LUND UNIVERSITY

EITG05 – Digital Communications

Week 1, Lecture 2

Signal Constellations (p. 31–55)

Michael Lentmaier
Thursday, August 31, 2017

Week 1, Lecture 2

Chapter 2: Model of a Digital Communication System

- 2.4 Signal constellations
  - 2.4.1 Pulse amplitude modulation (PAM)
  - 2.4.2 Phase shift keying (PSK)
  - 2.4.3.1 Frequency shift keying (FSK)
  - 2.4.4 Pulse position modulation (PPM)
  - 2.4.5 Quadrature amplitude modulation (QAM)
  - 2.4.6 Pulse width modulation (PWM)
  - 2.4.7.1 Multitone signaling: OFDM

Pages 31 – 55 (excluding 2.4.3.2)

Exercises: Problems 2.11, 2.12, 2.13, 2.14a, 2.28, 2.15

Signal constellations

- In case of M-ary signaling, one of $M = 2^k$ messages $m[i]$ is transmitted by its corresponding signal alternative

$$s_\ell(t) \in \{s_0(t), s_1(t), \dots, s_{M-1}(t)\}$$

- When the message equals $m[i] = j$ then $s_j(t - iT_s)$ is sent

$$s(t) = s_{m[0]}(t) + s_{m[1]}(t - T_s) + s_{m[2]}(t - 2T_s) + \cdots$$

- The signal constellation is the set of possible signal alternatives
- The mapping defines which message is assigned to which signal

Question: how should we choose the different signals?

Pulse Amplitude Modulation (PAM)

- In pulse amplitude modulation the message is mapped into the amplitude only:

$$s_\ell(t) = A_\ell\, g(t)\ , \quad \ell = 0, 1, \dots, M-1$$

- PAM is a natural generalization of binary on-off signaling and antipodal signaling, which are special cases for $M = 2$
- A common choice are equidistant amplitudes located symmetrically around zero:

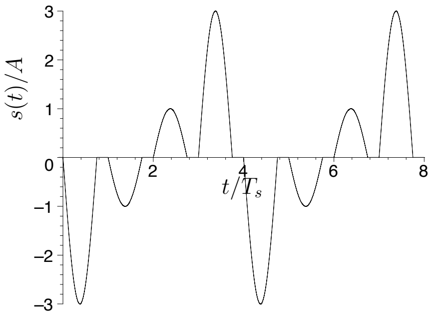
$$A_\ell = -M + 1 + 2\ell\ , \quad \ell = 0, 1, \dots, M-1$$

- **Example:** $M = 4$, $A_0 = -3$, $A_1 = -1$, $A_2 = +1$, $A_3 = +3$
- The same constellation

$$\{A_\ell\}_{\ell=0}^{M-1} = \{\pm 1, \pm 3, \pm 5, \dots, \pm(M-1)\}$$

could also be used with other mappings

Example of 4-ary PAM



$$A_\ell = -M + 1 + 2\ell\ , \quad \ell = 0, 1, \ldots, M-1$$

What is the message sequence $m[i]$?

Symbol Energy of PAM

- The symbol energy of a PAM signal is

$$E_\ell = \int_0^{T_s} s_\ell^2(t)\ dt = \int_0^{T_s} A_\ell^2\ g^2(t)\ dt$$

- Using

$$E_g = \int_0^{T_s} s_\ell^2(t)\ dt$$

we can write the average symbol energy as

$$\overline{E}_s = E_g \sum_{\ell=0}^{M-1} P_\ell\ A_\ell^2$$

- Often the messages are equally likely, i.e., $P_\ell = \frac{1}{M} = 2^{-k}$, and for the symmetric constellation from above we get

$$\overline{E}_s = E_g \frac{M^2-1}{3}\ .$$

Euclidean distances of PAM signals

- The squared Euclidean distance between two PAM signal alternatives is

$$D_{i,j}^2 = \int_0^{T_s} (s_i(t) - s_j(t))^2\ dt = E_g\ (A_i - A_j)^2$$

- With $P_\ell = \frac{1}{M}$ and $A_\ell = -M + 1 + 2\ell$ this becomes

$$D_{i,j}^2 = 4E_g\ (i-j)^2$$

Compare this with Example 2.7 on page 28

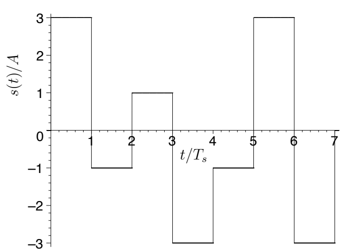
- We will later see that the minimum Euclidean distance $\min_{i,j} D_{i,j}$ strongly influences the error probability of the receiver
- For this reason, equidistant constellations are often used

