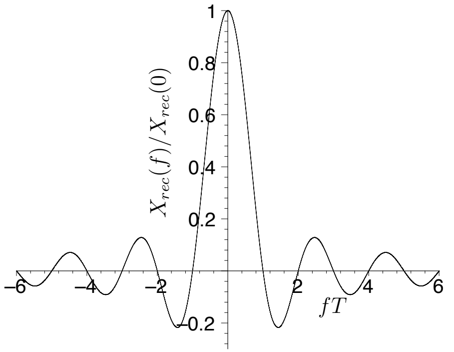
- The original signal $x(t)$ can then be expressed in terms of the inverse Fourier transform as
$$x(t) = \mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f)\ e^{+j2\pi f t}\ df = \int_{-\infty}^{\infty} |X(f)|\ e^{+j(2\pi f t + \varphi(f))}\ df$$
- Assuming $x(t)$ is a real-valued signal this can be written as
$$x(t) = 2\int_{0}^{\infty} |X(f)|\ \cos(2\pi f t + \varphi(f))\ df$$
- **Interpretation:** any signal $x(t)$ can be decomposed into sinusoidal components at different frequencies and phase offsets
- The magnitude $|X(f)|$ measures the strength of the signal component at frequency f

Example: rectangular pulse

- Let us compute the Fourier transform of the following signal:
$$x_{rec}(t) = \begin{cases} A & -\frac{T}{2} \le t \le \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}$$
- We get
$$X_{rec}(f) = \mathcal{F}\{x_{rec}(t)\} = \int_{-\infty}^{\infty} x_{rec}(t)\ e^{-j2\pi f t}\ dt$$
$$= \int_{-T/2}^{+T/2} A\ e^{-j2\pi f t}\ dt\ = \left[-\frac{Ae^{-j2\pi f t}}{j2\pi f}\right]_{-T/2}^{+T/2}$$
$$= \frac{A}{\pi f}\ \frac{e^{j\pi f T} - e^{-j\pi f T}}{2j} = AT\ \frac{\sin(\pi f T)}{\pi f T}$$
- We have found that
$$x_{rec}(t) \longleftrightarrow AT\ \frac{\sin(\pi f T)}{\pi f T} = AT\,\mathrm{sinc}(fT)$$

Notation: $x(t) \longleftrightarrow \mathcal{F}\{x(t)\}$

Example 2.17: sketch of $X_{rec}(f)$



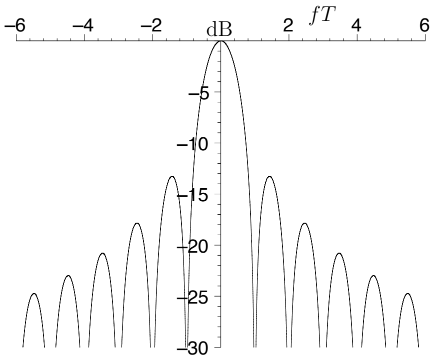
- the Fourier transform $X(f)$ is centered around $f=0$: baseband
- we observe a main-lobe and several side-lobes
- Note: $fT=2$ means that $f=2\cdot 1/T$

Sketch the function for $T=10^{-6}\,s$ and $T=2\cdot 10^{-6}\,s$

Example 2.17: sketch of $|X_{rec}(f)|^2$

- Consider now the normalized energy spectrum in dB

$$10\log_{10}\left(\frac{|X_{rec}(f)|^2}{E_x T}\right)=10\log_{10}\left(\frac{\sin(\pi f T)}{\pi f T}\right)^2$$



$\Rightarrow$ most energy is contained in the main-lobe (90.3 %)

Fourier transform of time-shifted signals

- Did you notice the difference between $x_{rec}(t)$ in this example and the elementrary pulse $g_{rec}(t)$ which we used last week?

$$x_{rec}(t)=\begin{cases}A & -\frac{T}{2}\le t\le\frac{T}{2}\\ 0 & \text{otherwise}\end{cases}\,,\qquad g_{rec}(t)=\begin{cases}A & 0\le t\le T\\ 0 & \text{otherwise}\end{cases}$$

- The pulse $g_{rec}(t)=x_{rec}(t-T/2)$ is a time-shifted version of $x_{rec}(t)$
- In general, the Fourier transform of a signal $y(t)=x(t-t_d)$ with a constant delay t_d becomes