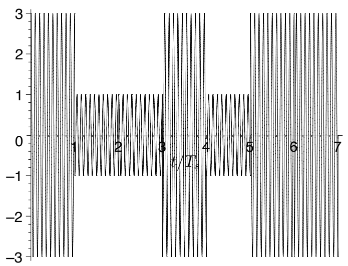
Bandpass M-ary PAM

- In many applications we want to transmit signals at high frequencies, centered around a carrier frequency f_c
- To achieve this, we can multiply the original PAM signal $s(t)$ with a sinusoidal signal (more about this we see in Chap. 3)

$$s_{bp}(t) = s(t) \cdot \cos(2\pi\, f_c\ t) = \sum_{i=0}^{\infty} A_{m[i]}\ g(t - i\ T_s) \cdot \cos(2\pi\, f_c\ t)$$

Example:





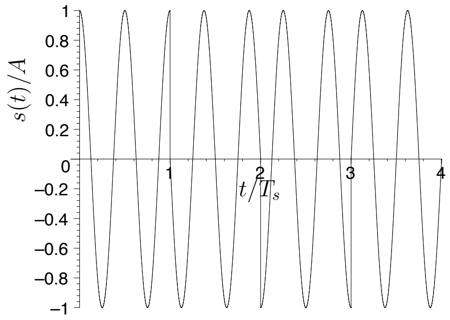
Phase Shift Keying (PSK)

- We have seen that with PAM signaling the message modulates the amplitude A_ℓ of the signal $s_\ell(t)$
- The idea of phase shift keying signaling is to modulate instead the phase ν_ℓ of $s_\ell(t)$

$$s_\ell(t) = g(t)\ \cos(2\pi f_c\ t\ +\ \nu_\ell)\ , \quad \ell = 0, 1, \ldots, M-1\ ,$$

- If we choose
$$f_c = n\ R_s$$
for some positive integer n, then n full cycles of the carrier wave are contained within a symbol interval T_s
- $M = 2$**:** binary PSK (BPSK) with $\nu_0 = 0$ and $\nu_1 = \pi$ is equivalent to binary PAM with $A_0 = +1$ and $A_1 = -1$
- $M = 4$**:** 4-ary PSK is also called quadrature PSK (QPSK)

Example of QPSK



$$f_c = 2\ R_s\ , \quad \nu_0 = 0,\ \nu_1 = \pi/2,\ \nu_2 = \pi,\ \text{and}\ \nu_3 = 3\pi/2$$

What is the message sequence $m[i]$?

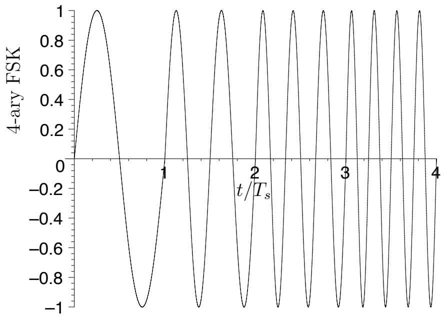
Symmetric M-ary PSK

- Normally, the phase alternatives are located symmetrically on a circle
$$\nu_\ell = \frac{2\pi\ \ell}{M}\ +\ \nu_{const}\ , \quad \ell = 0, 1, \ldots, M-1\ ,$$
where ν_{const} is a contant phase offset value
- If $P_\ell = \frac{1}{M}$, and $f_c \gg R_s$, then the average symbol energy is
$$\overline{E}_s = \frac{E_g}{2}$$
and
$$D^2_{i,j} = E_g\ (1 - \cos(\nu_i - \nu_j))$$
- PSK has a constant symbol energy

Frequency Shift Keying (FSK)

- Instead of amplitude and phase, the message can modulate the frequency f_ℓ
$$s_\ell(t) = A\ \cos(2\pi f_\ell\ t\ +\ \nu)\ , \quad \ell = 0, 1, \ldots, M-1$$
- Amplitude A and phase ν are constants
- In many applications the frequency alternatives f_ℓ are chosen such that the signals are orthogonal, i.e.,
$$\int_0^{T_s} s_i(t)\ s_j(t)\ dt = 0\ , \quad i \neq j$$
- If $\nu = 0$ or $\nu = -\pi/2$ (often used), then we can choose
$$f_\ell = n_0 \frac{R_s}{2} + \ell\ I\ \frac{R_s}{2} \stackrel{\text{def}}{=} f_0 + \ell\ f_\Delta\ , \quad \ell = 0, 1, \ldots, M-1\ ,$$
where n_0 and I are positive integers

Example of 4-ary FSK



$$v = -\frac{\pi}{2}\ , \quad f_0 = R_s,\ f_1 = 2R_s,\ f_2 = 3R_s,\ \text{and}\ f_3 = 4R_s$$

What is the message sequence $m[i]$?

Energy and Distance of M-ary FSK

- Choosing $f_\ell \gg R_s$ and the orthogonal frequency alternatives form above we get

$$\overline{E}_s = \frac{A^2\ T_s}{2}$$

$$D_{i,j}^2 = E_i + E_j = A^2\ T_s$$

- Observe that $\overline{E}_s$ is the same as for M-ary PSK with $E_g = A^2\ T_s$
- A special property of FSK is that the Euclidean distance $D_{i,j}$ is the same for any pair (i,j) of signals
- This means that we can increase M (and thus the bit rate R_b) without increasing the error probability

What happens with the bandwidth W if M increases?

Pulse Position Modulation (PPM)

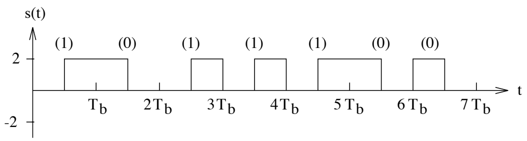
- In pulse position modulation the message modulates the position of a short pulse $c(t)$ within the symbol interval T_s

$$s_\ell(t) = c\left(t - \ell\ \frac{T_s}{M}\right)\ , \quad \ell = 0, 1, \ldots, M-1$$

- The duration T of the pulse $c(t)$ has to satisfy $T \leq T_s/M$
- The pulses are orthogonal and we get

$$\overline{E}_s = E_c\ , \qquad D_{i,j}^2 = E_i + E_j = 2\ E_c$$

Example:



Used for low-power optical links (e.g. IR remote controls)

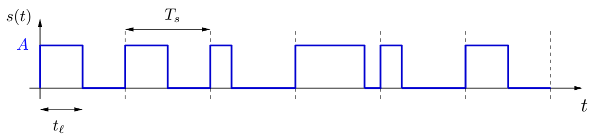
Pulse Width Modulation (PWM)

- In pulse width modulation the message modulates the duration T of a pulse $c(t)$ within the symbol interval T_s

$$s_\ell(t) = c\left(\frac{t}{t_\ell}\right)\ , \quad \ell = 0, 1, \ldots, M-1$$

- The duration of the pulse $c(t)$ is equal to $T = 1$
- It follows that $s_\ell(t)$ is zero outside the interval $0 \leq t \leq t_\ell$
- It is assumed that $t_\ell < T_s$
- Average symbol energy: $\overline{E}_s = E_c\ \overline{t}_\ell$

Example:



Used in control applications, not much for data transmission (e.g., speed of CPU fan, LED intensity)

Quadrature Amplitude Modulation (QAM)

- With QAM signaling the message modulates the amplitudes of two orthogonal signals (inphase and quadrature component)

$$s_\ell(t) = A_\ell \; g(t) \; \cos(2\pi f_c \; t) - B_\ell \; g(t) \; \sin(2\pi f_c \; t) \; , \quad \ell = 0, 1, \ldots, M-1$$

- We can interpret $s_\ell(t)$ as the sum of two bandpass PAM signals
- **Motivation:** We can transmit two signals independently using the same carrier frequency and bandwidth
- The signal $s_\ell(t)$ can also be expressed as

$$s_\ell(t) = g(t)\sqrt{A_\ell^2 + B_\ell^2} \; \cos(2\pi f_c \; t \; + \; \nu_\ell)$$

- It follows that QAM is a generalization of PSK:

selecting $A_\ell^2 + B_\ell^2 = 1$ we can put the information into ν_ℓ and get

$$A_\ell = \cos(\nu_\ell) \; , \quad B_\ell = \sin(\nu_\ell)$$

Energy and Distance of M-ary QAM

- Choosing $f_c \gg R_s$ it can be shown that

$$E_\ell = \left(A_\ell^2 + B_\ell^2\right) \frac{E_g}{2}$$

$$D_{i,j}^2 = \left((A_i - A_j)^2 + (B_i - B_j)^2\right) \frac{E_g}{2}$$

- A common choice are equidistant amplitudes located symmetrically around zero: (two $\sqrt{M}$-ary PAM with $k/2$ bits each)

$$\{A_\ell\}_{\ell=0}^{\sqrt{M}-1} = \{B_\ell\}_{\ell=0}^{\sqrt{M}-1} = \left\{\pm 1, \pm 3, \pm 5, \ldots, \pm\left(\sqrt{M}-1\right)\right\}$$

- For equally likely messages $P_\ell = \frac{1}{M}$, this results in the average energy

$$\overline{E}_s = \sum_{\ell=0}^{M-1} \frac{1}{M} \; E_\ell = \frac{2(M-1)}{3} \; \frac{E_g}{2}$$