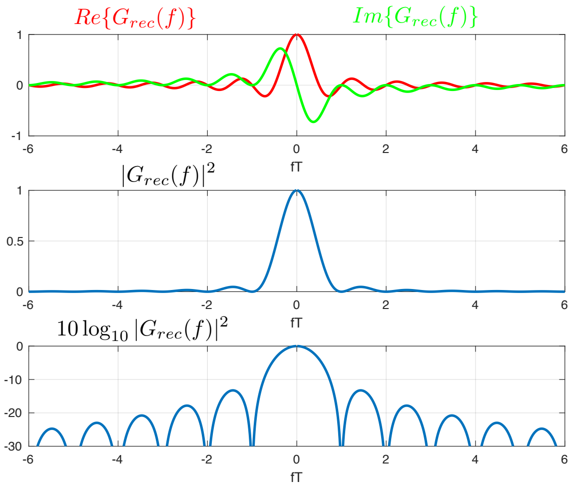
$$Y(f)=\int_{-\infty}^{\infty}x(t-t_d)\,e^{-j2\pi f t}\,dt=\int_{-\infty}^{\infty}x(\tau)\,e^{-j2\pi f(\tau+t_d)}\,d\tau=X(f)\,e^{-j2\pi f t_d}$$

- **Observe:** the delay t_d changes only the phase of $Y(f)$
- The energy spectrum is not affected by time-shifts

$$|X_{rec}(f)|^2=|G_{rec}(f)|^2 \quad \text{(compare App. D.1)}$$

A simple Matlab exercise

Let us plot the spectrum of the pulse $g_{rec}(t)$



A simple Matlab exercise

And this is how it was done:

```
% Example: rect pulse spectrum


x=-6:0.01:6;
G=sin(pi.*x)./(pi.*x).*exp(-j*pi*x); % T=1

figure(2)
subplot(3,1,1);
plot(x,real(G),'r',x,imag(G),'g'); xlabel('fT');
grid on;

subplot(3,1,2);
plot(x,abs(G).^2); xlabel('fT');
grid on;

subplot(3,1,3);
plot(x,10.*log10(abs(G).^2)); xlabel('fT');
set(gca,'YLim',[-30 0]);
grid on;
```

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Fourier transform of other pulses

- The Fourier transforms $G(f)$ and sketches of the energy spectra $|G(f)|^2$ are given for a number of different elementrary pulses $g(t)$ in Appendix D
- **Example:** half cycle sinusoidal pulse

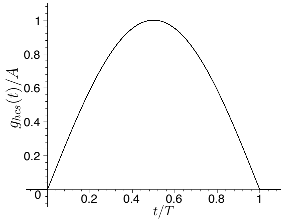


Figure D.7: $g_{hcs}(t)/A$.

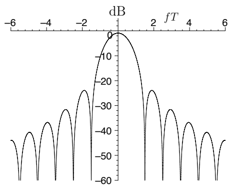


Figure D.8: $\frac{|G_{hcs}(f)|^2}{E_g T}$ in dB.

$$g_{hcs}(t) = \begin{cases} A\sin(\pi t/T) & , \quad 0 \le t \le T \\ 0 & , \quad \text{otherwise} \end{cases}$$

$$E_g = A^2 T/2$$

$$G_{hcs}(f) = \mathcal{F}\{g_{hcs}(t)\} = \frac{2AT}{\pi}\,\frac{\cos(\pi f T)}{1-(2fT)^2}\,e^{-j\pi f T}$$

$$G_{hcs}(f = \pm 1/2T) = \mp jAT/2$$

$$G_{hcs}(n/T) = 0 \text{ if } n = \pm 3/2, \pm 5/2, \pm 7/2, \ldots$$

Frequency shift operations

- We have seen the effect of a time shift on the Fourier transform

$$g(t-t_d) \longleftrightarrow G(f)\, e^{-j2\pi f t_d}$$

- In a similar way we can characterize a frequency shift f_c by

$$g(t)\, e^{j2\pi f_c t} \longleftrightarrow G(f-f_c)$$

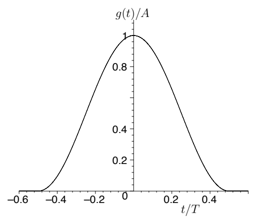
- Let us make use of the relation $e^{j2\pi f_c t} = \cos(2\pi f_c t) + j\, \sin(2\pi f_c t)$
- We can now express this in terms of cosine and sine functions,