Geometric interpretation

- It is possible to describe QAM signals as two-dimensional vectors in a so-called signal space
- For this the signal

$$s_\ell(t) = A_\ell \; g(t) \; \cos(2\pi f_c \; t) - B_\ell \; g(t) \; \sin(2\pi f_c \; t)$$

is written as

$$s_\ell(t) = s_{\ell,1} \; \phi_1(t) + s_{\ell,2} \; \phi_2(t)$$

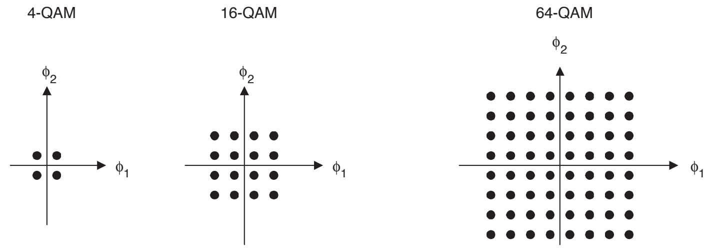
- Here $s_{\ell,1} = A_\ell\sqrt{E_g/2}$ and $s_{\ell,2} = B_\ell\sqrt{E_g/2}$ are the coordinates
- The functions $\phi_1(t)$ and $\phi_2(t)$ form an orthonormal basis of a vector space that spans all possible transmit signals:

$$\phi_1(t) = \frac{g(t) \; \cos(2\pi f_c \; t)}{\sqrt{E_g/2}} \; , \quad \phi_2(t) = \frac{g(t) \; \sin(2\pi f_c \; t)}{\sqrt{E_g/2}}$$

This looks abstract, but can be very useful!

Signal space representation of QAM

- Now we can describe each signal alternative $s_\ell(t)$ as a point with coordinates $(s_{\ell,1}, s_{\ell,2})$ within a constellation diagram

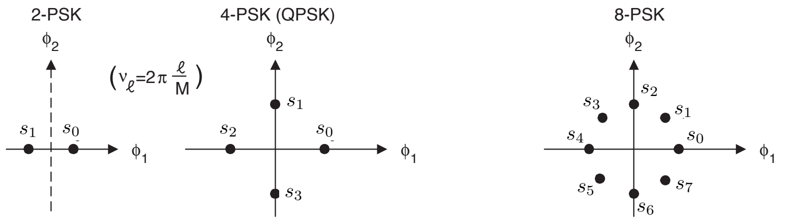


$$s_{\ell,1} = A_\ell\sqrt{E_g/2} \; , \quad s_{\ell,2} = B_\ell\sqrt{E_g/2}$$

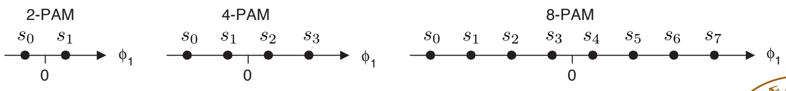
- The signal energy E_ℓ and the Euclidean distance $D_{i,j}^2$ can be determined in the signal space

Signal space representation of PSK and PAM

- PSK and PAM can be seen as a special cases of QAM:



$$s_{\ell,1} = \cos(\nu_\ell)\sqrt{E_g/2}\ , \qquad s_{\ell,2} = \sin(\nu_\ell)\sqrt{E_g/2}$$



$$s_{\ell,1} = (-M+1+2\,\ell)\,\sqrt{E_g}$$

Multitone Signaling: OFDM

- With FSK signaling, orthogonal signal alternatives are transmitted at different frequencies
- Disadvantage: only one frequency can be used at the same time
- **Orthogonal Frequency Division Multiplexing (OFDM)**: use QAM at N orthogonal frequencies and transmit the sum
- OFDM is widely used in modern communication systems: WLAN, LTE, DAB (radio), DVB (TV), DSL

Example:

$N = 4096$
64-ary QAM at each frequency (carrier)

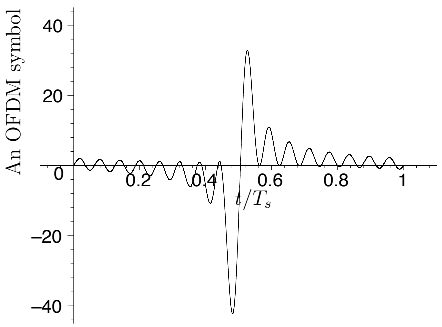
Then an OFDM signal carries $4096 \cdot 6 = 24\,576$ bits

How does a typical OFDM signal look like?

How can such a system be realized in practice?
$\Rightarrow$ OFDM will be explained in detail in the advanced course

Example of an OFDM symbol

$N = 16$, 16-ary QAM in each subcarrier (p. 52)



$$x(t) = \sum_{n=0}^{N-1} \left(a_I[n]\; g(t)\; \cos(2\pi f_n\, t) - a_Q[n]\; g(t)\; \sin(2\pi f_n\, t)\right)\ , \qquad 0 \le t \le T_s$$

In this example the symbol $x(t)$ carries $16 \cdot 4 = 64$ bits