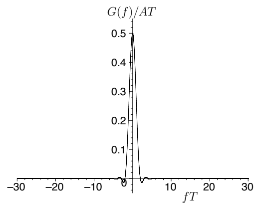
$$g(t)\, \cos(2\pi f_c t) \longleftrightarrow \frac{G(f+f_c)+G(f-f_c)}{2}$$

$$g(t)\, \sin(2\pi f_c t) \longleftrightarrow j\,\frac{G(f+f_c)-G(f-f_c)}{2}$$

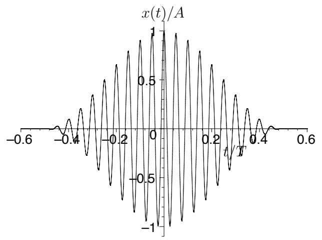
$\Rightarrow$ by simply changing the carrier frequency f_c we can move our signals to a suitable location along the frequency axis

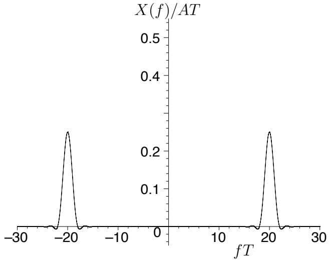
Example: time raised cosine pulse





$$x(t) = g(t)\cdot\cos(2\pi f_c t) = g_{rc}(t+T/2)\cdot\cos(2\pi f_c t)\ , \quad f_c = 20/T$$





Back to the transmitted signal

- We have seen how the Fourier transform can be used to calculate the energy spectrum $|X(f)|^2$ of a given signal $x(t)$
- Let us now look at the transmitted signal for M-ary modulation

$$s(t) = s_{m[0]}(t) + s_{m[1]}(t-T_s) + s_{m[2]}(t-2T_s) + \cdots = \sum_{i=0}^{\infty} s_{m[i]}(t-iT_s)$$

- Message $m[i]$ selects the signal alternative to be sent at time iT_s
- Since the information bit stream is random, the transmitted signal $s(t)$ consists of a sequence of random signal alternatives

How can we determine the bandwidth W of the transmitted signal?

Power Spectral Density

- Since the signal has no predefined length the energy is not a good measure (could be infinite according to our model)
- On the other hand, we know that the signal has finite power
- The power spectral density $R(f)$ shows how the average signal power $\overline{P}$ is distributed along the frequency axis on average

$$\overline{P} = \overline{E}_b R_b = \int_{-\infty}^{\infty} R(f)\, df$$

- Most of the average signal power $\overline{P}$ $[\mathrm{V}^2]$ will be contained within the main-lobe of $R(f)$ $[\mathrm{V}^2/\mathrm{Hz}]$

  $\Rightarrow$ we can determine the signal bandwidth from $R(f)$

Our aim is to find $R(f)$ for a given modulation order M and set of M signal alternatives (constellation)

Power Spectral Density

- The random M-ary sequence of messages $m[i]$ consists of independent, identically distributed (i.i.d) M-ary symbols
- The probability for each of the $M = 2^k$ symbols (messages) is denoted by $P_\ell, \ell = 0, 1, \ldots, M-1$
- All signal alternatives $s_\ell(t)$ in the constellation have finite energy
- The average signal over all signal alternatives is denoted $a(t)$, i.e.,

$$a(t) = \sum_{\ell=0}^{M-1} P_\ell \, s_\ell(t)$$

and its Fourier transform is

$$A(f) = \sum_{n=0}^{M-1} P_n \, S_n(f)$$

Remark:
Source coding (compression) can be used to remove or reduce correlations in the information stream

$R(f)$: Main Result

- The power spectral density $R(f)$ can be divided into a continuous part $R_c(f)$ and a discrete part $R_d(f)$

$$R(f) = R_c(f) + R_d(f)$$

- The general expression for the continuous part is

$$R_c(f) = \frac{1}{T_s} \sum_{n=0}^{M-1} P_n \, |S_n(f) - A(f)|^2 = \left(\frac{1}{T_s} \sum_{n=0}^{M-1} P_n \, |S_n(f)|^2 \right) - \frac{|A(f)|^2}{T_s}$$

- For the discrete part we have

$$R_d(f) = \frac{|A(f)|^2}{T_s^2} \sum_{n=-\infty}^{\infty} \delta(f - n/T_s)$$

$R(f)$: Main Result

- Assume now that the average signal $a(t) = 0$ for all t
- It follows that $A(f) = 0$ for all f
- This simplifies the result to

$$R(f) = R_c(f) = R_s \sum_{n=0}^{M-1} P_n \, |S_n(f)|^2 = R_s \, E\{|S_{m[n]}(f)|^2\}$$

- These **general results** can also be used to study the consequences that technical errors or impairments in the transmitter can have on the frequency spectrum
- We will now consider various **special cases** used in practice

$R(f)$: Binary Signaling

- In the general binary case, i.e., $M = 2$, we have

$$A(f) = P_0 \, S_0(f) + P_1 \, S_1(f)$$

- This simplifies the expression for the power spectral density to

$$R(f) = R_c(f) + R_d(f) = \frac{P_0 P_1}{T_b} |S_0(f) - S_1(f)|^2 + \frac{|P_0 \, S_0(f) + P_1 \, S_1(f)|^2}{T_b^2} \sum_{n=-\infty}^{\infty} \delta(f - n/T_b)$$

(derivation in Ex. 2.20)

- We will now consider some examples from the compendium

Example 2.21

Assume equally likely antipodal signal alternatives, such that

$$s_1(t) = -s_0(t) = g(t)$$

where $g(t) = g_{rec}(t)$, and $g_{rec}(t)$ is given in (D.1). Assume also that $T \leq T_b$.

i) Calculate the power spectral density $R(f)$.

*ii) Calculate **the bandwidth W defined as the one-sided width of the mainlobe of $R(f)$**, if the information bit rate is 10 [kbps], and if $T = T_b/2$. Calculate also the bandwidth efficiency ρ.*

iii) Estimate the attenuation in dB of the first sidelobe of $R(f)$ compared to $R(0)$.

- $M = 2$ with equally likely antipodal signaling $s_1(t) = -s_0(t) = g(t)$
- With $P_0 = P_1 = 1/2$ and $S_1(f) = -S_0(f) = G(f)$ we get

$$R(f) = R_b \, |S_1(f)|^2 = R_b \, |S_0(f)|^2 = R_b \, |G(f)|^2$$

- Details for the pulse in Appendix D

Example 2.23

Assume equally likely antipodal signal alternatives below. Assume that $s_1(t) = -s_0(t) = g_{rc}(t)$, where the time raised cosine pulse $g_{rc}(t)$ is defined in (D.18). Assume also that $T = T_b$.

Find an expression for the power spectral density $R(f)$. Calculate the bandwidth W, defined as the one-sided width of the mainlobe of $R(f)$, if R_b is 10 [kbps]. Calculate also the bandwidth efficiency ρ.

- Same as Example 2.21, but with $g_{rc}(t)$ pulse
- Analogously we get

$$R(f) = R_b \, |G_{rc}(f)|^2$$

- From the one-sided main-lobe we get

$$W = 2/T \text{ [Hz]}$$

- Bandwidth efficiency $\rho = 1/2$ [bps/Hz] is the same (why?)

Example 2.24

Assume $P_0 = P_1$ *and that,*

$$s_1(t) = -s_0(t) = g_{rc}(t)\cos(2\pi f_c t)$$

with $T = T_b$, *and* $f_c \gg 1/T$. *Hence, a version of binary PSK signaling is considered here (alternatively binary antipodal bandpass PAM). Calculate* ***the bandwidth W, defined as the double-sided width of the mainlobe around the carrier frequency*** $\boldsymbol{f_c}$. *Assume that the information bit rate is 10* [kbps]. *Calculate also the bandwidth*

- This corresponds to the bandpass case
- Let $g_{hf}(t)$ denote the high-frequency pulse

$$g_{hf}(t) = g_{rc}(t)\cos(2\pi f_c t) \quad \text{and} \quad R(f) = R_b\,|G_{hf}(f)|^2$$

- Using shift operations we get

$$R(f) = R_b\left|\frac{G_{rc}(f+f_c)}{2} + \frac{G_{rc}(f-f_c)}{2}\right|^2$$

- From the two-sided main-lobe we get

$$W = 4/T \text{ [Hz]}$$